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Models of Elastic Plates with Piezoelectric Inclusions Part I: Models without Homogenization

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Abstract—In this paper, we present models of elastic plates, dielectric plates, and elastic plates with piezoelectric inclusions. Single layer plate models and multilayered plate models are presented. They are studied in view of the active controlled structures design. Various sorts of boundary conditions for the piezoelectric cells are considered. They represent different ways of controlling the structure: voltage, current, or voltage/current control. The derivation of these models is based on asymptotic methods.

Keywords—Electrical circuits, Piezoelectricity, Singular perturbations, Plate models.

1. INTRODUCTION

For more than ten years, the field of Smart Materials has been an area of intensive research. The first principle of Smart Materials consists of replacing mass by energy. Energy is easier to carry than material. Smart Materials are equipped with transductors which are transforming this carriable energy into mechanical energy, and with a system transporting this energy. The more usual examples are the elastic thin structures (plates, shells, or beams) with piezoelectric inclusions. The second principle of Smart Materials consists of including a certain form of smartness. With the progress in analogical electronics, one may think that it is now possible to design distributed analogical electrical circuits included in the structures. The electrical circuit may link the piezoelectric inclusions and acts as a command.

On the one hand, numerous authors have obtained different models of structures coupling elastic and piezoelectric materials [1–6]. On the other hand, for fifteen years, classical models of thin elastic structures have been justified by an asymptotic approach [7–11], for example. The first goal of this work is to justify some plate models including piezoelectric transductors using an asymptotic method based on [9,10].

By another way, homogenization techniques [12–15] lead to models of composite materials with periodic structures. Plates including periodically distributed heterogeneity have been derived in [8,16–18]. The second goal of this work is to derive models of elastic plates including small piezoelectric inclusions. This will be done in Part II.

The third goal is the treatment of several boundary conditions for the inclusions. Three classes of boundary conditions are considered: Dirichlet boundary conditions when the electrical potential is controlled, Neumann boundary conditions when current is controlled, and mixed boundary

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conditions when the inclusions are connected to an analogical circuit with R-L-C devices. Two sorts of mixed boundary conditions are considered: the case where the upper and lower surfaces of each inclusion are connected (but with no connection between inclusions) and the case where the circuit links several inclusions. In our models, an inclusion may be connected only with its direct neighbours. These neighbours can be in the in-plane (considered only in Part II) or in the out-plane direction. This leads to nonlocal boundary conditions. In practice, the piezoelectric faces are covered with a conductive metal. This implies that the tangent electrical field vanishes on each metallized face. Then, the electrical potential is constant on such faces. Models concerning both metallized and nonmetallized faces are derived.

Last, let us point out our technique of models derivation. The approach [10] is based on a stress-displacement formulation of the elasticity equations. The approach of [9,11] are based on a displacement formulation. In this paper, we follow the displacement approach concerning *a priori* estimates. But the model derivation, which is obtained by eliminating some of the fields, is done in the space of the gradients of the solutions. It results in some projection operations. This leads to fully algebraic computations. This procedure simplifies significantly the model derivation (which otherwise needs rather long formulas). The resulting formulae are especially interesting in view of numerical computation. They are based on sums, products, inverses of matrices, and on projection maps. Even for well-known models, our approach leads to appreciable simplifications in their formulation.

This first part of our work is organized as follows. Section 2 is devoted to the derivation of the elastic plate model. This model is classical, but the formulation and convergence properties established for this derivation are used in Section 4. Section 3 is devoted to the derivation of models of dielectric thin plates with various boundary conditions. In Section 4, models of elastic thin plates with piezoelectric inclusions are derived. The proofs in Section 4 are based on the results of Sections 2 and 3. For the sake of clarity, we choose to present them only for single layered plates. The models of multilayered plates are stated. Their proofs are not reported, but they are very similar to those of the single layered plate models. Discussion of models is done on the multilayered case because in practice they are the most often considered.

Even if some models of Sections 2 and 3 are not new, we present them for two reasons. The first one is, as we said before, that the difficulties in the proofs of Section 4 (due to notations) become simple to understand after the reading of Sections 2 and 3. The second reason is that the presentation with the same formalism of all these models allows easy comparisons.

2. THE ELASTIC THIN PLATE MODEL

2.1. The Single Layered Plate Model

2.1.1. Statement of the three-dimensional plate geometry and equations

First, we introduce the notations for geometry and equations relative to thin elastic plates. They are taken from [9]. The plate is represented by a cylindrical domain $\Omega^a = \omega \times]-a,a[$, where ω is the mean section of the plate, its boundary is $\partial\omega$.

- The thickness of the plate is equal to $2a$ and a is considered as a small parameter which is intended to tend to zero.
- $x^a = (x_1^a, x_2^a, x_3^a)$ is the current point in Ω^a, $(x_1^a, x_2^a) \in \omega$, and $x_3^a \in]-a,a[$.
- The lateral boundary $\Gamma^a = \partial\omega \times]-a,a[$ of Ω^a is divided into two parts $\Gamma_0^a = \gamma_0 \times]-a,a[$ and $\Gamma_1^a = \gamma_1 \times]-a,a[$ such that meas$(\gamma_0) \neq 0$.
- Γ^{a+} and Γ^{a-} represent, respectively, the upper and the lower faces of the plate $\Gamma^{a\pm} = \Gamma^{a+} \cup \Gamma^{a-}$.
- The stiffness tensor $\mathbf{R} = (R_{ijkl})_{i,j,k,l=1,\dots,3} \in (L^\infty(\Omega^a))^{81}$ is assumed to be independent of x_3 and verifies the following symmetry:

$$R_{ijkl} = R_{klij} = R_{jikl}, \qquad \forall\, i,j,k,l = 1,\dots,3, \tag{1}$$

and the ellipticity property:

$$R_{ijkl}K_{ij}K_{kl} \geq CK_{ij}^2, \qquad \forall\ (K_{ij})_{i,j=1,\dots,3} \in \mathbb{R}^9 \text{ such that } K_{ij} = K_{ji}, \tag{2}$$

where C is a positive constant. In all the paper, the Einstein convention of summation on repeated indexes is used, with summation from one to three for Latin indexes and from one to two for Greek indexes.

- The volume forces are denoted by $\mathbf{f}^a = (f_i^a)_{i=1,\dots,3} \in (L^2(\Omega^a))^3$. Surface forces $\mathbf{g}^a = (g_i^a)_{i=1,\dots,3} \in L^2(\Gamma^{a\pm} \cup \Gamma_1^a)$ are applied on $\Gamma^{a\pm} \cup \Gamma_1^a$. The plate is clamped on Γ_0^a.
- The mechanical displacements are denoted by $\mathbf{u}^a = (u_i)_{i=1,\dots,3}$. The linear strains associated with a field of displacements $\mathbf{v} = (v_i)_{i=1,\dots,3}$ are denoted by $s_{ij}(\mathbf{v}) = (1/2)(\partial_i v_j + \partial_j v_i)$. Here, ∂_i denotes the derivation in the direction x_i. The mechanical stress is $\sigma^a = (\sigma_{ij}^a)_{i,j=1,\dots,3}$. The linear elasticity equations are given by the Hooke law, the equilibrium equations and the boundary conditions

$$-\partial_j \sigma_{ij}^a = f_i^a \text{ in } \Omega^a, \qquad \sigma_{ij}^a n_j = g_i^a \text{ on } \Gamma_1^a \cup \Gamma^{a\pm}, \qquad \text{and} \quad \mathbf{u}^a = 0 \text{ on } \Gamma_0^a. \tag{3}$$

2.1.2. Scaling and statement of the theorem

In order to reformulate (3) on the reference domain $\Omega = \omega \times]-1,1[$, we use the transformation $\mathbf{F}^a$:

$$\mathbf{F}^a : \begin{array}{l} \Omega^a \to \Omega \\ \mathbf{x}^a \mapsto \mathbf{F}^a(\mathbf{x}^a) = \mathbf{x} = (x_1, x_2, x_3) = \left(x_1^a, x_2^a, \frac{1}{a}, x_3^a\right). \end{array} \tag{4}$$

The geometric characteristics of Ω are deduced from those of Ω^a with the transformation $\mathbf{F}^a$. The notations relative to Ω are the same as those relative to Ω^a except that the index a is removed.

The following scaling on volume forces, surface forces, and displacements is classical [9]:

$$\begin{aligned} \widehat{\mathbf{u}}^a(\mathbf{x}) &= \left(u_1^a(\mathbf{x}^a), u_2^a(\mathbf{x}^a), a u_3^a(\mathbf{x}^a)\right) \text{ in } \Omega, \\ \widehat{\mathbf{f}}^a(\mathbf{x}) &= \left(f_1^a(\mathbf{x}^a), f_2^a(\mathbf{x}^a), \frac{f_3^a(\mathbf{x}^a)}{a}\right) \text{ in } \Omega, \\ \widehat{\mathbf{g}}^a(\mathbf{x}) &= \left(g_1^a(\mathbf{x}^a), g_1(\mathbf{x}^a), \frac{g_3^a(\mathbf{x}^a)}{a}\right) \text{ on } \Gamma_1, \\ \widehat{\mathbf{g}}^a(\mathbf{x}) &= \frac{1}{a}\left(g_1^a(\mathbf{x}^a), g_1(\mathbf{x}^a), \frac{g_3^a(\mathbf{x}^a)}{a}\right) \text{ on } \Gamma^{\pm}. \end{aligned} \tag{5}$$

The forces are assumed to satisfy

$$\begin{gathered} \left\|\widehat{\mathbf{f}}^a\right\|_{L^2(\Omega)^3}^2 + \|\mathbf{g}^a\|_{L^2(\Gamma_1 \cup \Gamma^\pm)^3}^2 \leq C, \\ \widehat{\mathbf{f}}^a \rightharpoonup \mathbf{f} \text{ in } \left(L^2(\Omega)\right)^3 \text{ weak} \quad \text{and} \quad \widehat{\mathbf{g}}^a \rightharpoonup \mathbf{g} \text{ in } \left(L^2\left(\Gamma_1 \cup \Gamma^\pm\right)\right)^3 \text{ weak}, \end{gathered} \tag{6}$$

when a vanishes. Here $\widehat{\mathbf{f}}^a$ and $\widehat{\mathbf{g}}^a_{|\Gamma_1^a}$ are independent of x_3 (this assumption is done in order to obtain a simpler strong formulation of the plate model). It can easily be released.

For every $\mathbf{v} \in (H^1(\Omega))^3$, the functions $(K_{ij}^a)_{i,j=1,\dots,3}$ are defined by

$$K_{\alpha\beta}^a(\mathbf{v}) = s_{\alpha\beta}(\mathbf{v}), \qquad K_{3\alpha}^a(\mathbf{v}) = K_{\alpha 3}^a(\mathbf{v}) = \frac{1}{a}\, s_{\alpha 3}(\mathbf{v}), \qquad K_{33}^a(\mathbf{v}) = \frac{1}{a^2}\, s_{33}(\mathbf{v}).$$

Let $\mathbf{K}^a(\mathbf{v}) = (K_{ij}^a(\mathbf{v}))_{i,j=1,\dots,3}$ be decomposed as $\mathbf{K}^a(\mathbf{v}) = \mathbf{K}^0(\mathbf{v}) + (1/a)\mathbf{K}^{-1}(\mathbf{v}) + (1/a^2)\,\mathbf{K}^{-2}(\mathbf{v})$, which defines the operators $\mathbf{K}^0$, $\mathbf{K}^{-1}$, $\mathbf{K}^{-2}$. The admissible Love-Kirchhoff displacements space is

$$\begin{aligned} V_{KL} &= \left\{\mathbf{u} \in V_{\text{ad}};\ \mathbf{K}^{-1}(\mathbf{u}) = \mathbf{K}^{-2}(\mathbf{u}) = 0\right\} \\ &= \Big\{\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3) \in V_{\text{ad}}, \\ &\qquad \text{where } (\overline{u}_\alpha)_{\alpha=1,2} \in \left(H^1(\omega)\right)^2 \text{ and } u_3 \in H^2(\omega)\Big\}. \end{aligned} \tag{7}$$

Define a format on the fields of $(L^2(\Omega))^7$ by

$$\left\{\left((K_{\alpha\beta})_{\alpha,\beta=1,\dots,2},(K_{\alpha 3})_{\alpha=1,\dots,2},K_{33}\right)\in(L^2(\Omega))^7;\ K_{12}=K_{21}\right\}. \tag{8}$$

Then, define $\mathbb{K}^0$, $\mathbb{K}^{-1}$, $\mathbb{K}^{-2}$, and $\mathbb{K}$, the subspaces of $(L^2(\Omega))^7$ stored accordingly with the format (8)

$$\begin{aligned}
\mathbb{K}^0 &= \{(K_{\alpha\beta}(\mathbf{v})_{\alpha,\beta=1,2},\mathbf{0},0);\ \mathbf{v}\in V_{KL}\},\\
\mathbb{K}^{-1} &= \left\{(\mathbf{0},(K_{\alpha 3})_{\alpha=1,2},0);\ K_{\alpha 3}\in\left(L^2(\Omega)\right)^2\right\},\\
\mathbb{K}^{-2} &= \{(\mathbf{0},\mathbf{0},K_{33});\ K_{33}\in L^2(\Omega)\},\qquad\text{and}\\
\mathbb{K} &= \mathbb{K}^0\oplus\mathbb{K}^{-1}\oplus\mathbb{K}^{-2}.
\end{aligned} \tag{9}$$

REMARK. Each element of $\mathbb{K}^0$ is associated with a unique element of V_{KL}.

The stiffness tensor $\mathbf{R}$ is stored in a format compatible with $\mathbb{K}$:

$$\mathcal{R}=\begin{pmatrix}(R_{\alpha\beta\gamma\delta})_{\alpha,\beta,\gamma,\delta=1,2} & (2R_{\alpha\beta\gamma 3})_{\alpha,\beta,\gamma=1,2} & (R_{\alpha\beta 33})_{\alpha,\beta=1,2}\\ (2R_{\alpha 3\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (4R_{\alpha 3\gamma 3})_{\alpha,\gamma=1,2} & (2R_{\alpha 333})_{\alpha=1,2}\\ (R_{33\gamma\delta})_{\gamma,\delta=1,2} & (2R_{33\gamma 3})_{\gamma=1,2} & R_{3333}\end{pmatrix}. \tag{10}$$

Let us denote by Π the projector from $(L^2(\Omega))^7$ to $\mathbb{K}^{-1}\oplus\mathbb{K}^{-2}$. Remark that ${}^t\Pi=\Pi$. Let

$$\mathbf{T}=-\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R}\quad\text{ and }\quad\mathbf{Q}=\left(\mathbf{Id}+{}^t\mathbf{T}\right)\mathcal{R}\left(\mathbf{Id}+\mathbf{T}\right). \tag{11}$$

Note that (1) and (2) guarantee that $\Pi\mathcal{R}\Pi$ is invertible on $\mathbb{K}^{-1}\oplus\mathbb{K}^{-2}$.

Define last

$$\begin{pmatrix}\mathbf{Q}^{11} & \mathbf{Q}^{12}\\ \mathbf{Q}^{21} & \mathbf{Q}^{22}\end{pmatrix}=\int_{-1}^{1}\begin{pmatrix}\mathbf{Q} & -x_3\mathbf{Q}\\ -x_3\mathbf{Q} & x_3^2\mathbf{Q}\end{pmatrix}dx_3.$$

THEOREM 2.1. *Assume that (1), (2), and (6) hold. Then the sequence $(\widehat{u}^a)_{a>0}$ associated with the weak solutions $(u^a)_{a>0}$ of (3) by formulae (5) converges weakly in $H^1(\Omega)$ to the unique weak solution $\mathbf{u}\in V_{KL}$ of*

$$\begin{aligned}
-\partial_\beta\left(Q^{11}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{u})+Q^{12}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right) &= \int_{-1}^{+1} f_\alpha\,dx_3+g_\alpha(1)+g_\alpha(-1)\ \text{in }\omega,\\
\partial^2_{\alpha\beta}\left(Q^{21}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{u})+Q^{22}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right) &= \int_{-1}^{+1}\left(x_3\partial_\alpha f_\alpha+f_3\right)dx_3\\
&\quad+\left(\partial_\alpha g_\alpha\right)(1)-\left(\partial_\alpha g_\alpha\right)(-1)+g_3(1)+g_3(-1)\ \text{in }\omega,
\end{aligned} \tag{12}$$

with the following boundary conditions:

$$\begin{aligned}
&\left(Q^{11}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{\mathbf{u}})+Q^{12}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right).n_\beta=\int_{-1}^{+1} g_\alpha\,dx_3\ \text{on }\gamma_1,\\
&-\frac{\partial}{\partial\tau}\left(\left(Q^{21}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{\mathbf{u}})+Q^{22}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right).n_\alpha\tau_\beta\right)-\partial_\alpha\left(Q^{21}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{\mathbf{u}})+Q^{22}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right).n_\beta\\
&=\int_{-1}^{1}\frac{\partial}{\partial\tau}(\mathbf{g}.\tau)\,x_3+x_3\,(\mathbf{f}.\mathbf{n})\,dx_3-(\mathbf{g}.\mathbf{n})\,(+1)+(\mathbf{g}.\mathbf{n})\,(-1)\ \text{on }\gamma_1,\\
&\left(Q^{21}_{\alpha\beta\gamma\delta}s_{\gamma\delta}(\overline{u})+Q^{22}_{\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right).n_\alpha n_\beta=-\int_{-1}^{1}\mathbf{g}.\mathbf{n}x_3\,dx_3\ \text{on }\gamma_1,\\
&\overline{u}_1=\overline{u}_2=u_3=\frac{\partial u_3}{\partial\mathbf{n}}=0\ \text{on }\gamma_0,
\end{aligned} \tag{13}$$

where $\mathbf{n}$ and τ denote the normal and tangent vectors to the boundary of ω.

REMARK. Since the coefficients R_{ijkl} are independent of x_3 and since the plate has a single layer ($\int_{-1}^{1} x_3\, dx_3 = 0$), the coefficients Q^{12} and Q^{21} vanish.

2.1.3. Proof of Theorem 2.1

In the subsequence of this proof, the hats $\widehat{\ }$ are suppressed on the functions defined on the reference domain Ω. This simplification will be done in the proof of every subsequent theorem, and this will not be repeated. The variational formulation of (3) is

$$V_{\mathrm{ad}}^a = \left\{ \mathbf{v} \in \left(H^1\left(\Omega^a\right)\right)^3 ;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0^a \right\},$$
$$\int_{\Omega^a} R_{ijkl} s_{kl}\left(\mathbf{u}^a\right) s_{ij}\left(\mathbf{v}\right) d\mathbf{x}^a = \int_{\Omega^a} f_i^a\, v_i\, d\mathbf{x}^a + \int_{\Gamma_1^a \cup \Gamma^{a\pm}} g_i^a\, v_i\, ds^a, \tag{14}$$
$$\forall \mathbf{v} \in V_{\mathrm{ad}}^a, \text{ where } \mathbf{u}^a \in V_{\mathrm{ad}}^a.$$

The application of the scaling (5) and the choice of test functions of the form $(v_1(\mathbf{x}), v_2(\mathbf{x}), (v_3/a)(\mathbf{x}))$ in the above variational formulation lead to (remember that the hat has been suppressed on the scaled functions)

$$V_{\mathrm{ad}} = \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^3 ;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \right\},$$
$$\int_{\Omega} R_{ijkl} K_{kl}^a\left(\mathbf{u}^a\right) K_{ij}^a\left(\mathbf{v}\right) d\mathbf{x} = l_u^a\left(\mathbf{v}\right), \qquad \forall \mathbf{v} \in V_{\mathrm{ad}}, \text{ where } \mathbf{u}^a \in V_{\mathrm{ad}}, \tag{15}$$
$$l_u^a\left(\mathbf{v}\right) = \int_{\Omega} f_i^a\, v_i\, d\mathbf{x} + \int_{\Gamma_1 \cup \Gamma^{\pm}} g_i^a\, v_i\, d\mathbf{s}.$$

In particular, for $\mathbf{v} = \mathbf{u}^a$,

$$\int_{\Omega} R_{ijkl} K_{kl}^a\left(\mathbf{u}^a\right) K_{ij}^a\left(\mathbf{u}^a\right) dx = l_u^a\left(\mathbf{u}^a\right),$$

which using (2) and (6) leads to

$$\sum_{i,j=1}^{3} \left\| K_{ij}^a\left(\mathbf{u}^a\right) \right\|_{L^2(\Omega)} \leq C.$$

Then, the *a priori* estimate

$$\left\| K_{ij}^a\left(\mathbf{u}^a\right) \right\|_{L^2(\Omega)} \leq C,$$

holds for every $i, j = 1, \ldots, 3$. From the Korn inequality, for every $i, j = 1, \ldots, 3$, an extracted subsequence of $(K_{ij}^a(\mathbf{u}^a), \mathbf{u}^a)$ converges weakly to some $(K_{ij}, \mathbf{u})$ in $L^2(\Omega) \times (H^1(\Omega))^3$. Now, we derive the equations satisfied by the limit $\mathbf{K} = (K_{ij})_{i,j=1,\ldots,3} \in (L^2(\Omega))^9$.

In the sequel, the fields denoted by $\mathbf{K}$ will be considered on the above format. The variational formulation (15) above may be rewritten as

$$\int_{\Omega} {}^t\mathbf{K}^a\left(\mathbf{v}\right) \mathcal{R} \mathbf{K}^a\left(\mathbf{u}^a\right) dx = l_u^a\left(\mathbf{v}\right).$$

Multiplying this equation successively by a^2, a, and 1, and passing to the limit when a vanishes, the three variational formulations are obtained:

$$\begin{aligned} &\int_{\Omega} {}^t\mathbf{K}^{-2}(\mathbf{v}) \mathcal{R}\mathbf{K}\, d\mathbf{x} = 0, && \forall \mathbf{v} \in V_{\mathrm{ad}}, \\ &\int_{\Omega} {}^t\mathbf{K}^{-1}(\mathbf{v}) \mathcal{R}\mathbf{K}\, d\mathbf{x} = 0, && \forall \mathbf{v} \in V_{\mathrm{ad}} \text{ such that } \mathbf{K}^{-2}(\mathbf{v}) = \mathbf{0}, \\ &\int_{\Omega} {}^t\mathbf{K}^{0}(\mathbf{v}) \mathcal{R}\mathbf{K}\, d\mathbf{x} = l_u(\mathbf{v}), && \forall \mathbf{v} \in V_{\mathrm{ad}} \text{ such that } \mathbf{K}^{-2}(\mathbf{v}) = \mathbf{K}^{-1}(\mathbf{v}) = \mathbf{0}, \end{aligned} \tag{16}$$

where for $\mathbf{v} \in V_{KL}$,

$$l_u(\mathbf{v}) = \int_{\Omega} \left(f_\alpha\left(\overline{v}_\alpha - x_3 \partial_\alpha v_3\right) + f_3 v_3\right) d\mathbf{x} + \int_{\Gamma_1 \cup \Gamma^{\pm}} \left(g_\alpha\left(\overline{v}_\alpha - x_3 \partial_\alpha v_3\right) + g_3 v_3\right) ds. \tag{17}$$

To compute $\mathbf{K}$, we need the following lemmas.

LEMMA 2.1.

(i) *The subspaces of* $\mathbb{K}$, $\{\mathbf{K}^{-2}(\mathbf{v}); \mathbf{v} \in V_{\text{ad}}\}$, $\{\mathbf{K}^{-1}(\mathbf{v}); \mathbf{v} \in V_{\text{ad}}$ *and* $\mathbf{K}^{-2}(\mathbf{v}) = \mathbf{0}\}$, *are dense in* $\mathbb{K}^{-2}$ *and in* $\mathbb{K}^{-1}$, *respectively.*

(ii) $\{\mathbf{K}^0(\mathbf{v});\ \mathbf{v} \in V_{\text{ad}}, \mathbf{K}^{-2}(\mathbf{v}) = \mathbf{K}^{-1}(\mathbf{v}) = \mathbf{0}\} = \mathbb{K}^0$.

PROOF. For every $K_{33} \in \mathcal{D}(\Omega)$, there exists $v_3 = \int_0^{x_3} K_{33}(x_1, x_2, t)\, dt \in V_{\text{ad}}$ such that $\partial_3 v_3 = K_{33}$. Thus, $\mathcal{D}(\Omega) \subset \{\mathbf{K}^{-2}(\mathbf{v});\ \mathbf{v} \in V_{\text{ad}}\} \subset \mathbb{K}^{-2}$. It follows that the set $\{\mathbf{K}^{-2}(\mathbf{v}); \mathbf{v} \in V_{\text{ad}}\}$ is dense in $\mathbb{K}^{-2}$. The proof is similar for $\mathbb{K}^{-1}$. Part (ii) is just a restatement of the definition of $\mathbb{K}^0$. ∎

Applying Lemma 2.1, we deduce from (16) that

$$\begin{aligned}
&\int_\Omega {}^t\tilde{\mathbf{K}}^{-2}\mathcal{R}\mathbf{K}\, d\mathbf{x} = 0, && \forall \tilde{\mathbf{K}}^{-2} \in \mathbb{K}^{-2},\\
&\int_\Omega {}^t\tilde{\mathbf{K}}^{-1}\mathcal{R}\mathbf{K}\, d\mathbf{x} = 0, && \forall \tilde{\mathbf{K}}^{-1} \in \mathbb{K}^{-1},\\
&\int_\Omega {}^t\tilde{\mathbf{K}}^{0}\mathcal{R}\mathbf{K}\, d\mathbf{x} = l_u(\mathbf{v}), && \forall \tilde{\mathbf{K}}^{0} \in \mathbb{K}^{0}.
\end{aligned} \tag{18}$$

In the last equation, $\mathbf{v} \in V_{KL}$ is the unique vector field associated with $\tilde{\mathbf{K}}^0$. Since $\mathbb{K} = \mathbb{K}^{-2} \oplus \mathbb{K}^{-1} \oplus \mathbb{K}^0$, (18) is equivalent to

$$\int_\Omega {}^t\tilde{\mathbf{K}}\mathcal{R}\mathbf{K}\, d\mathbf{x} = l_u(\mathbf{v}), \tag{19}$$

for every $\tilde{\mathbf{K}} \in \mathbf{K}$, where $\mathbf{v} \in V_{KL}$ is the vector field associated with $\tilde{\mathbf{K}}$.

LEMMA 2.2.

(i) *The limit* $\mathbf{K}$ *of* $\mathbf{K}^a(\mathbf{u}^a)$ *belongs to* $\mathbb{K}$.

(ii) *Under assumptions (1) and (2), the variational formulation (19) has a unique solution.*

PROOF. The *a priori* estimate $\|K^a_{ij}(\mathbf{u}^a)\|_{L^2(\Omega)} \le C$ for every $i, j = 1, \dots, 3$, implies that the weak limit $\mathbf{u}$ of $\mathbf{u}^a$ belongs to V_{KL}. Thus, the weak limit $K_{\alpha\beta}$ of $s_{\alpha\beta}(\mathbf{u}^a)$ is equal to $s_{\alpha\beta}(\mathbf{u})$ and belongs to $\mathbb{K}^0$. The functions $(1/a)s_{\alpha 3}(\mathbf{u}^a)$ and $(1/a^2)s_{33}(\mathbf{u}^a)$ converges weakly in $L^2(\Omega)$ to some limits $K_{\alpha 3}$ and K_{33}, and $(\mathbf{0}, (K_{\alpha 3})_{\alpha=1,2}, 0) \in \mathbb{K}^{-1}$ and $(\mathbf{0}, 0, K_{33}) \in \mathbb{K}^{-2}$. In conclusion, $\mathbf{K}^a(\mathbf{u}^a)$ converges to $\mathbf{K} = ((s_{\alpha\beta}(\mathbf{u}))_{\alpha,\beta=1,2}, (K_{\alpha 3})_{\alpha=1,2}, K_{33}) \in \mathbb{K}$. The existence and uniqueness of the solution of (19) result of the Lax-Milgram lemma and of assumptions (1) and (2). ∎

Now, from (18) or (19), we derive the plate model. The sum of the first two equations of (18) leads to

$$\int_\Omega {}^t\tilde{\mathbf{K}}\,\mathcal{R}\,\mathbf{K}\, d\mathbf{x} = 0, \qquad \forall \tilde{\mathbf{K}} \in \mathbb{K}^{-1} \oplus \mathbb{K}^{-2}. \tag{20}$$

Hence, $\mathcal{R}\,\mathbf{K}$ is orthogonal to $\mathbb{K}^{-1} \oplus \mathbb{K}^{-2}$ or equivalently $\Pi\mathcal{R}\mathbf{K} = 0$. Let us introduce the decomposition $\mathbf{K} = \Pi\mathbf{K} + \mathbf{K}^0$, where $\mathbf{K}^0 \in \mathbb{K}^0$. Then $\Pi\mathcal{R}\Pi\mathbf{K} = -\Pi\mathcal{R}\mathbf{K}^0$, which leads to the expression of $\Pi\mathbf{K}$ with respect to $\mathbf{K}^0 : \Pi\mathbf{K} = \mathbf{T}\,\mathbf{K}^0$. Using this expression in (19) and choosing $\tilde{\mathbf{K}} \in \mathbb{K}$ on the form $\tilde{\mathbf{K}} = (\mathbf{Id} + \mathbf{T})\,\tilde{\mathbf{K}}^0$, where $\tilde{\mathbf{K}}^0 \in \mathbb{K}^0$, then

$$\int_\Omega {}^t\tilde{\mathbf{K}}^0\mathbf{Q}\mathbf{K}^0\, dx = l_u(\mathbf{v}), \qquad \forall \tilde{\mathbf{K}}^0 \in \mathbb{K}^0, \tag{21}$$

where $\mathbf{v} \in V_{KL}$ is the field associated with $\tilde{\mathbf{K}}^0$. Taking account of that $\tilde{\mathbf{K}}^0$ and $\mathbf{K}^0 \in \mathbb{K}^0$, then $K^0_{\alpha\beta} = s_{\alpha\beta}(\overline{\mathbf{u}}) - x_3\partial^2_{\alpha\beta}u_3$ and $\tilde{K}^0_{\alpha\beta} = s_{\alpha\beta}(\overline{\mathbf{v}}) - x_3\partial^2_{\alpha\beta}v_3$. Thus, (21) is equivalent to

$$\int_\Omega {}^t(s_{\alpha\beta}(\overline{\mathbf{v}}) - x_3\partial^2_{\alpha\beta}v_3)Q_{\alpha\beta\gamma\delta}\left(s_{\gamma\delta}(\overline{\mathbf{u}}) - x_3\partial^2_{\gamma\delta}u_3\right)\, dx = l_u(\mathbf{v}), \qquad \text{for all } \mathbf{v} \in V_{KL},$$

which is in turn equivalent to $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3,\ \overline{u}_2 - x_3\partial_2 u_3,\ u_3) \in V_{KL}$ and

$$\int_\omega \left(s_{\alpha\beta}(\overline{\mathbf{v}}),\ \partial^2_{\alpha\beta}v_3\right)\begin{pmatrix} Q^{11}_{\alpha\beta\gamma\delta} & Q^{12}_{\alpha\beta\gamma\delta} \\ Q^{21}_{\alpha\beta\gamma\delta} & Q^{22}_{\alpha\beta\gamma\delta}\end{pmatrix}\begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}}) \\ \partial^2_{\gamma\delta}u_3\end{pmatrix} dx = l_u(\mathbf{v}), \tag{22}$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL}$. This is the variational formulation of (12) and (13). ∎

2.2. The Multilayered Plates Model

Notations and assumptions are identical to those in the preceding section except those explicitly specified.

For multilayered plates, the domain Ω^a is divided into N layers $(\Omega^a_\xi)_{\xi=1,\dots,N} = (\omega\times\,]a^a_\xi, b^a_\xi[\,)$, where $-a = a^a_1 < b^a_1 = a^a_2 < b^a_2 \cdots a^a_N < b^a_N = a$. Upper and lower faces of the layer number ξ are denoted by Γ^{a+}_ξ and Γ^{a-}_ξ. The stiffness tensor is supposed to be independent of x_3 in each layer. Its value in the layer ξ is denoted by $(R^\xi_{ijkl})_{i,j,k,l=1,\dots,3}$. The equilibrium equation (3) holds in each layer. Continuity equations are written between the layers

$$(u^a_i)_{\Gamma^+_\xi} = (u^a_i)_{\Gamma^-_{\xi+1}} \quad \text{and} \quad \left(\sigma^a_{ij}.n_j\right)_{\Gamma^{a+}_\xi} = -\left(\sigma^a_{ij}.n_j\right)_{\Gamma^{a-}_{\xi+1}}, \qquad \text{for } \xi = 1,\dots,N-1, \tag{23}$$

where $\mathbf{n} = (n_j)_{j=1,\dots,3}$ represents the vector in the normal direction of $\partial\Omega^a_\xi$. The layer Ω^a_ξ scaled by (4) is $\Omega_\xi = (\omega\times\,]a_\xi, b_\xi[\,)$, where a_ξ and b_ξ are defined by $a_\xi = a^{-1}a^a_\xi$ and $b_\xi = a^{-1}b^a_\xi$.

THEOREM 2.2. *Under assumptions (1), (2), and (6), the sequence of weak solutions to (3) and (23) scaled by formula (5) converges weakly towards the unique weak solution of equations (12) and (13).*

PROOF. The variational formulation (14) still holds. The proof is the same as the proof of Theorem 2.1, but, since $\mathcal{R}$ is different in each layer, the coefficients Q^{12} and Q^{21} do not vanish anymore.

(ii) From a practical point of view, the computation of the stiffness tensor $\begin{pmatrix} Q^{11}_{\alpha\beta\gamma\delta} & Q^{12}_{\alpha\beta\gamma\delta} \\ Q^{21}_{\alpha\beta\gamma\delta} & Q^{22}_{\alpha\beta\gamma\delta} \end{pmatrix}$ can be carried out independently in each layer. The tensor $\mathcal{R}$ is constant in each layer

$$\mathcal{R} = \left(\mathcal{R}^\xi\right)_{\xi=1,\dots,N} = \begin{pmatrix} \left(R^\xi_{\alpha\beta\gamma\delta}\right)_{\alpha,\beta,\gamma,\delta=1,2} & \left(2R^\xi_{\alpha\beta\gamma3}\right)_{\alpha,\beta,\gamma=1,2} & \left(R^\xi_{\alpha\beta33}\right)_{\alpha,\beta=1,2} \\ \left(2R^\xi_{\alpha3\gamma\delta}\right)_{\alpha,\gamma,\delta=1,2} & \left(4R^\xi_{\alpha3\gamma3}\right)_{\alpha,\gamma=1,2} & \left(2R^\xi_{\alpha333}\right)_{\alpha=1,2} \\ \left(R^\xi_{33\gamma\delta}\right)_{\gamma,\delta=1,2} & \left(2R^\xi_{33\gamma3}\right)_{\gamma=1,2} & R^\xi_{3333} \end{pmatrix}_{\xi=1,\dots,N}.$$

Let us denote by Π^ξ the restriction of Π to $(L^2(\Omega_\xi))^7$, $T^\xi = -(\Pi^\xi\mathcal{R}^\xi\Pi^\xi)^{-1}\Pi^\xi\mathcal{R}^\xi$, and $Q^\xi = (\mathbf{Id} + {}^t\mathbf{T}^\xi)\mathcal{R}^\xi(\mathbf{Id} + \mathbf{T}^\xi)$. Then,

$$\begin{pmatrix} Q^{11}_{\alpha\beta\gamma\delta} & Q^{12}_{\alpha\beta\gamma\delta} \\ Q^{21}_{\alpha\beta\gamma\delta} & Q^{22}_{\alpha\beta\gamma\delta} \end{pmatrix} = \sum_{\xi=1}^{N} \int_{a_\xi}^{b_\xi} \begin{pmatrix} Q^\xi_{\alpha\beta\gamma\delta} & -x_3 Q^\xi_{\alpha\beta\gamma\delta} \\ -x_3 Q^\xi_{\alpha\beta\gamma\delta} & x_3^2 Q^\xi_{\alpha\beta\gamma\delta} \end{pmatrix} dx_3.$$

3. THE DIELECTRIC PLATE MODELS

3.1. Statement of the Problem

3.1.1. The plate geometry

The geometry of single and multilayered plates has already been defined in Sections 2.1 and 2.2. Here, the same notations are used. In addition, the domain Ω^a is divided into two parts Ω^a_1 and Ω^a_2. They are, respectively, filled up with a dielectric and an elastic material. These two subdomains are cylindrical: $\Omega^a_1 = \omega_1\times\,]-a,a[$ and $\Omega^a_2 = \omega_2\times\,]-a,a[$, $\overline{\omega}_1 \cup \omega_2$, being a partition of ω, $\omega_1 \subset\subset \omega$.

First, consider the single layer case. The lateral, upper and lower boundaries of the inclusions Ω^a_1 are denoted by $\Gamma^a_{\text{inc}} = \gamma_{\text{inc}}\times\,]-a,a[$, Γ^{a+}_{inc}, and Γ^{a-}_{inc}. Index inc on geometrical elements refers to inclusions.

The dielectrical domain ω_1 is divided into several simply connected subdomains which are called inclusions. The inclusions are indexed by $M = (i_1, i_2) \in \mathcal{I}$, where $\mathcal{I}$ represents a set of couple of integer indexes. The restriction of a function to an inclusion M is indexed by M. For a function $f \in L^1(\Gamma_{\text{inc}}^{a+})$ and $M \in \mathcal{I}$, $\langle f \rangle_M$ represents the mean value of f on the upper face of inclusion M, and $\langle f \rangle$ represents the function defined on Γ_{inc}^{a+} such that its restriction to the upper face of the inclusion M is $\langle f \rangle_M$.

It has to be mentioned that mechanical effects and electrical effects are discoupled. Then, the electrical phenomena can be studied in Ω_1^a independently of the mechanical phenomena occurring in Ω. Except in the case of the boundary condition which couples the different inclusions, each inclusion can be treated independently. However, the problem is treated with several inclusions in view of the derivation of the piezoelectric plate models in Section 4.

3.1.2. Equations

The electrical potential φ^a is governed by the equations

$$D_i^a = c_{ij} e_j (\varphi^a) \quad \text{and} \quad - \partial_i D_i^a = 0 \text{ in } \Omega_1^a, \tag{24}$$

where the components of the electrical field $(e_i(\varphi^a))_{i=1,\dots,3}$ are the derivative of φ^a with respect to $(x_i)_{i=1,\dots,3}$, the field $(D_i)_{i=1,\dots,3}$ represents the electrical displacement, and $\mathbf{c} = (c_{ij})_{i,j=1,\dots,3} \in (L^\infty(\Omega_1))^9$ represents the tensor of permitivity. It is supposed to be constant in the thickness of the plate and strictly elliptic. It means that there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{3} c_{ij} \eta_i \eta_j \geq C \|\eta\|^2, \qquad \forall \eta = (\eta_i)_{i=1,\dots,3} \in \mathbb{R}^3. \tag{25}$$

Subsequently, C represents a constant independent of a. The boundary conditions on the lateral faces are

$$\mathbf{D}^a.\mathbf{n} = h^a \text{ on } \Gamma_{\text{inc}}^a, \quad \text{with } \|h^a\|^2_{L^2(\Gamma_{\text{inc}}^a)} \leq C, \qquad \text{for every } a > 0, \tag{26}$$

where $(h^a)_a$ is a given sequence of functions.

3.1.3. Boundary conditions on the upper and lower faces

Three types of boundary conditions are considered on the upper and lower faces Γ_{inc}^{a+} and Γ_{inc}^{a-}. In practice, each of the upper and lower faces of a piezoelectric patch are covered with a conductive film. Then, the electrical potential is constant on each of these faces. If an electrical circuit is linked to one of these faces, the electrical displacement of the charges flowing in this circuit is equal to $\langle \mathbf{D}^a.\mathbf{n} \rangle_M$. For each boundary condition described below, two cases are distinguished: the case where the lower and upper faces of the piezoelectric patches are covered with a conductive film and the case where they are not. For the sake of simplicity, we assume that, in each model, every face is metallized or every face is not metallized. The functional space

$$H_c^1(\Omega^a) = \left\{ \psi \in H^1(\Omega_1^a);\, \psi_{M|\Gamma_{\text{inc}}^{a+}} \text{ is constant for every } M \in \mathcal{I}, \text{ if the face } M \text{ is metallized} \right\}$$

is of constant use in the sequel.

Now, we define the three sorts of boundary conditions on $\Gamma_{\text{inc}}^{a+} \cup \Gamma_{\text{inc}}^{a-}$. For each of them, there are two equations. In the three cases, the electrical potential is imposed on the lower face Γ_{inc}^{a-}. This leads to the Dirichlet condition. Then, the boundary conditions differ only on Γ_{inc}^{a-}. The names of the boundary conditions refers, therefore, to the condition on Γ_{inc}^{a+}.

- Neumann boundary condition:

$$\begin{aligned} &\varphi^a = \varphi_m^a \text{ on } \Gamma_{\text{inc}}^{a-} \quad \text{and} \quad \mathbf{D}^a.\mathbf{n} = h^a \text{ on } \Gamma_{\text{inc}}^{a+}, && \text{for nonmetallized faces,} \\ &\varphi^a = \varphi_m^a \text{ on } \Gamma_{\text{inc}}^{a-} \quad \text{and} \quad \langle \mathbf{D}^a.\mathbf{n} \rangle = h^a \text{ on } \Gamma_{\text{inc}}^{a+}, && \text{for metallized faces,} \\ &\quad \text{with } \|\varphi_m^a\|^2_{H^1(\Gamma_{\text{inc}}^{a-})} + \|h^a\|^2_{L^2(\Gamma_{\text{inc}}^{a+})} \leq C, && \text{for every } a > 0, \end{aligned} \tag{27}$$

where h^a and φ^a_m are given functions. If the faces are metallized, h^a and φ^a_m are constant on each inclusion.

- Dirichlet boundary condition:

$$\begin{aligned}&\varphi^a = \varphi^a_m + a\varphi^a_c \text{ on } \Gamma^{a+}_{\text{inc}} \quad \text{and} \quad \varphi^a = \varphi^a_m - a\varphi^a_c \text{ on } \Gamma^{a-}_{\text{inc}},\\ &\text{with } \|\varphi^a_m\|_{H^1(\omega_1)} + \left\|\sqrt{a}\varphi^a_c\right\|_{H^{1/2}(\omega_1)} + \|\varphi^a_c\|_{L^2(\omega_1)} \leq C, \qquad \text{for every } a > 0,\end{aligned} \tag{28}$$

where φ^a_m and φ^a_c are given functions. If the faces are metallized, then φ^a_m and φ^a_c are constant on each inclusion.

- Mixed boundary conditions: if the upper and lower faces of each inclusion are linked by an electric circuit of admittance G^a, and if the faces are metallized, then the current which flows out from the upper face of each inclusion is $I_M = -\frac{d}{dt}\langle \mathbf{D}^a.\mathbf{n}\rangle_M$, where $\frac{d}{dt}$ represents the time derivative. If there is a source h^a of current, the equation of the circuit is then $I_M = -\frac{d}{dt}\langle \mathbf{D}^a.\mathbf{n}\rangle_M = G^a\overline{\varphi}^a_M + h^a$, where $\overline{\varphi}^a_M = \varphi_M - \varphi^a_{|\Gamma^{a-}_{\text{inc}}}$. Since this paper does not treat the evolution problem, we consider a stationary version of this condition

$$\begin{aligned}&\varphi^a = \varphi^a_m \text{ on } \Gamma^{a-}_{\text{inc}} \quad \text{and} \quad \langle \mathbf{D}^a.\mathbf{n}\rangle_M = G^a\overline{\varphi}^a_M, && \text{for every } M \in \mathcal{I} \text{ for metallized faces,}\\ &\varphi^a = \varphi^a_m \text{ on } \Gamma^{a-}_{\text{inc}} \quad \text{and} \quad \mathbf{D}^a.\mathbf{n}_{|\Gamma^{a+}_{\text{inc}}} = G^a\overline{\varphi}^a, && \text{for nonmetallized faces,}\\ &\quad\text{with } \|\varphi^a_m\|^2_{H^1(\Gamma^{a-}_{\text{inc}})} + \|h^a\|^2_{L^2(\Gamma^{a+}_{\text{inc}})} \leq C, && \text{for every } a > 0,\end{aligned} \tag{29}$$

where h^a and φ^a_m are given functions. If the faces are metallized, h^a and φ^a_m are constant on each inclusion.

3.1.4. Multilayered plates

Let us consider multilayered plates. The description of the domain Ω^a has been done in Section 2.2. In addition, the restriction of Ω^a_1 to the layer ξ is denoted by $\Omega^a_{1\xi} = \omega_{1\xi} \times]a^a_\xi, b^a_\xi[$. The lateral, lower, and upper boundaries of inclusions are $\Gamma^a_{\text{inc}} = \cup_{\xi=1}^N \Gamma^a_{\text{inc}\xi}$, $\Gamma^{a-}_{\text{inc}} = \cup_{\xi=1}^N \Gamma^{a-}_{\text{inc}\xi}$, and $\Gamma^{a+}_{\text{inc}} = \cup_{\xi=1}^N \Gamma^{a+}_{\text{inc}\xi}$, where $\Gamma^a_{\text{inc}\xi}$, $\Gamma^{a+}_{\text{inc}\xi}$, and $\Gamma^{a+}_{\text{inc}\xi}$ are their restrictions to the layer ξ.

The domains $\omega_{1\xi}$ may be different in each layer. A consequence is that dielectric inclusions are not necessarily superposed. In particular, some layers may be without inclusion. In this case, $\omega_{1\xi}$ is void. The restriction to the layer number ξ of each function defined on Ω^a_1 is indexed by ξ. The set $\mathcal{I}$ of inclusions indexes for the layer number ξ depends on ξ. It is, therefore, denoted by $\mathcal{I}_\xi$. The inclusions are indexed by M_ξ, where $M_\xi \in \mathcal{I}_\xi$.

For every $f \in L^1(\Gamma^{a+}_{\text{inc}})$, $\langle f\rangle_{M_\xi}$ is the mean value of f in the cell M_ξ. $\langle f\rangle$ is the function defined on Γ^{a+}_{inc} such that its restriction to the inclusion M_ξ is equal to $\langle f\rangle_{M_\xi}$.

The equations satisfied by the electrical potential are

$$D^a_{\xi i} = c_{\xi ij}e_j(\varphi^a) \quad \text{and} \quad -\partial_i D^a_{\xi i} = 0 \text{ in } \Omega^a_{1\xi}, \qquad \text{for } \xi = 1, \dots, N. \tag{30}$$

On the lateral, upper, and lower faces of the inclusions, the boundary conditions are the same as in the single layer case. They have to be considered independently for each layer. They are written as in (26)–(29).

A fourth condition which couples consecutive layers is considered. Since this kind of condition uses the finite difference approximation of $(\partial_3 e_i(\varphi^a))_{i=1,\dots,3}$, it may be useful when an approximate of the derivatives $(\partial_3 e_i(\varphi^a))_{i=1,\dots,3}$ is needed for the controller design (for example, for the design of a plate reflecting or absorbing acoustic waves). For this condition, every $\omega_{1\xi}$ for $\xi = 1, \dots, N$ are identical. Upper and lower faces of each inclusion are linked by an electrical circuit characterized by its admittance G^a_1, and upper faces of consecutive inclusions (in the thickness direction) are linked by another electrical circuit of admittance G^a_2. If there is a current source in each layer, the Kirchhoff current and voltage laws (see [19]) lead to the boundary conditions

$$
\begin{aligned}
\varphi^a &= \varphi^a_m \text{ on } \Gamma^{a-}_{\text{inc}}, && \text{for metallized and nonmetallized faces,}\\
-\left\langle \mathbf{D}^a.\mathbf{n}\right\rangle_{M\xi} &= G^a_1 \overline{\varphi}^a_{M\xi} \\
&\quad - G^a_2\left(\overline{\varphi}^a_{M\xi+1} - 2\overline{\varphi}^a_{M\xi} + \overline{\varphi}^a_{M\xi-1}\right) + h^a_\xi, && \text{for } \xi = 2,\dots,N-1,\\
-\left\langle \mathbf{D}^a.\mathbf{n}\right\rangle_{M1} &= G^a_1 \overline{\varphi}^a_{M1} \\
&\quad - G^a_2\left(\overline{\varphi}^a_{M2} - \overline{\varphi}^a_{M1}\right) + h^a_1,\\
-\left\langle \mathbf{D}^a.\mathbf{n}\right\rangle_{MN} &= G^a_1 \overline{\varphi}^a_{MN} \\
&\quad - G^a_2\left(-\overline{\varphi}^a_{MN} + \overline{\varphi}^a_{MN-1}\right) + h^a_N, && \text{for metallized faces,}\\
-\mathbf{D}^a.\mathbf{n}_{|\Gamma^{a+}_{\text{inc}\xi}} &= G^a_1 \overline{\varphi}^a_\xi \\
&\quad - G^a_2\left(\overline{\varphi}^a_{\xi+1} - 2\overline{\varphi}^a_\xi + \overline{\varphi}^a_{\xi-1}\right) + h^a_\xi, && \text{for } \xi = 2,\dots,N-1,\\
-\mathbf{D}^a.\mathbf{n}_{|\Gamma^{a+}_{\text{inc}1}} &= G^a_1 \overline{\varphi}^a_1 \\
&\quad - G^a_2\left(\overline{\varphi}^a_2 - \overline{\varphi}^a_1\right) + h^a_1,\\
-\mathbf{D}^a.\mathbf{n}_{|\Gamma^{a+}_{\text{inc}N}} &= G^a_1 \overline{\varphi}^a_N \\
&\quad - G^a_2\left(-\overline{\varphi}^a_N + \overline{\varphi}^a_{N-1}\right) + h^a_N, && \text{for nonmetallized faces,}\\
&\quad \text{with } \|\varphi^a_m\|^2_{H^1(\Gamma^{a-}_{\text{inc}})} + \|h^a\|^2_{L^2(\Gamma^{a+}_{\text{inc}})} \le C, && \text{for every } a > 0,
\end{aligned}
\tag{31}
$$

where $\overline{\varphi}^a_{M\xi} = \varphi_{M\xi} - \varphi^a_{|\Gamma^{a-}_{\text{inc}\xi}}$, and h^a and φ^a_m are given functions. If the faces are metallized, h^a and φ^a_m are constant on the restriction of Γ^{a+}_{inc} and Γ^{a-}_{inc} to each inclusion. In the following sections, the models relative to these boundary conditions are derived.

3.2. Neumann Conditions

The scaling on h^a, φ^a, and φ^a_m is

$$
\begin{aligned}
\widehat{\varphi}^a(x) &= \varphi^a(x^a), && \text{for } x \in \Omega^a_1,\\
\widehat{h}^a(x) &= h^a(x^a), && \text{for } x \in \Gamma_{\text{inc}} \cup \Gamma^+_{\text{inc}}, \quad \text{and}\\
\widehat{\varphi}^a_m(x) &= \varphi^a_m(x^a), && \text{for } x \in \Gamma^-_{\text{inc}}.
\end{aligned}
\tag{32}
$$

Hypotheses (27_3) and (26_2) then become

$$
\|\widehat{\varphi}^a_m\|^2_{H^1(\Gamma^-_{\text{inc}})} + \left\|\widehat{h}^a\right\|^2_{L^2(\Gamma_{\text{inc}}\cup\Gamma^+_{\text{inc}})} \le C. \tag{33}
$$

In addition, we assume that

$$
\widehat{\varphi}^a_m \rightharpoonup \varphi_m \text{ in } H^1\left(\Gamma^-_{\text{inc}}\right), \qquad \widehat{h}^a \rightharpoonup h_{\text{lat}} \text{ in } L^2\left(\Gamma_{\text{inc}}\right), \qquad \text{and } \widehat{h}^a \rightharpoonup h \text{ in } L^2\left(\Gamma^+_{\text{inc}}\right). \tag{34}
$$

We denote also by h and φ_m the functions defined in Ω_1, independent of x_3 and such that $h = h_{|\Gamma^+_{\text{inc}}}$ and $\varphi_m = \varphi_{m|\Gamma^-_{\text{inc}}}$.

THEOREM 3.1. *Under assumptions (25), (26_2), (27_3), and (34),*

(i) *the weak solution $\widehat{\varphi}^a$ of (24), (26), and ($27_{1,2}$), scaled by formula (32) converges weakly towards $\varphi = \varphi_m$ in $H^1(\Omega_1)$,*

(ii) *$\frac{\partial_3 \widehat{\varphi}^a}{a}$ converges weakly in $L^2(\Omega_1)$ towards $L_3 = (h - c_{3\alpha} e_\alpha(\varphi_m))/c_{33}$.*

PROOF. The variational formulation related to (24), (26), and ($27_{1,2}$) is

$$
\begin{aligned}
&\Psi^a_{\text{ad}}\left(\varphi^a_m\right) = \left\{\psi \in H^1_c\left(\Omega^a_1\right);\ \psi = \varphi^a_m \text{ on } \Gamma^{a-}_{\text{inc}}\right\},\\
&\int_{\Omega^a_1} c_{ij} e_j\left(\varphi^a\right) e_i(\psi)\, dx = \int_{\Gamma^a_{\text{inc}}\cup\Gamma^{a+}_{\text{inc}}} h^a\, \psi\, ds,\\
&\quad \text{for every } \psi \in \Psi^a_{\text{ad}}(0), \qquad \text{with } \varphi^a \in \Psi^a_{\text{ad}}\left(\varphi^a_m\right).
\end{aligned}
\tag{35}
$$

For every $\psi \in H^1_c(\Omega_1)$, the vectors $\mathbf{L}^a(\psi) = (L^a_i(\psi))_{i=1,\dots,3}$, $\mathbf{L}^0(\psi)$, $\mathbf{L}^{-1}(\psi)$ are defined by

$$\mathbf{L}^a(\psi) = \left(e_1(\psi), e_2(\psi), \frac{1}{a}\, e_3(\psi)\right) = \mathbf{L}^0(\psi) + \frac{1}{a}\,\mathbf{L}^{-1}(\psi).$$

The scaling (32) applied to the variational formulation (35) leads to $\varphi^a \in \Psi_{\text{ad}}(\varphi^a_m)$ and

$$\begin{gathered}\Psi_{\text{ad}}\left(\varphi^a_m\right) = \left\{\psi \in H^1_c(\Omega_1);\ \psi = \varphi^a_m \text{ on } \Gamma^-_{\text{inc}}\right\},\\ \int_{\Omega_1} c_{ij} L^a_j\left(\varphi^a\right) L^a_i(\psi)\, dx = \int_{\Gamma_{\text{inc}}} h^a \psi\, ds + l^a_\varphi\left(L^a_3(\psi)\right), \qquad \forall\, \psi \in \Psi_{\text{ad}}(0),\end{gathered} \tag{36}$$

where

$$l^a_\varphi\left(L^a_3(\psi)\right) = \frac{1}{a}\int_{\Gamma^+_{\text{inc}}} h^a\, \psi\, ds = \int_{\Omega_1} h^a L^a_3(\psi)\, dx.$$

LEMMA 3.1. *If $\varphi^a_m \in H^1(\Gamma^-_{\text{inc}})$ satisfies $\|\varphi^a_m\|_{H^1(\Gamma^-_{\text{inc}})} \le C$ uniformly in a, then there exists an extension $\tilde\varphi^a \in H^1_c(\Omega_1)$ such that $\tilde\varphi^a = \varphi^a_m$ on Γ^{a-}_{inc} and $\|\mathbf{L}^a(\tilde\varphi^a)\|_{(L^2(\Omega_1))^3} \le C$.*

PROOF. Consider the function $\tilde\varphi^a = \varphi^a_m$ constant in the thickness of the plate. Then $L^a_3(\tilde\varphi^a) = 0$ and $L^a_\alpha(\tilde\varphi^a) = L^a_\alpha(\varphi^a_m)$, for $\alpha = 1, 2$. Thus, $\|\mathbf{L}^a(\tilde\varphi^a)\|_{(L^2(\Omega_1))^3} \le 2\|\varphi^a_m\|_{H^1(\Gamma^+_{\text{inc}})}$. ∎

Let us state $\overline{\varphi}^a = \varphi^a - \tilde\varphi^a$. Then $\overline{\varphi}^a \in \Psi_{\text{ad}}(0)$ is the solution of

$$\begin{aligned}\int_{\Omega_1} c_{ij} L^a_j\left(\overline{\varphi}^a\right) L^a_i(\psi)\, dx = &-\int_{\Omega_1} c_{ij} L^a_j\left(\tilde\varphi^a\right) L^a_i(\psi)\, dx\\ &+\int_{\Gamma_{\text{inc}}} h^a \psi\, ds + l^a_\varphi\left(L^a_3(\psi)\right), \qquad \forall\, \psi \in \Psi_{\text{ad}}(0).\end{aligned}$$

State $\psi = \overline{\varphi}^a$, using (25), (33), (35) and Lemma 3.1, then

$$\left\|\mathbf{L}^a\left(\overline{\varphi}^a\right)\right\|_{(L^2(\Omega_1))^3} \le C \quad\text{and}\quad \left\|\mathbf{L}^a\left(\varphi^a\right)\right\|_{(L^2(\Omega_1))^3} \le C.$$

It follows that for each $i = 1, \dots, 3$, there exists an extracted subsequence of $L^a_i(\varphi^a)$ which converges weakly towards a limit L_i in $L^2(\Omega_1)$. From the Poincaré inequality, up to the extraction of another subsequence, the function $\overline{\varphi}^a$ and φ^a converges weakly in $H^1(\Omega_1)$. The limit of φ^a is denoted by φ, then $L_\alpha = e_\alpha(\varphi)$. But when $a \to 0$, $e_3(\varphi^a) \to 0$, then $e_3(\varphi) = 0$, $\varphi = \varphi_m$, and $L_\alpha = e_\alpha(\varphi_m)$. This proves Part (i).

The space $(L^2(\Omega_1))^3$ is decomposed on the form $((L_\alpha)_{\alpha=1,2}, L_3)$, where $L_i \in L^2(\Omega_1)$. Let us define the subspaces of $(L^2(\Omega_1))^3$:

$$\mathbb{L}^{-1} = \left\{(\mathbf{0}, L_3)\,;\ L_3 \in L^2(\Omega_1)\right\}. \tag{37}$$

The following lemma is trivial.

LEMMA 3.2.

(i) *The space $\{\mathbf{L} = \mathbf{L}^{-1}(\psi);\ \psi \in \Psi_{\text{ad}}(0)\}$ is dense in $\mathbb{L}^{-1}$.*

(ii) $\mathbf{L} = (L_i)_{i=1,\dots,3} \in \mathbb{L}^{-1} + ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$.

Consider $\mathcal{C}$, the permitivity matrix stored on a format compatible with the above decomposition of $(L^2(\Omega_1))^3$:

$$\mathcal{C} = \begin{pmatrix} (c_{\alpha\beta})_{\alpha,\beta=1,2} & (c_{\alpha 3})_{\alpha=1,2} \\ (c_{3\alpha})_{\alpha=1,2} & c_{33}\end{pmatrix}.$$

After multiplication by a and 1, successively, we take the limit of (35) when a vanishes. This leads to

$$\begin{aligned}&\int_{\Omega_1} {}^t\mathbf{L}^{-1}(\psi)\,\mathcal{C}\mathbf{L}\, dx = \int_{\Gamma^+_{\text{inc}}} h\psi\, ds && \forall\, \psi \in \Psi_{\text{ad}}(0),\\ &\int_{\Omega_1} {}^t\mathbf{L}^{0}(\psi)\,\mathcal{C}\mathbf{L}\, dx = \int_{\Gamma_{\text{inc}}} h_{\text{lat}}\psi\, ds, && \text{for every } \psi \in \Psi_{\text{ad}}(0) \text{ such that } \mathbf{L}^{-1}(\psi) = 0.\end{aligned} \tag{38}$$

In the second equation, the fact that $\psi \in \Psi_{\text{ad}}(0)$ and $\mathbf{L}^{-1}(\psi) = 0$ leads to $\psi = 0$. Thus, only the first equation plays a role in the model. For $\psi \in \Psi_{\text{ad}}(0)$, $\psi_{|\Gamma^+_{\text{inc}}} = \int_{-1}^{1} \partial_3 \psi \, dx_3$, using Lemma 3.1, it follows that the first equation of (38) is equivalent to

$$\int_{\Omega_1} {}^t\tilde{\mathbf{L}} \mathcal{C} \mathbf{L} \, dx = l_\varphi \left(\tilde{L}_3 \right), \qquad \text{for every } \tilde{\mathbf{L}} = \left(0, 0, \tilde{L}_3\right) \in \mathbb{L}, \tag{39}$$

where

$$l_\varphi \left(\tilde{L}_3 \right) = \int_{\Gamma^+_{\text{inc}}} h \int_{-1}^{1} \tilde{L}_3 \, dx_3 \, ds = \int_{\Omega_1} h \, \tilde{L}_3 \, dx.$$

From the Lax-Milgram lemma and Lemma 3.2 (ii), this variational problem has a unique solution $\mathbf{L} \in \mathbb{L}^{-1} \oplus ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$. This formulation is equivalent to $\mathcal{C}_{3i} L_i = h$. In conclusion using Part (i),

$$\varphi = \varphi_m \quad \text{and} \quad L_3 = \frac{h - c_{3\alpha} e_\alpha(\varphi_m)}{c_{33}} \text{ in } \omega_1.$$

This completes the plate model. ∎

3.3. Dirichlet Conditions

Let us scale the boundary conditions,

$$\begin{aligned} &\widehat{h}^a(x) = h^a(x^a), && \text{for } x \in \Gamma_{\text{inc}}, \\ &\widehat{\varphi}^a_m(x) = \varphi^a_m(x^a) \quad \text{and} \quad \widehat{\varphi}^a_c(x) = \varphi^a_c(x^a), && \text{for every } x \in \omega_1. \end{aligned} \tag{40}$$

Hypotheses (28_2) and (26_2) then become

$$\|\widehat{\varphi}^a_c\|^2_{L^2(\Gamma^+_{\text{inc}})} + \left\|\sqrt{a}\widehat{\varphi}^a_c\right\|^2_{H^{1/2}(\Gamma^+_{\text{inc}})} + \|\widehat{\varphi}^a_m\|^2_{H^1(\Gamma^-_{\text{inc}})} + \left\|\widehat{h}^a\right\|^2_{L^2(\Gamma_{\text{inc}})} \le C. \tag{41}$$

In addition, we assume that

$$\widehat{\varphi}^a_c \rightharpoonup \varphi_c \text{ in } L^2\left(\Gamma^+_{\text{inc}}\right), \quad \widehat{\varphi}^a_m \rightharpoonup \varphi_m \text{ in } H^1\left(\Gamma^-_{\text{inc}}\right), \quad \text{and } \widehat{h}^a \rightharpoonup h_{\text{lat}} \text{ in } L^2\left(\Gamma_{\text{inc}}\right). \tag{42}$$

THEOREM 3.2. *Under assumptions (25), (26_2), (28_2), and (42),*

(i) *the solution $\widehat{\varphi}^a$ of (24), (26_1), and (28_1) scaled by formula (40) converges weakly towards $\varphi = \varphi_m$ in $H^1(\Omega_1)$,*

(ii) *$\frac{\partial_3 \widehat{\varphi}^a}{a}$ converges weakly in $L^2(\Omega_1)$ towards φ_c.*

First, let us prove the following lemma.

LEMMA 3.3. *Let φ^a_m and φ^a_c be two functions defined on ω_1 such that*

$$\|\varphi^a_m\|_{H^1(\omega_1)} + \left\|\sqrt{a}\varphi^a_c\right\|_{H^{1/2}(\omega_1)} \le C, \tag{43}$$

then there exists $\tilde{\varphi}^a \in H^1(\Omega_1)$ such that

$$\tilde{\varphi}^a = \varphi^a_m + a\varphi^a_c \text{ on } \Gamma^+_{\text{inc}}, \quad \tilde{\varphi}^a = \varphi^a_m - a\varphi^a_c \text{ on } \Gamma^-_{\text{inc}}, \quad \text{and} \quad \|\mathbf{L}^a(\tilde{\varphi}^a)\|_{L^2(\Omega_1)} \le C. \tag{44}$$

REMARK. If the faces are metallized, since $\tilde{\varphi}^a$ is constant on each inclusion, $\mathbf{L}^a(\tilde{\varphi}^a) = 0$.

PROOF. Define $\tilde{\varphi}^a$ by $\tilde{\varphi}^a(x_1, x_2, x_3) = \varphi^a_m(x_1, x_2) + \tilde{\varphi}^a_c(x_1, x_2, x_3)$, where $\tilde{\varphi}^a_c(x_1, x_2, x_3) = \tilde{\varphi}_c(x_1, x_2, (x_3/a))$, $\tilde{\varphi}_c$ being defined on Ω^a_1 by

$$-\Delta\tilde{\varphi}_c = 0 \text{ in } \Omega^a_1, \quad \nabla\tilde{\varphi}_c.n = 0 \text{ on } \Gamma^a_{\text{inc}}, \quad \tilde{\varphi}_c = a\varphi^a_c \text{ on } \Gamma^{a+}_{\text{inc}}, \quad \text{and} \quad \tilde{\varphi}_c = -a\varphi^a_c \text{ on } \Gamma^{a-}_{\text{inc}}.$$

The solution $\tilde{\varphi}_c$ of the above problem is the unique solution of the minimization problem, $\inf_\psi \int_{\Omega_1^a} |\nabla\psi|^2\,dx$, where $\psi \in H^1(\Omega_1^a)$, $\psi = a\varphi_c^a$ on Γ_{inc}^{a+}, and $\psi = -a\varphi_c^a$ on Γ_{inc}^{a-}. Then, by definition, $\|\tilde{\varphi}_c\|^2_{H^{1/2}(\Gamma_{\text{inc}}^{a+}\cup\Gamma_{\text{inc}}^{a-})} = |\nabla\tilde{\varphi}_c|^2$ or equivalently $|\nabla\tilde{\varphi}_c|^2 = 2a^2\|\varphi_c^a\|^2_{H^{1/2}(\omega_1)}$. Application of the scaling then leads to $\|\mathbf{L}^a(\tilde{\varphi}_c)\|^2_{(L^2(\Omega_1))^3} \le Ca\|\varphi_c^a\|^2_{H^{1/2}(\omega_1)}$. Thus, $\|\mathbf{L}^a(\tilde{\varphi}^a)\|^2_{(L^2(\Omega_1))^3} \le C\left(a\|\varphi_c^a\|^2_{H^{1/2}(\omega_1)} + \|\varphi_m^a\|^2_{H^1(\omega_1)}\right)$. ∎

PROOF OF THEOREM 3.2. The variational formulation of (24), (26_1), and (28_1) scaled by formula (40) is

$$\Psi_{\text{ad}}(\varphi_m^a,\varphi_c^a) = \left\{\psi \in H_c^1(\Omega_1);\ \psi = \varphi_m^a + a\varphi_c^a \text{ on } \Gamma_{\text{inc}}^+ \text{ and } \psi = \varphi_m^a - a\varphi_c^a \text{ on } \Gamma_{\text{inc}}^-\right\},$$
$$\varphi^a \in \Psi_{\text{ad}}^a(\varphi_m^a,\varphi_c^a), \tag{45}$$
$$\int_{\Omega_1} c_{ij}e_j(\varphi^a)\,e_i(\psi)\,dx = \int_{\Gamma_{\text{inc}}^a} h^a\psi\,ds, \qquad \forall\,\psi \in \Psi_{\text{ad}}(0,0).$$

Define $\overline{\varphi}^a$ by $\overline{\varphi}^a = \varphi^a - \tilde{\varphi}^a$. Then $\overline{\varphi}^a \in \Psi_{\text{ad}}(0)$ satisfies

$$\int_{\Omega_1} c_{ij}L_j^a(\overline{\varphi}^a)\,L_i^a(\psi)\,dx = -\int_{\Omega_1} c_{ij}L_j^a(\tilde{\varphi}^a)\,L_i^a(\psi)\,dx + \int_{\Gamma_{\text{inc}}} h^a\psi\,ds, \qquad \text{for every } \psi \in \Psi_{\text{ad}}(0).$$

Choosing $\psi = \overline{\varphi}^a$, using (25),($41_2$) leads to $\|L_i^a(\overline{\varphi}^a)\|_{L^2(\Omega_1)} \le C$, and thus to $\|L_i^a(\varphi^a)\|_{L^2(\Omega_1)} \le C$, for $i = 1,\dots,3$. Hence, there exists a subsequence of $\mathbf{L}^a(\varphi^a)$ which converges weakly towards $\mathbf{L} \in (L^2(\Omega_1))^3$. As in the case of Neumann conditions, the function φ^a converges weakly towards φ_m in $H^1(\Omega_1)$. This is Point (i).

Define the operators $\mathcal{M}$, $\mathcal{N}$ by

$$\mathcal{M}(\mathbf{L}) = \frac{1}{2}\int_{-1}^{1}\mathbf{L}\,dx_3, \qquad \text{for every } \mathbf{L} \in \left(L^1(\Omega_1)\right)^3 \text{ and } \mathcal{N} = \mathbf{Id} - \mathcal{M}. \tag{46}$$

LEMMA 3.4. *The space* $\{\mathbf{L} = \mathbf{L}^{-1}(\psi);\ \psi \in \Psi_{\text{ad}}(0,0)\}$ *is dense in* $\mathcal{N}(\mathbb{L}^{-1})$.

The proof is straightforward.

LEMMA 3.5. *The limit* $\mathbf{L} \in \mathcal{N}(\mathbb{L}^{-1}) + ((e_\alpha(\varphi_m))_{\alpha=1,2},\varphi_c)$.

PROOF. Because $\int_{-1}^1 L_3\,dx_3 = \lim_{a\to 0}\int_{-1}^1 (1/a)\partial_3\varphi^a\,dx_3 = \lim_{a\to0} 2\varphi_c^a = 2\varphi_c$, this means that $\int_{-1}^1 (L_3 - \varphi_c)\,dx_3 = 0$, then $\mathbf{L}-((e_\alpha(\varphi_m))_{\alpha=1,2},\varphi_c) \in \mathcal{N}(\mathbb{L}^{-1})$. ∎

Passing to the limit in (45) and applying Lemma 3.4, the variational formulation satisfied by $\mathbf{L}$ is

$$\int_{\Omega_1} {}^t\tilde{\mathbf{L}}\mathcal{C}\mathbf{L}\,dx = 0, \qquad \text{for every } \tilde{\mathbf{L}} = \left(\mathbf{0},\tilde{L}_3\right) \in \mathbb{L}^{-1}. \tag{47}$$

From the Lax-Milgram lemma, this variational problem has a unique solution. This formulation is equivalent to $\partial_3(c_{33}L_3 + c_{3\alpha}e_\alpha(\varphi_m)) = 0$ in Ω_1. Then L_3 is independent of x_3. From Lemma 3.5, $L_3 = \varphi_c$. This proves Point (ii). ∎

3.4. Mixed Conditions

The scaling on the data is (32) and

$$\widehat{G} = aG^a, \tag{48}$$

where $\widehat{G}$ is a positive constant independent of a.

THEOREM 3.3. *Under assumptions (25), (26_2), (29_3), and (34),*

(i) *the weak solution* $\widehat{\varphi}^a$ *of (24), ($29_{1,2}$), and (26_1) scaled by formulae (32),(48) converges weakly towards* $\varphi = \varphi_m$ *in* $H^1(\Omega_1)$,

(ii) $\frac{\partial_3\widehat{\varphi}^a}{a}$ *converges weakly in* $L^2(\Omega_1)$ *towards* $L_3 = (c_{33} + 2\widehat{G})^{-1}(h - c_{3\alpha}e_\alpha(\varphi_m))$.

PROOF. The variational formulation in Ω_1^a verified by $\varphi^a \in \Psi_{\mathrm{ad}}^a(\varphi_m^a)$ solution of (24), $(29_{1,2})$, and (26_1) is

$$\Psi_{\mathrm{ad}^a}(\varphi_m) = \left\{\psi \in H_c^1(\Omega_1^a);\ \psi = \varphi_m \text{ on } \Gamma_{\mathrm{inc}}^{a-}\right\},$$

$$\int_{\Omega_1^a} c_{ij} e_j(\varphi^a)\, e_i(\psi)\, dx^a + \int_{\omega_1} G^a \overline{\varphi}^a \psi\, ds = \int_{\Gamma_{\mathrm{inc}}^{a+}} h^a\, \psi\, ds + \int_{\Gamma_{\mathrm{inc}}^{a}} h^a\, \psi\, ds,$$
$$\text{for every } \psi \in \Psi_{\mathrm{ad}}^a(0).$$

Consider the extension $\tilde{\varphi}^a \in H_c^1(\Omega_1)$ of the boundary condition φ_m^a defined in Lemma 3.1, the decomposition of the solution in the form $\varphi^a = \tilde{\varphi}^a + \overline{\varphi}^a$ and the operator $\mathcal{M}$ defined in (46). After application of the scaling (32),(48), the variational formulation satisfied by $\overline{\varphi}^a \in \Psi_{\mathrm{ad}}(0)$ is

$$\Psi_{\mathrm{ad}}(0) = \left\{\psi \in H_c^1(\Omega_1);\ \psi = 0 \text{ on } \Gamma_{\mathrm{inc}}^-\right\},$$

$$\begin{aligned}\int_{\Omega_1} c_{ij} L_j^a(\overline{\varphi}^a)\, L_i^a(\psi)\, dx &+ \int_{\Omega_1} 2G\, \mathcal{M}\left(L_3^a(\overline{\varphi}^a)\right) L_3^a(\psi)\, dx \\ &= \int_{\Gamma_{\mathrm{inc}}} h^a \psi\, ds + l_\varphi^a\left(L_3^a(\psi)\right) - \int_{\Omega_1} c_{ij} L_j^a(\tilde{\varphi}^a)\, L_i^a(\psi)\, dx, \qquad \forall\, \psi \in \Psi_{\mathrm{ad}}(0),\end{aligned} \tag{49}$$

where $l_\varphi^a(L_3^a(\psi))$ is defined in (36). Choose $\psi = \overline{\varphi}^a$, then

$$\begin{aligned}\int_{\Omega_1} c_{ij} L_j^a(\overline{\varphi}^a)\, L_i^a(\overline{\varphi}^a)\, dx &+ \int_{\Omega_1} 2G\mathcal{M}\left(L_3^a(\overline{\varphi}^a)\right)^2 ds \\ &= \int_{\Gamma_{\mathrm{inc}}} h^a \overline{\varphi}^a\, ds + l_\varphi^a\left(L_3^a(\overline{\varphi}^a)\right) - \int_{\Omega_1} c_{ij} L_j^a(\tilde{\varphi}^a)\, L_i^a(\overline{\varphi}^a)\, dx.\end{aligned}$$

We deduce that $\|\mathcal{M}(L_3^a(\overline{\varphi}^a))\|_{L^2(\Omega_1)} + \|\mathbf{L}(\overline{\varphi}^a)\|_{(L^2(\Omega_1))^3} \le C$. Then, there exists an extracted subsequence of $\mathbf{L}^a(\varphi^a)$ which converges weakly to some $\mathbf{L} \in (L^2(\Omega_1))^3$. For the same reason that in Section 3.2, φ^a converges to φ weakly in $H^1(\Omega_1)$, $L_\alpha = e_\alpha(\varphi_m)$, and the electrical field L_3 is independent of x_3. $\mathbf{L} \in \mathbb{L}^{-1} \oplus ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$. The derivation of the model is similar to that in the case of Neumann boundary conditions. From Lemma 3.2(i), the resulting variational formulation is

$$\int_{\Omega_1} {}^t\tilde{\mathbf{L}}\mathbf{C}\mathbf{L}\, dx + \int_{\Omega_1} 2G\, \tilde{L}_3\, \mathcal{M}(L_3)\, ds = l_\varphi\left(\tilde{L}_3\right), \qquad \text{for every } \tilde{\mathbf{L}} \in \mathbb{L}^{-1}, \tag{50}$$

where $l_\varphi(\tilde{L}_3)$ is defined in (39). We used the fact that $\mathcal{M}(L_3^a(\varphi^a)) = \mathcal{M}(L_3^a(\overline{\varphi}^a)) + \mathcal{M}(L_3^a(\tilde{\varphi}^a))$ and $\mathcal{M}(L_3^a(\tilde{\varphi}^a)) = 0$. From the Lax-Milgram lemma, this variational problem has a unique solution. The equation (50) leads to $C_{3i}L_i + 2G\mathcal{M}(L_3) = h$. Thus,

$$L_3 = (c_{33} + 2G)^{-1}\left(h - c_{3\alpha} e_\alpha(\varphi_m)\right) \text{ in } \omega_1. \tag{51}$$

This is Point (ii). ∎

3.5. Multilayered Plates

Since the models for multilayered plates are derived in the same way as those of single layer plates, only the cases of Neumann conditions and conditions coupling the layers are detailed. Models relative to other conditions may be derived by a similar method. The space $\mathbb{L}^{-1}$ is defined in (37). The operators $\mathcal{M}$ and $\mathcal{N}$ are defined by their restriction on each layer $\Omega_{1\xi}$: $\mathcal{M}_{|\Omega_{1\xi}}(\mathbf{L}) = \mathcal{M}_\xi(\mathbf{L}) = (1/b_\xi - a_\xi)\int_{a_\xi}^{b_\xi} \mathbf{L}\, dx_3$ and $\mathcal{N}_{|\Omega_{1\xi}}(\mathbf{L}) = \mathcal{N}_\xi(\mathbf{L}) = \mathbf{L} - \mathcal{M}_\xi(\mathbf{L})$, for any $\mathbf{L} \in (L^2(\Omega_1))^3$.

3.5.1. Neumann boundary conditions

The scaling is,

$$\begin{aligned}\widehat{h}^a_\xi(x) &= h^a_\xi(x^a), && \text{for } x \in \Gamma^+_{\text{inc}\xi} \cup \Gamma_{\text{inc}\xi}, \\ \widehat{\varphi}^a_{m\xi}(x) &= \varphi^a_{m\xi}(x^a), && \text{for } x \in \Gamma^-_{\text{inc}\xi}, \qquad \text{for every } \xi = 1, \dots, N.\end{aligned} \tag{52}$$

The scaled data satisfy

$$\begin{aligned}&\left\|\widehat{h}^a_\xi\right\|^2_{L^2(\Gamma_{\text{inc}\xi})} + \left\|\widehat{h}^a_\xi\right\|^2_{L^2(\Gamma^+_{\text{inc}\xi})} + \left\|\widehat{\varphi}^a_{m\xi}\right\|_{H^1(\Gamma^-_{\text{inc}\xi})} \le C, \qquad \text{for every } \xi = 1, \dots, N, \\ &\widehat{h}^a_\xi \rightharpoonup h_\xi \text{ in } L^2\left(\Gamma^+_{\text{inc}\xi}\right), \qquad \widehat{h}^a_\xi \rightharpoonup h_{\xi\text{lat}} \text{ in } L^2\left(\Gamma_{\text{inc}\xi}\right), \\ &\widehat{\varphi}^a_{m\xi} \rightharpoonup \varphi_{m\xi} \text{ in } H^1\left(\Gamma^-_{\text{inc}\xi}\right), \qquad \text{for every } \xi = 1, \dots, N.\end{aligned} \tag{53}$$

THEOREM 3.4. *Under assumptions (53) and (54),*

(i) *the solution $\widehat{\varphi}^a$ of (30) and (26) with Neumann boundary conditions on $\Gamma^{a+}_{\xi i}$, scaled by formula (52) converges weakly towards $\varphi = \varphi_m$ in $H^1(\cup_{\xi=1}^N \Omega_{1\xi})$, and*

(ii) $\frac{\partial_3 \widehat{\varphi}^a}{a}_{|\Omega_{1\xi}}$ *converges weakly in $L^2(\Omega_{1\xi})$ towards $L_{3\xi} = (h_\xi - c_{3\alpha\xi} e_\alpha(\varphi_{m\xi}))/c_{33\xi}$.*

PROOF. $\tilde{\varphi}^a_\xi$ being the extension of $\varphi^a_{m\xi}$ defined in each layer as in Lemma 3.1, letting $\overline{\varphi}^a_\xi = \varphi^a_\xi - \tilde{\varphi}^a_\xi$, then $\overline{\varphi}^a \in \Psi_{\text{ad}}(0)$ is the solution of the variational formulation

$$\begin{aligned}&\Psi_{\text{ad}}(\varphi_m) = \left\{\psi \in H^1_c\left(\cup_{\xi=1}^N \Omega_{1\xi}\right); \psi_\xi = \varphi^a_{m\xi} \text{ on } \Gamma^-_{\text{inc}\xi}\right\}, \\ &\sum_{\xi=1}^N \int_{\Omega_{1\xi}} c_{ij} L^a_j\left(\overline{\varphi}^a_\xi\right) L^a_i(\psi_\xi)\, dx = -\sum_{\xi=1}^N \int_{\Omega_{1\xi}} c_{ij} L^a_j\left(\tilde{\varphi}_\xi\right) L^a_i(\psi_\xi)\, dx \\ &\quad + \sum_{\xi=1}^N \int_{\Gamma_{\text{inc}\xi}} h^a_\xi \psi\, ds + \frac{1}{a}\sum_{\xi=1}^N \int_{\Gamma^+_{\text{inc}\xi}} h^a_\xi \psi\, ds, \qquad \forall \psi \in \Psi_{\text{ad}}(0).\end{aligned} \tag{54}$$

For every $\psi \in H^1_c(\cup_{\xi=1}^N \Omega_{1\xi})$, the vector $\mathbf{L}^a(\psi)$ is defined in $\cup_{\xi=1}^N \Omega_{1\xi}$ by $\mathbf{L}^a(\psi) = (e_1(\psi), e_2(\psi), (1/a)e_3(\psi)) = \mathbf{L}^0(\psi) + (1/a)\mathbf{L}^{-1}(\psi)$. The choice $\psi = \overline{\varphi}^a$ in (54) implies that $\|\mathbf{L}^a(\overline{\varphi}^a)\|_{(L^2(\cup_{\xi=1}^N \Omega_{1\xi}))^3} \le C$, and therefore, that $\|\mathbf{L}^a(\varphi^a)\|_{(L^2(\cup_{\xi=1}^N \Omega_{1\xi}))^3} \le C$. Then, for each $i = 1, \dots, 3$, there exists a subsequence $(L^a_i(\varphi^a))_{a>0}$ weakly converging towards a limit L_i in $L^2(\Omega_1)$. As in Section 3.2, the sequence φ^a converges in $H^1(\cup_{\xi=1}^N \Omega_{1\xi})$ towards φ_m and $\mathbf{L} \in \mathbb{L}^{-1} \oplus ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$. The model related to a single layer plate is thus valid for each layer

$$\varphi_\xi = \varphi_{m\xi} \quad \text{and} \quad L_{3\xi} = \frac{h_\xi - c_{3\alpha\xi} e_\alpha(\varphi_{m\xi})}{c_{33\xi}} \text{ in } \omega_{1\xi}, \qquad \text{for every } \xi = 1, \dots, N. \tag{55}$$ ∎

3.5.2. Boundary conditions coupling the layers

The scaling of the admittances is

$$\widehat{G}_1 = aG^a_1 \quad \text{and} \quad \widehat{G}_2 = aG^a_2, \tag{56}$$

$\widehat{G}_1$ and $\widehat{G}_2$ being positive constants. Define two $N \times N$ matrices,

$$\widehat{\mathbf{A}} = \begin{pmatrix} r_1\left(\widehat{G}_2 + \widehat{G}_1\right) & -\widehat{G}_2 r_2 & 0 & 0 & . & . & 0 \\ -\widehat{G}_2 r_1 & r_2\left(\widehat{G}_2 + \widehat{G}_1\right) & -\widehat{G}_2 r_3 & 0 & . & . & 0 \\ 0 & -\widehat{G}_2 r_2 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & -\widehat{G}_2 r_N \\ 0 & 0 & 0 & 0 & 0 & -\widehat{G}_2 r_{N-1} & r_N\left(\widehat{G}_2 + \widehat{G}_1\right) \end{pmatrix}, \tag{57}$$

where $r_\xi = b_\xi - a_\xi$ and I the identity matrix.

THEOREM 3.5. *Under assumptions (53) and (25),*

(i) *the weak solution* $\widehat{\varphi}^a$ *of (30), (31), and (26) scaled by formulae (52) and (56) converges weakly towards* φ_m *in* $H^1(\cup_{\xi=1}^N \Omega_{1\xi})$,

(ii) $\frac{\partial_3 \widehat{\varphi^a}}{a}$ *converges weakly in* $L^2(\cup_{\xi=1}^N \Omega_{1\xi})$ *towards* $L_{3\xi} = \sum_{\eta=1}^N ((c_{33}I + \widehat{A})^{-1})_{\xi\eta}(-c_{3\alpha\eta}e_\alpha(\varphi_{m\eta}) + h_\eta)$ *for* $\xi = 1, \dots, N$.

PROOF. After application of the scaling, suppression of the hats on scaled functions and construction of the extension $\widetilde{\varphi}^a$ of the Dirichlet boundary conditions on $\Gamma^-_{\mathrm{inc}\xi}$ as in Section 3.5.1, the variational formulation is similar to (54) with the additional bilinear form $a(L_3^a(\overline{\varphi}^a), L_3^a(\psi))$ on the left-hand side, where

$$a(p,q) = \sum_{\xi=1}^{N} \int_{\omega_{1\xi}} r_\xi^2 G_1 \mathcal{M}_\xi(p)\mathcal{M}_\xi(q)\,dx + \sum_{\xi=1}^{N-1} \int_{\omega_1} G_2\left(r_{\xi+1}\mathcal{M}_{\xi+1}(p) - r_\xi \mathcal{M}_\xi(p)\right) \times \left(r_{\xi+1}\mathcal{M}_{\xi+1}(q) - r_\xi\mathcal{M}_\xi(q)\right) ds, \qquad \text{for every } p,\ q \in L^2(\Omega_1). \tag{58}$$

The same estimate is obtained as for Neumann boundary conditions. In addition, $a(L_3^a(\overline{\varphi}^a), L_3^a(\overline{\varphi}^a)) \le C$. Then, for each $i = 1, \dots, 3$, there exists a subsequence $(L_i^a(\varphi^a))_a$ weakly converging towards a limit L_i in $L^2(\Omega_1)$. As in Section 3.5.1, the sequence φ^a converges in $H^1(\cup_{\xi=1}^N \Omega_{1\xi})$ towards φ_m and $\mathbf{L} \in \mathbb{L}^{-1} \oplus ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$, which proves (i). Arguing as in Sections 3.2 and 3.4, the variational formulation of the limit problem is

$$\int_{\Omega_1} \tilde{\mathbf{L}}\mathcal{C}\mathbf{L}\,dx + a\left(L_3, \tilde{L}_3\right) = l_\varphi\left(\tilde{L}_3\right), \qquad \text{for every } \tilde{\mathbf{L}} = \left(0, 0, \tilde{L}_3\right) \in \mathbb{L}^{-1}, \tag{59}$$

where

$$l_\varphi\left(\tilde{L}_3\right) = \int_{\cup_{\xi=1}^N \Omega_{1\xi}} h_\xi \mathcal{M}\left(\tilde{L}_3\right) ds.$$

From the Lax Milgram lemma, this variational problem has a unique solution $\mathbf{L} = \mathbb{L}^{-1} \oplus ((e_\alpha(\varphi_m))_{\alpha=1,2}, 0)$. Equation (59) is equivalent to

$$\sum_{\eta=1}^{N} \left(c_{33\eta} + \widehat{A}_{\xi\eta}\right) L_{3\eta} + c_{3\alpha\xi} e_\alpha\left(\varphi_{m\xi}\right) = h_\xi, \qquad \text{for } \xi = 1, \dots, N, \tag{60}$$

which proves (ii). Point (iii) follows from (i) and (ii). ∎

4. THE ELASTIC PLATE WITH PIEZOELECTRIC INCLUSIONS

First, let us consider the single layer plate. Now the inclusions located in Ω_1^a are made of piezoelectric material and Ω_2^a contains an elastic material. Let us state the equations of piezoelectricity in statics. The tensor of piezoelectricity $(d_{ijk})_{i,j,k=1,\dots,3}$ satisfies the symmetries

$$d_{ijk} = d_{ikj}, \qquad \text{for every } i,j,k = 1, \dots, 3, \tag{61}$$

and is assumed not to depend on x_3. The tensor $(d_{ijk})_{i,j,k=1,\dots,3}$ vanishes in Ω_2^a. We assume that Ω_1^a is electrically insulated of Ω_2^a (see condition (64_3)). Then, the electrical field in Ω_2^a neither affects the electrical field in Ω_1^a nor the mechanical field in Ω^a. For this reason, the equation governing the electrical potential is not considered in Ω_2^a. The strong form of the piezoelectricity equations results of the expression of the stress tensor and of the electrical displacement vector (see [20])

$$\sigma_{ij}^a = R_{ijkl} s_{kl}\left(\mathbf{u}^a\right) + d_{kij} e_k\left(\varphi^a\right) \text{ in } \Omega^a \quad \text{and} \quad D_k^a = -d_{kij} s_{ij}\left(\mathbf{u}^a\right) + c_{ki} e_i\left(\varphi^a\right) \text{ in } \Omega_1^a, \tag{62}$$

of the elastostatic and electrostatic equations

$$-\partial_j \sigma_{ij}^a = f_i^a \text{ in } \Omega^a \quad \text{and} \quad -\partial_i D_i^a = 0 \text{ in } \Omega_1^a. \tag{63}$$

Subsequently, the permitivity tensor $\mathbf{c}$ is considered as a tensor defined on Ω^a but equal to zero on Ω_2^a. In the previous section, we have shown that the source term h^a on Γ_{inc}^a does not contribute in the plate models. For simplicity of the derivation, it is therefore taken equal to zero. The boundary conditions are

$$\sigma_{ij}^a n_j = g_i^a \text{ on } \Gamma_1^a \cup \Gamma^{a\pm}, \quad \mathbf{u}^a = \mathbf{0} \text{ on } \Gamma_0^a, \quad \text{and} \quad \mathbf{D}^a.\mathbf{n} = 0 \text{ on } \Gamma_{\text{inc}}^a. \tag{64}$$

The boundary conditions on $\Gamma_{\text{inc}}^{a+} \cup \Gamma_{\text{inc}}^{a-}$ are one of (27)–(29). They lead to different models considered in the following sections.

Let us consider multilayered plates. The equations (62)–(64) have to be written in each layer Ω_ξ^a and $\Omega_{1\xi}^a$:

$$\begin{gathered}
\sigma_{ij}^a = R_{ijkl} s_{kl}(\mathbf{u}^a) + d_{kij} e_k(\varphi^a) \text{ in } \Omega_\xi^a \quad \text{and} \quad D_k^a = -d_{kij} s_{ij}(\mathbf{u}^a) + c_{ki} e_i(\varphi^a) \text{ in } \Omega_{1\xi}^a, \\
-\partial_j \sigma_{ij}^a = f_i^a \text{ in } \Omega_\xi^a \quad \text{and} \quad -\partial_i D_i^a = 0 \text{ in } \Omega_{1\xi}^a, \\
\sigma_{ij}^a n_j = g_i^a \text{ on } \Gamma_{1\xi}^a \cup \Gamma_\xi^{a\pm}, \quad \mathbf{u}^a = \mathbf{0} \text{ on } \Gamma_{0\xi}^a, \quad \text{and} \quad \mathbf{D}^a.\mathbf{n} = 0 \text{ on } \Gamma_{\text{inc}\xi}^a,
\end{gathered} \tag{65}$$

for every $\xi = 1, \dots, N$. The three tensors $\mathbf{R}$, $\mathbf{d}$, and $\mathbf{c}$ are independent of x_3 in each layer. The continuity conditions between the layers are

$$(u_i^a)_{\Gamma_\xi^+} = (u_i^a)_{\Gamma_{\xi+1}^-} \quad \text{and} \quad (\sigma_{ij}^a.n_j)_{\Gamma_\xi^+} = -(\sigma_{ij}^a.n_j)_{\Gamma_{\xi+1}^-}, \qquad \text{for } \xi = 1, \dots, N-1. \tag{66}$$

As for dielectric multilayered plates, the layers are electrically insulated, so there are no continuity conditions relative to the electrical displacement or to the electrical potential. The boundary condition on $\Gamma_{\text{inc}\xi}^{a+} \cup \Gamma_{\text{inc}\xi}^{a-}$ is one of (27)–(29) or (31).

4.1. Neumann Boundary Conditions

For $\mathbf{V} = (\mathbf{v}, \psi) \in (H^1(\Omega))^3 \times H^1(\Omega_1)$, define

$$\mathbf{M}^a(\mathbf{V}) = \left((s_{\alpha\beta}(\mathbf{v}))_{\alpha,\beta=1,2}, \frac{1}{a} (s_{\alpha 3}(\mathbf{v}))_{\alpha=1,2}, \frac{1}{a^2} s_{33}(\mathbf{v}), (e_\alpha(\psi))_{\alpha=1,2}, \frac{1}{a} e_3(\psi) \right), \tag{67}$$

which provides a decomposition of a subspace of $(L^2(\Omega))^{10}$. The tensor $\mathcal{R}$ composed of the stiffness tensor, piezoelectricity tensor, and permitivity tensor is built in a format compatible with (67):

$$\mathcal{R} = \begin{pmatrix}
(R_{\alpha\beta\gamma\delta})_{\alpha,\beta,\gamma,\delta=1,2} & (2R_{\alpha\beta\gamma 3})_{\alpha,\beta,\gamma=1,2} & (R_{\alpha\beta 33})_{\alpha,\beta=1,2} & (d_{\gamma\alpha\beta})_{\alpha,\beta,\gamma=1,2} & (d_{3\alpha\beta})_{\alpha\beta=1,2} \\
(2R_{\alpha 3\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (4R_{\alpha 3\gamma 3})_{\alpha,\gamma=1,2} & (2R_{\alpha 333})_{\alpha=1,2} & (2d_{\gamma\alpha 3})_{\alpha,\gamma=1,2} & (2d_{3\alpha 3})_{\alpha=1,2} \\
(R_{33\gamma\delta})_{\gamma,\delta=1,2} & (2R_{33\gamma 3})_{\gamma=1,2} & R_{3333} & (d_{\gamma 33})_{\gamma=1,2} & d_{333} \\
(-d_{\alpha\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (-2d_{\alpha\gamma 3})_{\alpha,\gamma=1,2} & (-d_{\alpha 33})_{\alpha=1,2} & (c_{\alpha\gamma})_{\alpha,\gamma=1,2} & (c_{\alpha 3})_{\alpha=1,2} \\
(-d_{3\gamma\delta})_{\gamma,\delta=1,2} & (-2d_{3\gamma 3})_{3,\gamma=1,2} & -d_{333} & (c_{3\gamma})_{\gamma=1,2} & c_{33}
\end{pmatrix}.$$

Now, introduce the spaces

$$\begin{aligned}
\mathbb{M} &= \mathbb{M}^0 \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}, \\
\mathbb{M}^0 &= \left\{ \mathbf{M} = (\mathbf{K}, \mathbf{0}_3) \in \mathbb{K}^0 \times \left(L^2(\Omega_1)\right)^3 \right\}, \\
\mathbb{M}^{-1} &= \left\{ \mathbf{M} = (\mathbf{K}, \mathbf{L}) \in \mathbb{K}^{-1} \times \mathbb{L}^{-1} \right\}, \\
\mathbb{M}^{-2} &= \left\{ \mathbf{M} = (\mathbf{K}, \mathbf{0}_3) \in \mathbb{K}^{-2} \times \left(L^2(\Omega_1)\right)^3 \right\}, \\
\mathbb{L}^0 &= \left\{ \mathbf{M} = (\mathbf{0}_7, \mathbf{L}) \in \left(L^2(\Omega)\right)^7 \times \left\{ \left((e_\alpha(\psi))_{\alpha=1,2}, 0 \right); \text{ where } \psi \in H^1(\Omega_1),\ \partial_3 \psi = 0 \right\} \right\}, \\
\mathbb{E} &= \left\{ \mathbf{M} = (\mathbf{K}, \mathbf{L}) \in \left(L^2(\Omega)\right)^7 \times \left(L^2(\Omega_1)\right)^3 \right\},
\end{aligned} \tag{68}$$

$\mathbb{K}$, $\mathbb{K}^0$, $\mathbb{K}^{-1}$, $\mathbb{K}^{-2}$, and $\mathbb{L}^{-1}$ are defined in (9) and (37).

The spaces $\mathbb{K}^i$ and $\mathbb{L}^i$ being trivially identified to subspaces of $\mathbb{E}$, for every such subspace X of $\mathbb{E}$, we denote by $\mathcal{P}_X$ the identification operator

$$\mathcal{P}_X : \begin{array}{l} X \to \mathbb{E} \\ \mathbf{x} \mapsto \mathcal{P}_X \mathbf{x} \end{array}$$

such that the restriction of $\mathcal{P}_X\mathbf{x}$ to X is equal to $\mathbf{x}$, and its other components are null. For example, for $\mathbf{L} = (\mathbf{0}, L_3) \in \mathbb{L}$, $\mathcal{P}_{\mathbb{L}}\mathbf{L} = (\mathbf{0}_{\mathbb{K}}, \mathbf{L})$. For convenience, since no confusion is possible, we use in fact the abusive notation $\mathcal{P}_{\mathbb{L}^{-1}} L_3$ instead of $\mathcal{P}_{\mathbb{L}^{-1}}(\mathbf{0}, L_3)$. The similar abusive notation is being used for the other subspaces $\mathbb{K}^i$, $\mathbb{L}^i$.

We also use the following notations:

$$\begin{aligned}
&\Pi \text{ and } \Pi_1 \text{ are the projector from } \mathbb{E} \text{ onto } \mathbb{M}^{-1} \oplus \mathbb{M}^{-2} \text{ and onto } \mathbb{L}^{-1},\\
&\mathbf{T} = -\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R},\\
&\mathbf{Q} = \left(\mathbf{Id} + {}^t\mathbf{T}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}\right),\\
&\mathbf{H} = \left({}^t\Pi_1\mathbf{T}\Pi_1 - \left(\mathbf{Id} + {}^t\mathbf{T}\right)\mathcal{R}\left(\Pi\mathcal{R}\Pi\right)^{-1}\right)\mathcal{P}_{\mathbb{L}^{-1}}h,\\
&\mathbf{F} = 2\left(\mathbf{H} - \mathbf{Q}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha\left(\varphi_m\right)\right)_{\alpha=1,2}\right),\\
&Q_{\alpha\beta\gamma\delta}, F^0_{\alpha\beta} \text{ are the components on } \mathbb{K}^0 \text{ of } \mathbf{Q} \text{ and } \mathbf{F}.
\end{aligned}$$

V_{KL} and $l_u(\mathbf{v})$ are defined in (7) and (17).

THEOREM 4.1. *Under assumptions (1), (2), (6), (25), (27_3), (61), and ($34_{1,3}$),*

(i) *the solution $(\widehat{\mathbf{u}}^a, \widehat{\varphi}^a)$ of (62)–(64), and (27) scaled by formulae (5) and (32) converges weakly towards $(\mathbf{u}, \varphi_m)$ in $H^1(\Omega)^3 \times H^1(\Omega_1)$, where $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3,\ \overline{u}_2 - x_3\partial_2 u_3,\ u_3) \in V_{KL}$ is the unique solution of*

$$\int_\omega \left(s_{\alpha\beta}\left(\overline{\mathbf{v}}\right)\ \partial^2_{\alpha\beta}v_3\right)\begin{pmatrix} 2Q_{\alpha\beta\gamma\delta} & 0 \\ 0 & \dfrac{2}{3}Q_{\alpha\beta\gamma\delta}\end{pmatrix}\begin{pmatrix} s_{\gamma\delta}\left(\overline{\mathbf{u}}\right) \\ \partial^2_{\gamma\delta}u_3\end{pmatrix} d\mathbf{x} = l_u\left(\mathbf{v}\right) + \int_{\omega_1} s_{\alpha\beta}\left(\overline{\mathbf{v}}\right)F_{\alpha\beta}\,d\mathbf{x}, \tag{69}$$
$$\textit{for every } \mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL},$$

(ii) $\frac{\partial_3\widehat{\varphi}^a}{a}$ *converges weakly in $L^2(\Omega_1)$ towards L_3 such that $\mathcal{P}_{\mathbb{L}^{-1}}(L_3) = \Pi_1(\mathbf{T}(\mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}) + (\Pi\mathcal{R}\Pi)^{-1}\mathcal{P}_{\mathbb{L}^{-1}}h)$, where $\mathbf{M}^0 = (((s_{\alpha\beta}(\mathbf{u}))_{\alpha,\beta=1,2}, \mathbf{0}, 0, \mathbf{0}, 0)$.*

PROOF. V^a_{ad} and $\Psi^a_{\text{ad}}(\varphi^a_m)$ are defined in (14) and (35). The variational formulation of (62)–(64) and (27) is

$$W^a_{\text{ad}}\left(\varphi^a_m\right) = V^a_{\text{ad}} \times \Psi^a_{\text{ad}}\left(\varphi^a_m\right),$$
$$\int_{\Omega^a}\left(R_{ijkl}s_{kl}\left(\mathbf{u}^a\right) + d_{kij}e_k\left(\varphi^a\right)\right)s_{ij}\left(\mathbf{v}\right) + \left(-d_{kij}s_{ij}\left(u^a\right) + c_{ki}e_i\left(\varphi^a\right)\right)e_k(\psi)\,dx$$
$$= \int_{\Omega^a} f^a_i v_i\,dx + \int_{\Gamma^a_1 \cup \Gamma^{a\pm}} g^a_i v_i\,ds + \int_{\Gamma^{a+}_{\text{inc}}} h^a\psi\,ds,$$
$$\forall\ (\mathbf{v}, \psi) \in W^a_{\text{ad}}(0), \qquad \text{with } \left(\mathbf{u}^a, \varphi^a\right) \in W^a_{\text{ad}}\left(\varphi^a_m\right).$$

(Recall that the tensors $\mathbf{d}$ and $\mathbf{c}$ vanish in Ω^a_2.)

Scaling the solution with (5) and (32), removing the hats on the scaled functions, V_{ad} and $\Psi_{\text{ad}}(\varphi^a_m)$ being defined in (15) and (36), $\tilde{\varphi}^a$ being the extension of φ^a_m defined in Lemma 3.1, letting $\overline{\varphi}^a = \varphi^a - \tilde{\varphi}^a$, and $\widetilde{\mathbf{U}}^a = (\mathbf{0}, \tilde{\varphi}^a)$ to which this is equivalent, $\overline{\mathbf{U}}^a = (\mathbf{u}^a, \overline{\varphi}^a) \in W_{\text{ad}}(0)$ is the solution of

$$W_{\text{ad}}(\varphi^a_m) = V_{\text{ad}} \times \Psi_{\text{ad}}\left(\varphi^a_m\right),$$
$$\int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\overline{\mathbf{U}}^a\right)dx = l^a_u(\mathbf{v}) + l^a_\varphi\left(L^a_3(\psi)\right) - \int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\widetilde{\mathbf{U}}^a\right)dx, \tag{70}$$
$$\text{for every } \mathbf{V} = (\mathbf{v}, \psi) \in W_{\text{ad}}(0),$$

where $l_u^a(\mathbf{v})$ and $l_\varphi^a(L_3^a(\psi))$ are defined in (15) and (36). From (1), (2), (25), and (61), one easily verifies that there exists $C > 0$ such that

$$^t\mathbf{M}.\mathcal{R}.\mathbf{M} \geq C\,\|\mathbf{M}\|^2, \qquad \text{for every } \mathbf{M} \in \mathbb{R}^{10} \text{ such that } M_2 = M_3. \tag{71}$$

Then, letting $\mathbf{V} = \overline{\mathbf{U}}^a$ in (70) implies that $\|\mathbf{M}^a(\overline{\mathbf{U}}^a)\|_{(L^2(\Omega))^7\times(L^2(\Omega_1))^3} \leq C$ and $\|\mathbf{M}^a(\mathbf{U}^a)\|_{(L^2(\Omega))^7\times(L^2(\Omega_1))^3} \leq C$. Thus, there exists an extracted subsequence of $M^a(\mathbf{U}^a)$ which converges weakly in $(L^2(\Omega))^7 \times (L^2(\Omega_1))^3$ towards a limit $\mathbf{M} = ((K_{\alpha\beta})_{\alpha\beta=1,2}, (K_{\alpha 3})_{\alpha=1,2}, K_{33}, (L_\alpha)_{\alpha=1,2}, L_3)$. Application of Korn and Poincaré inequalities leads to the weak convergence of $\mathbf{U}^a$ towards a limit $\mathbf{U} = (\mathbf{u}, \varphi_m)$ in $H^1(\Omega)^3 \times H^1(\Omega_1)$. Define the operators $\mathbf{M}^0(\mathbf{V})$, $\mathbf{M}^{-1}(\mathbf{V})$, and $\mathbf{M}^{-2}(\mathbf{V})$ on $\mathbf{V} = (\mathbf{v}, \psi) \in W_{\mathrm{ad}}(0)$ by

$$\mathbf{M}^0(\mathbf{V}) = \left(\mathbf{K}^0(\mathbf{v}), \mathbf{0}_3\right), \quad \mathbf{M}^{-1}(\mathbf{V}) = \left(\mathbf{K}^{-1}(\mathbf{v}), \mathbf{L}^{-1}(\psi)\right), \quad \text{and} \quad \mathbf{M}^{-2}(\mathbf{V}) = \left(\mathbf{K}^{-2}(\mathbf{v}), \mathbf{0}_3\right), \tag{72}$$

where operators $\mathbf{K}^0$, $\mathbf{K}^{-1}$, $\mathbf{K}^{-2}$ are defined in Section 2.2, $\mathbf{L}^{-1}$ is defined in Section 3.2 and $\mathbf{0}_3 = (0,0,0)$. Multiplication of (70) by a^2, a, and 1 successively implies

$$\begin{aligned}
&\int_\Omega {}^t\mathbf{M}^{-2}(\mathbf{V})\mathcal{R}\mathbf{M}\,dx = 0, && \forall \mathbf{V} = (\mathbf{v}, \psi) \in W_{\mathrm{ad}}(0), \\
&\int_\Omega {}^t\mathbf{M}^{-1}(\mathbf{V})\mathcal{R}\mathbf{M}\,dx = l_\varphi\left(L_3^{-1}(\psi)\right), && \forall \mathbf{V} = (\mathbf{v}, \psi) \in W_{\mathrm{ad}}(0) \cap \mathrm{Ker}\left(\mathbf{M}^{-2}\right), \\
&\int_\Omega {}^t\mathbf{M}^{0}(\mathbf{V})\mathcal{R}\mathbf{M}\,dx = l_u(\mathbf{v}), && \forall \mathbf{V} = (\mathbf{v}, \psi) \in W_{\mathrm{ad}}(0) \cap \mathrm{Ker}\left(\mathbf{M}^{-2}\right) \cap \mathrm{Ker}\left(\mathbf{M}^{-1}\right),
\end{aligned} \tag{73}$$

where $l_u(\mathbf{v})$ and $l_\varphi(L_3^{-1}(\psi))$ are defined in (17) and (39). Now, Lemmas 2.1 and 3.2 yield the following.

LEMMA 4.1. *The spaces $\mathbb{M}^{-2}$, $\mathbb{M}^{-1}$, and $\mathbb{M}^0$ being defined by (68),*

(i) *the spaces $\{\mathbf{M}^{-2}(\mathbf{V});\, V \in W_{\mathrm{ad}}(0)\}$, $\{\mathbf{M}^{-1}(\mathbf{V});\, \mathbf{V} \in W_{\mathrm{ad}}(0)$ and $\mathbf{M}^{-2}(\mathbf{V}) = 0\}$ are dense, respectively, in $\mathbb{M}^{-2}$ and $\mathbb{M}^{-1}$, and*

(ii) *$\{\mathbf{M}^0(\mathbf{V});\, \mathbf{V} = (\mathbf{v}, \psi) \in W_{\mathrm{ad}}(0)$ and $\mathbf{M}^{-2}(\mathbf{V}) = \mathbf{M}^{-1}(\mathbf{V}) = 0\} = \mathbb{M}^0$. In particular, for every $\mathbf{M}^0 \in \mathbb{M}^0$, there exists $\mathbf{V} = (\mathbf{v}, 0) \in W_{\mathrm{ad}}(0)$ such that $\mathbf{M}^{-2}(\mathbf{V}) = \mathbf{M}^{-1}(\mathbf{V}) = 0$ and $\mathbf{M}^0 = \mathbf{M}^0(\mathbf{V})$.*

Lemma 4.1 and (73) imply that

$$\begin{aligned}
&\int_\Omega {}^t\tilde{\mathbf{M}}^{-2}\mathcal{R}\mathbf{M}\,dx = 0, && \text{for every } \tilde{\mathbf{M}}^{-2} \in \mathbb{M}^{-2}, \\
&\int_\Omega {}^t\tilde{\mathbf{M}}^{-1}\mathcal{R}\mathbf{M}\,dx = l_\varphi\left(\tilde{L}_3\right), && \text{for every } \tilde{\mathbf{M}}^{-1} \in \mathbb{M}^{-1}, \\
&\int_\Omega {}^t\tilde{\mathbf{M}}^{0}\mathcal{R}\mathbf{M}\,dx = l_u(\mathbf{v}), && \text{for every } \tilde{\mathbf{M}}^{0} \in \mathbb{M}^{0},
\end{aligned}$$

where $\mathbf{v}$ is the vector associated with $\tilde{\mathbf{M}}^0$. Since $\mathbb{M} = \mathbb{M}^{-2} \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^0$, this is equivalent to

$$\int_\Omega \tilde{\mathbf{M}}\mathcal{R}\mathbf{M}\,dx = l_\varphi(\tilde{L}_3) + l_u(\mathbf{v}), \qquad \text{for every } \tilde{\mathbf{M}} \in \mathbb{M}. \tag{74}$$

LEMMA 4.2. *The limit $\mathbf{M}$ of $\mathbf{M}^a(\mathbf{U}^a)$ belongs to $\mathbb{M} + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}$ and is the unique solution of (74).*

PROOF. The fact that $\mathbf{M} \in \mathbb{M} + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}$ is shown as $\mathbf{K} \in \mathbb{K}$ in Lemma 2.2 and $\mathbf{L} \in \mathbb{L}^{-1} + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}$ in Section 3.2. Existence and uniqueness of (74) follow from the Lax-Milgram lemma on the Hilbert space $\mathbb{M}$ and of the ellipticity (71) of $\mathcal{R}$. ∎

REMARK. From Lemma 4.2, the limit $\mathbf{M} = (\mathbf{K}, \mathbf{L})$ satisfies $L_\alpha = e_\alpha(\varphi_m)$, $K_{\alpha\beta} = s_{\alpha\beta}(\mathbf{u})$, and $\mathbf{u}$ is a Love Kirchhoff field: $u_\alpha = \overline{u}_\alpha - x_3\partial_\alpha u_3$.

Now, we simplify as much as possible equation (74). The choice $\tilde{\mathbf{M}} \in \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$ in (74) leads to $\Pi\mathcal{R}\mathbf{M} = \mathcal{P}_{\mathbf{L}^{-1}}h$. Writing $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0 + \mathcal{P}_{\mathbf{L}^0}((e_\alpha(\varphi_m))_{\alpha=1,2})$, where $\mathbf{M}^0 \in \mathbb{M}^0$, then $\Pi\mathcal{R}\Pi M = \mathcal{P}_{\mathbf{L}^{-1}}h - \Pi\mathcal{R}(\mathbf{M}^0 + \mathcal{P}_{\mathbf{L}^0}((e_\alpha(\varphi_m))_{\alpha=1,2}))$ or equivalently $\Pi\mathbf{M} = \mathbf{T}(\mathbf{M}^0 + \mathcal{P}_{\mathbf{L}^0}((e_\alpha(\varphi_m))_{\alpha=1,2})) + (\Pi\mathcal{R}\Pi)^{-1}\mathcal{P}_{\mathbf{L}^{-1}}h$. This gives the expression of $\mathbf{M}$ with respect to $\mathbf{M}^0$:

$$\mathbf{M} = (\mathbf{Id} + \mathbf{T})\left(\mathbf{M}^0 + \mathcal{P}_{\mathbf{L}^0}\left((e_\alpha(\varphi_m))_{\alpha=1,2}\right)\right) + (\Pi\mathcal{R}\Pi)^{-1}\mathcal{P}_{\mathbf{L}^{-1}}h. \tag{75}$$

Choose $\tilde{\mathbf{M}}$ on the form $\tilde{\mathbf{M}} = (\mathbf{Id} + \mathbf{T})\tilde{\mathbf{M}}^0$, where $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$ in (74). Replacing $\mathbf{M}$ by its expression with respect to $\mathbf{M}^0$ and $\mathcal{P}_{\mathbf{L}^{-1}}\tilde{L}_3$ by $\Pi_1\tilde{\mathbf{M}} = \Pi_1(\Pi\tilde{\mathbf{M}}) = \Pi_1(\mathbf{Id} + \mathbf{T})\tilde{\mathbf{M}}_0$ leads to

$$\int_\Omega {}^t\tilde{\mathbf{M}}^0 (\mathbf{Id} + {}^t\mathbf{T})\,\mathcal{R}\left((\mathbf{Id} + \mathbf{T})\left(\mathbf{M}^0 + \mathcal{P}_{\mathbf{L}^0}\left((e_\alpha(\varphi_m))_{\alpha=1,2}\right)\right) + (\Pi\mathcal{R}\Pi)^{-1}\mathcal{P}_{\mathbf{L}^{-1}}h\right) dx$$
$$= l_u(\mathbf{v}) + \int_{\Omega_1} {}^t(\mathcal{P}_{\mathbf{L}^{-1}}h).\Pi_1\mathbf{T}\tilde{\mathbf{M}}_0\,dx, \tag{76}$$

or equivalently,

$$\int_\Omega {}^t\tilde{\mathbf{M}}^0\mathbf{Q}\mathbf{M}^0\,dx = \int_{\Omega_1} {}^t\tilde{\mathbf{M}}^0\left(\mathbf{H} - \mathbf{Q}\mathcal{P}_{\mathbf{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}\right)dx + l_u(\mathbf{v}), \tag{77}$$

for every $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$, where $\mathbf{v}$ is the vector of V_{KL} associated with $\tilde{\mathbf{M}}^0$. Since

$$\tilde{\mathbf{M}}^0 = \mathcal{P}_{\mathbb{K}^0}(s_{\alpha\beta}(\overline{\mathbf{v}}) - x_3\partial^2_{\alpha\beta}v_3)_{\alpha,\beta=1,2} \quad \text{and} \quad \mathbf{M}^0 = \mathcal{P}_{\mathbb{K}^0}\left(s_{\gamma\delta}(\overline{u}) - x_3\partial^2_{\gamma\delta}u_3\right)_{\gamma,\delta=1,2}, \tag{78}$$

this completes the proof of Point (i). Point (ii) follows directly from (77) and (78). ∎

4.2. Dirichlet Boundary Conditions

We use the same notations as in the preceding section excepted that here $\mathbb{M}^{-1}$ is defined by

$$\mathbb{M}^{-1} = \left\{\mathbf{M} = (\mathbf{K}, \mathbf{L}) \in \mathbb{K}^{-1} \times \mathcal{N}(\mathbb{L}^{-1})\right\}. \tag{79}$$

This changes the meaning of the preceding definitions of $\mathbb{M}$, Π, and Π_1. In addition, we need the operators

$$\begin{gathered}
\mathcal{M}(\mathbf{M}) = \frac{1}{2}\int_{-1}^{1}\mathbf{M}\,dx_3, \qquad \text{for every } \mathbf{M} \in \mathbb{M} \quad \text{and} \quad \mathcal{N} = \mathbf{Id}_{\mathbf{M}} - \mathcal{M},\\
\Pi_2 = \Pi - \Pi_1 \text{ the projector from } \mathbb{E} \text{ onto } \mathbb{M}^{-2} \oplus \left(\mathbb{K}^{-1} \times \{0_3\}\right),\\
\mathbf{T}_{\mathcal{N}} = -(\Pi\mathcal{R}\Pi)^{-1}\Pi\mathcal{R} \quad \text{and} \quad \mathbf{T}_{\mathcal{M}} = -(\Pi_2\mathcal{R}\Pi_2)^{-1}\Pi_2\mathcal{R},\\
\mathbf{Q}_{\mathcal{N}} = {}^t(\mathbf{Id} + \mathbf{T}_{\mathcal{N}})\mathcal{R}(\mathbf{Id} + \mathbf{T}_{\mathcal{N}}) \quad \text{and} \quad \mathbf{Q}_{\mathcal{M}} = {}^t(\mathbf{Id} + \mathbf{T}_{\mathcal{M}})\mathcal{R}(\mathbf{Id} + \mathbf{T}_{\mathcal{M}}),\\
\mathbf{F} = -2{}^t\mathbf{Q}_{\mathcal{M}}\left(\mathcal{P}_{\mathcal{M}(\mathbf{L}^{-1})}\varphi_c + \mathcal{P}_{\mathbf{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}\right),\\
Q_{\mathcal{M}\alpha\beta\gamma\delta},\ Q_{\mathcal{N}\alpha\beta\gamma\delta},\ F_{\alpha\beta} \text{ are the components on } \mathbb{K}^0 \text{ of } \mathbf{Q}_{\mathcal{M}},\ \mathbf{Q}_{\mathcal{N}},\ \mathbf{F},
\end{gathered} \tag{80}$$

and the definitions (7) and (17) of V_{KL} and $l_u(\mathbf{v})$.

THEOREM 4.2. *Under assumptions (1), (2), (6), (25), (26), (28₂), (42), and (61),*

(i) *the solutions $(\widehat{\mathbf{u}}^a, \widehat{\varphi}^a)$ of (62)–(64), and (28) scaled by formulae (5) and (40) converges weakly towards $(\mathbf{u}, \varphi_m)$ in $H^1(\Omega)^3 \times H^1(\Omega_1)$, where $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3,\ \overline{u}_2 - x_3\partial_2 u_3,\ u_3) \in V_{KL}$ is the unique solution of*

$$\int_\omega \left(s_{\alpha\beta}(\overline{\mathbf{v}}),\, \partial^2_{\alpha\beta}v_3\right)\begin{pmatrix} 2Q_{\mathcal{M}\alpha\beta\gamma\delta} & 0 \\ 0 & \frac{2}{3}Q_{\mathcal{N}\alpha\beta\gamma\delta}\end{pmatrix}\begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}}) \\ \partial^2_{\gamma\delta}u_3\end{pmatrix} dx$$
$$= l_u(\mathbf{v}) + \int_{\omega_1} s_{\alpha\beta}(\overline{\mathbf{v}})\,F_{\alpha\beta}\,dx,$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL}$*;*

(ii) $\frac{\partial_3\tilde{\varphi}^a}{a}$ *converges weakly in* $L^2(\Omega_1)$ *towards* L_3 *such that* $\mathcal{P}_{\mathbb{L}^{-1}}(L_3) = \mathcal{P}_{\mathbb{L}^{-1}}(\varphi_c) - x_3 \Pi_{\mathbb{L}^{-1}} \mathbf{T}_{\mathcal{N}}\mathcal{P}_{\mathbb{K}^0}(\partial^2_{\alpha\beta}u_3)_{\alpha,\beta=1,2}$.

PROOF. After scaling and suppression of the hats on scaled functions, if $\tilde{\varphi}^a$ is given by the Lemma 3.3, $\overline{\varphi}^a = \varphi^a - \tilde{\varphi}^a$, $\tilde{\mathbf{U}}^a = (0, \tilde{\varphi}^a)$, then $\overline{\mathbf{U}}^a = (\mathbf{u}^a, \overline{\varphi}^a) \in W_{\text{ad}}(0)$ is solution of the variational formulation on the scaled domain

$$
\begin{gathered}
W_{\text{ad}}\left(\varphi^a_m, \varphi^a_c\right) = V_{\text{ad}} \times \Psi_{\text{ad}}\left(\varphi^a_m, \varphi^a_c\right), \\
\int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\overline{\mathbf{U}}^a\right) dx = l^a_u(\mathbf{v}) - \int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\tilde{\mathbf{U}}^a\right) dx, \\
\text{for every } \mathbf{V} = (\mathbf{v}, \psi) \in W_{\text{ad}}(0),
\end{gathered} \tag{81}
$$

where V_{ad} and $\Psi_{\text{ad}}(\varphi^a_m, \varphi^a_c)$ are defined in (14) and (45). Choose $\mathbf{V} = \overline{\mathbf{U}}^a$ in (81). Taking into account assumptions (1), (2), (6), (25), (26), (28_2), and (61) and Lemma 3.3, then $\|\mathbf{M}^a(\mathbf{U}^a)\|_{(L^2(\Omega))^7\times(L^2(\Omega_1))^3} \leq C$. Then, $\mathbf{M}^a(\mathbf{U}^a)$ has the same properties of convergence as in the case of Neumann conditions. The variational formulation satisfied by $\mathbf{U}^a = (\mathbf{u}^a, \varphi^a) \in W_{\text{ad}}(\varphi^a_m, \varphi^a_c)$ is

$$
\int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\mathbf{U}^a\right) dx = l^a_u(\mathbf{v}), \qquad \text{for every } \mathbf{V} = (\mathbf{v}, \psi) \in W_{\text{ad}}(0). \tag{82}
$$

Multiplying successively by a^2, a, or 1 and passing in the limit in (82) leads to

$$
\begin{aligned}
&\int_\Omega {}^t\mathbf{M}^{-2}(\mathbf{V})\mathcal{R}\mathbf{M}\, dx = 0, && \text{for every } \mathbf{V} \in W_{\text{ad}}(0), \\
&\int_\Omega {}^t\mathbf{M}^{-1}(\mathbf{V})\mathcal{R}\mathbf{M}\, dx = 0, && \text{for every } \mathbf{V} \in W_{\text{ad}}(0) \cap \operatorname{Ker}\left(\mathbf{M}^{-2}\right), \\
&\int_\Omega {}^t\mathbf{M}^{0}(\mathbf{V})\mathcal{R}K\, dx = l_u(\mathbf{v}), && \text{for every } \mathbf{V} = (\mathbf{v}, \psi) \in W_{\text{ad}}(0) \cap \operatorname{Ker}\left(\mathbf{M}^{-2}\right) \cap \operatorname{Ker}\left(\mathbf{M}^{-1}\right).
\end{aligned} \tag{83}
$$

Similarly to Lemma 4.1, using Lemma 3.4, we have the following lemma.

LEMMA 4.3. *The spaces* $\mathbb{M}^{-2}$, $\mathbb{M}^{-1}$, *and* $\mathbb{M}^0$ *being defined by (68) and (79), the conclusions of Lemma 4.1 hold.*

Using Lemma 4.3 and $\mathbb{M} = \mathbb{M}^0 \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$, (83) is equivalent to

$$
\int_\Omega \tilde{\mathbf{M}}\mathcal{R}\mathbf{M}\, dx = l_u(\mathbf{v}), \qquad \text{for every } \tilde{\mathbf{M}} \in \mathbb{M}, \tag{84}
$$

where $\mathbf{v}$ is the vector associated with $\tilde{\mathbf{M}}^0 = \mathcal{P}_{\mathbb{M}^0}(\tilde{\mathbf{M}})$. From Theorem 3.2(i), $L_\alpha = e_\alpha(\varphi_m)$ and arguing as in the proof of Theorem 3.2(ii), $\mathcal{M}(\mathcal{P}_{\mathbb{L}^{-1}}L_3) = \mathcal{P}_{\mathbb{L}^{-1}}\varphi_c$. From Lemma 2.2(i), $\mathbf{K} \in \mathbb{K}$. Writing then $\mathcal{P}_{\mathbb{L}}L_3 = \mathcal{M}(\mathcal{P}_{\mathbb{L}^{-1}}L_3) + \mathcal{N}(\mathcal{P}_{\mathbb{L}^{-1}}L_3)$, where $\mathcal{N}(\mathcal{P}_{\mathbb{L}^{-1}}L_3) \in \mathcal{N}(\mathbb{L}^{-1})$, then $\mathbf{M} \in \mathbb{M} + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2} + \mathcal{P}_{\mathcal{M}(\mathbb{L}^{-1})}\varphi_c$. Then, the Lax Milgram lemma ensures the existence and uniqueness of $\mathbf{M}$ solution of (84). Now, choose $\tilde{\mathbf{M}} \in \mathcal{M}(\mathbb{M}^{-1} \oplus \mathbb{M}^{-2})$ in (84). Using the fact that $\mathcal{R}$ does not depend on x_3 and decomposing $\mathbf{M} = \mathcal{M}(\mathbf{M}) + \mathcal{N}(\mathbf{M})$ yields

$$
\int_\Omega \tilde{\mathbf{M}}\mathcal{R}\mathbf{M}\, dx = \int_\Omega \tilde{\mathbf{M}}\mathcal{R}\mathcal{M}(\mathbf{M})\, dx = 0,
$$

where $\int_{-1}^1 \mathcal{N}(\mathbf{M})\, dx_3 = 0$ has been used. Hence, $\Pi_2\mathcal{R}\mathcal{M}(\mathbf{M}) = 0$. Similarly, (84) implies that $\Pi\mathcal{R}\mathcal{N}(\mathbf{M}) = 0$. Writing $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2} + \mathcal{P}_{\mathcal{M}(\mathbb{L}^{-1})}\varphi_c$, where $\mathbf{M}^0 \in \mathbb{M}^0$, then

$$
\begin{aligned}
&\mathcal{N}(\Pi\mathbf{M}) = -\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R}\mathcal{N}\left(\mathbf{M}^0\right) \qquad \text{and} \\
&\mathcal{M}\left(\Pi_2\mathbf{M}\right) = -\left(\Pi_2\mathcal{R}\Pi_2\right)^{-1}\Pi_2\mathcal{R}\left(\mathcal{M}\left(\mathbf{M}^0\right) + \mathcal{P}_{\mathcal{M}(\mathbb{L}^{-1})}\varphi_c + \mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right).
\end{aligned} \tag{85}
$$

Thus,

$$\begin{aligned}&\mathcal{N}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{N}})\,\mathcal{N}(\mathbf{M}^0) \qquad \text{and}\\ &\mathcal{M}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{M}})\left(\mathcal{M}(\mathbf{M}^0) + \mathcal{P}_{\mathcal{M}(\mathbf{L}^{-1})}\varphi_c + \mathcal{P}_{\mathbf{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right).\end{aligned} \tag{86}$$

For $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$, choose $\tilde{\mathbf{M}} \in \mathbb{M}$ such that $\mathcal{N}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{N}})\mathcal{N}(\tilde{\mathbf{M}}^0)$ and $\mathcal{M}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{M}})\mathcal{M}(\tilde{\mathbf{M}}^0)$. Then, (84) yields

$$\begin{aligned}&\int_\Omega \left(\mathcal{M}\left(\tilde{\mathbf{M}}^0\right), \mathcal{N}\left(\tilde{\mathbf{M}}^0\right)\right)\begin{pmatrix}\mathbf{Q}_{\mathcal{M}} & 0\\ 0 & \mathbf{Q}_{\mathcal{N}}\end{pmatrix}\begin{pmatrix}\mathcal{M}(\mathbf{M}^0)\\ \mathcal{N}(\mathbf{M}^0)\end{pmatrix} dx\\ &\qquad = l_u(\mathbf{v}) - \int_{\Omega_1} \mathcal{M}\left(\tilde{\mathbf{M}}^0\right)\, \mathbf{Q}_{\mathcal{M}}\left(\mathcal{P}_{\mathcal{M}}\left(\mathbb{L}^{-1}\right)\varphi_c + \mathcal{P}_{\mathbf{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right) dx,\end{aligned} \tag{87}$$

for every $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$, where $\mathbf{v}$ is the vector associated with $\tilde{\mathbf{M}}^0$, since

$$\begin{aligned}\left(\mathcal{M}\left(\tilde{\mathbf{M}}^0\right), \mathcal{N}\left(\tilde{\mathbf{M}}^0\right)\right) &= \mathcal{P}_{\mathbf{K}^0}\left(s_{\alpha\beta}(\overline{\mathbf{v}}), -x_3\partial^2_{\alpha\beta}v_3\right)_{\alpha,\beta=1,2} \qquad \text{and}\\ \left(\mathcal{M}\left(\mathbf{M}^0\right), \mathcal{N}\left(\mathbf{M}^0\right)\right) &= \mathcal{P}_{\mathbf{K}^0}\left(s_{\gamma\delta}(\overline{\mathbf{u}}), -x_3\partial^2_{\gamma\delta}u_3\right)_{\gamma,\delta=1,2}.\end{aligned} \tag{88}$$

This ends the proof of (i). Point (ii) follows directly from (i) and (85)–(88) the model. ∎

4.3. The Mixed Boundary Conditions

The definitions and notations of Section 4.2 are used except the definition of $\mathbb{M}^{-1}$ is replaced by (68) and the definitions of $\mathbf{T}_{\mathcal{N}}$, $\mathbf{T}_{\mathcal{M}}$, and $(F_{\alpha\beta})_{\alpha,\beta=1,2}$ are replaced by

$$\begin{aligned}&\mathbf{T}_{\mathcal{N}} = (\Pi\mathcal{R}\Pi)^{-1}\,\Pi\mathcal{R} \quad \text{and} \quad \mathbf{T}_{\mathcal{M}} = (\Pi\mathcal{R}\Pi + 2\Pi_1 G\Pi_1)^{-1}\,\Pi\mathcal{R},\\ &\mathbf{F} = 2\left(\mathbf{H} - \mathbf{Q}_{\mathcal{M}}\mathcal{P}_{\mathbf{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right),\\ &\mathbf{H} = \left({}^t\mathbf{T}_{\mathcal{M}} - (\mathbf{Id}+{}^t\mathbf{T}_{\mathcal{M}})\,(\mathcal{R} + 2\Pi_1 G\Pi_1)\,(\Pi\mathcal{R}\Pi + 2\Pi_1 G\Pi_1)^{-1}\right)\mathcal{P}_{\mathbf{L}^{-1}}h,\\ &\qquad F_{\alpha\beta} \text{ are the components of } \mathbf{F} \text{ on } \mathbb{K}^0.\end{aligned}$$

THEOREM 4.3. *Under assumptions (1), (2), (6), (25), (26_2), (29_3), (34), and (61), if G is positive, then*

(i) *the solution $(\hat{\mathbf{u}}^a, \hat{\varphi}^a)$ of (62)–(64) and ($26_{1,2}$) scaled by formulae (5), (32), and (48) converges weakly towards $(\mathbf{u}, \varphi_m)$ in $H^1(\Omega)^3 \times H^1(\Omega_1)$ where $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3) \in V_{KL}$ is the unique solution of the equation*

$$\begin{aligned}&\int_\omega \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \partial^2_{\alpha\beta}v_3\right)\begin{pmatrix}2Q_{\mathcal{M}\alpha\beta\gamma\delta} & 0\\ 0 & \frac{2}{3}Q_{\mathcal{N}\alpha\beta\gamma\delta}\end{pmatrix}\begin{pmatrix}s_{\gamma\delta}(\overline{\mathbf{u}})\\ \partial^2_{\gamma\delta}u_3\end{pmatrix} dx\\ &\qquad = l_u(\mathbf{v}) + \int_{\omega_1} s_{\alpha\beta}(\overline{\mathbf{v}})\, F_{\mathcal{M}\alpha\beta}\, dx,\end{aligned} \tag{89}$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL}$;

(ii) $\frac{\partial_3\hat{\varphi}^a}{a}$ *converges weakly in $L^2(\Omega_1)$ towards L_3 such that* $\mathcal{P}_{\mathbf{L}^{-1}}(L_3) = \Pi_1(\mathbf{T}_{\mathcal{M}}(\mathcal{P}_{\mathbf{K}^0}(s_{\gamma\delta}(\overline{u}))_{\gamma,\delta=1,2} + \mathcal{P}_{\mathbf{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}) + (\Pi\mathcal{R}\Pi + 2\Pi_1 G\Pi_1)^{-1}\mathcal{P}_{\mathbf{L}^{-1}}h) - x_3\Pi_1\mathbf{T}_{\mathcal{N}}\mathcal{P}_{\mathbf{K}^0}(\partial^2_{\gamma\delta}u_3)_{\gamma,\delta=1,2}$.

PROOF. After scaling and suppression of the hats on scaled functions, if $\tilde{\varphi}^a$ is given by Lemma 3.1, and $\overline{\varphi}^a = \varphi^a - \tilde{\varphi}^a$, state $\tilde{\mathbf{U}}^a = (0, \tilde{\varphi}^a)$ and $\overline{\mathbf{U}}^a = (\mathbf{u}^a, \overline{\varphi}^a)$. The spaces V_{ad} and Ψ_{ad} being defined in (15) and (36), $\overline{\mathbf{U}}^a = (\mathbf{u}^a, \overline{\varphi}^a) \in W_{\text{ad}}(0)$ is the solution of the variational formulation on the scaled domain

$$W_{\text{ad}}\left(\varphi_m^a\right) = V_{\text{ad}} \times \Psi_{\text{ad}}\left(\varphi_m^a\right),$$

$$\int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\overline{\mathbf{U}}^a\right) dx + \frac{1}{a^2}\int_{\omega_1} G\overline{\varphi}^a\psi\, dx = -\int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}\mathbf{M}^a\left(\tilde{\mathbf{U}}^a\right) dx$$
$$+ l_u^a(\mathbf{v}) + l_\varphi^a\left(L_3^a(\psi)\right), \qquad \text{for every } \mathbf{V} = (\mathbf{v},\psi) \in W_{\text{ad}}(0),$$

where $l_u^a(\mathbf{v})$ and $l_\varphi^a(L_3^a(\psi))$ are defined in (15) and (36). Choose $\mathbf{V} = \overline{\mathbf{U}}^a$; as for Neumann conditions, it implies that $\|\mathbf{M}^a(\overline{\mathbf{U}}^a)\|_{(L^2(\Omega))^7\times(L^2(\Omega_1))^3} \leq C$, thus $\|\mathbf{M}^a(\mathbf{U}^a)\|_{(L^2(\Omega))^7\times(L^2(\Omega_1))^3} \leq C$. The same convergences of $\mathbf{M}^a(\mathbf{U}^a)$ as in the Neumann boundary conditions hold. The variational formulation for $\mathbf{U}^a = (\mathbf{u}^a, \varphi^a) \in W_{\text{ad}}(\varphi_m^a)$ is

$$\int_\Omega \left({}^t\mathbf{M}^a(\mathbf{V})\,\mathcal{R}\mathbf{M}^a(\mathbf{U}^a) + 2GL_3^a(\psi)\mathcal{M}(L_3^a)\right) dx$$
$$= l_u^a(\mathbf{v}) + l_\varphi^a\left(L_3^a(\psi)\right), \qquad \text{for every } \mathbf{V} = (\mathbf{v},\psi) \in W_{\text{ad}}(0). \tag{90}$$

Multiplying (90) by a^2, a, and 1 successively, and passing to the limit yields

$$\int_\Omega {}^t\mathbf{M}^{-2}(\mathbf{V})\,\mathcal{R}\mathbf{M}\, dx = 0, \qquad \text{for every } \mathbf{V} = (\mathbf{v},\psi) \in W_{\text{ad}}(0),$$
$$\int_\Omega \left({}^t\mathbf{M}^{-1}(\mathbf{V})\mathcal{R}\mathbf{M} + 2G\tilde{L}_3\,\mathcal{M}(L_3)\right) dx = l_\varphi(L_3), \qquad \text{for every } \mathbf{V} \in W_{\text{ad}}(0) \cap \operatorname{Ker}\left(\mathbf{M}^{-2}\right),$$
$$\int_\Omega {}^t\mathbf{M}^0(\mathbf{V})\mathcal{R}K\, dx = l_u(\mathbf{v}) \tag{91}$$
$$\text{for every } \mathbf{V} = (\mathbf{v},\psi) \in W_{\text{ad}} \cap \operatorname{Ker}\left(\mathbf{M}^{-2}\right) \cap \operatorname{Ker}\left(\mathbf{M}^{-1}\right).$$

As for the case of Neumann boundary conditions, (91) is equivalent to

$$\int_\Omega \left(\tilde{\mathbf{M}}\mathcal{R}\mathbf{M} + 2G\tilde{L}_3\mathcal{M}(L_3)\right) dx = l_u(\mathbf{v}) + l_\varphi\left(\mathcal{M}\left(\tilde{L}_3\right)\right), \qquad \text{for every } \tilde{\mathbf{M}} \in \mathbb{M}. \tag{92}$$

Because G is supposed to be a positive constant, the Lax-Milgram lemma applies to (92). As in the case of Neumann boundary conditions, $\mathbf{M} \in \mathbb{M} + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}$. Choose $\tilde{\mathbf{M}} \in \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$, and decompose $\tilde{\mathbf{M}} = \mathcal{N}(\tilde{\mathbf{M}}) + \mathcal{M}(\tilde{\mathbf{M}})$, then

$$\int_\Omega \mathcal{N}\left({}^t\tilde{\mathbf{M}}\right)\mathcal{R}\mathcal{N}(\mathbf{M}) + \mathcal{M}\left({}^t\tilde{\mathbf{M}}\right)\left(\mathcal{R} + 2{}^t\Pi_1 G\Pi_1\right)\mathcal{M}(\mathbf{M})\, dx = l_u(\mathbf{v}) + l_\varphi\left(\mathcal{M}\left(\tilde{L}_3\right)\right). \tag{93}$$

Arguing as in Theorem 4.2, (93) implies that $\Pi\mathcal{R}_G\mathcal{M}(\mathbf{M}) = \mathcal{P}_{\mathbb{L}^{-1}}h$ and $\Pi\mathcal{R}\mathcal{N}(\mathbf{M}) = 0$ where $\mathcal{R}_G = \mathcal{R} + 2{}^t\Pi_1 G\Pi_1$. Writing $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}$ with $\mathbf{M}^0 \in \mathbb{M}^0$, $\mathcal{N}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}\Pi)^{-1}\Pi\mathcal{R}\mathcal{N}(\mathbf{M}^0)$ and $\mathcal{M}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}_G\Pi)^{-1}(\Pi\mathcal{R}(\mathcal{M}(\mathbf{M}^0) + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi_m))_{\alpha=1,2}) - \mathcal{P}_{\mathbb{L}^{-1}}h)$. Then

$$\begin{aligned} &\mathcal{N}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{N})\,\mathcal{N}\left(\mathbf{M}^0\right) \qquad \text{and} \\ &\mathcal{M}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{M})\left(\mathcal{M}\left(\mathbf{M}^0\right) + \mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right) + (\Pi\mathcal{R}_G\Pi)^{-1}\mathcal{P}_{\mathbb{L}^{-1}}h. \end{aligned} \tag{94}$$

For every $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$, we define $\tilde{\mathbf{M}} \in \mathbb{M}$ by $\mathcal{N}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_\mathcal{N})\mathcal{N}\left(\tilde{\mathbf{M}}^0\right)$ and $\mathcal{M}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_\mathcal{M})\mathcal{M}(\tilde{\mathbf{M}}^0)$. Hence, from (92),

$$\int_\Omega \left(\mathcal{M}\left({}^t\tilde{\mathbf{M}}^0\right), \mathcal{N}\left({}^t\tilde{\mathbf{M}}^0\right)\right)\begin{pmatrix}\mathbf{Q}_\mathcal{M} & 0 \\ 0 & \mathbf{Q}_\mathcal{N}\end{pmatrix}\begin{pmatrix}\mathcal{M}\left(\mathbf{M}^0\right) \\ \mathcal{N}\left(\mathbf{M}^0\right)\end{pmatrix} dx$$
$$= l_u(\mathbf{v}) + \int_{\Omega_1} \mathcal{M}\left({}^t\tilde{\mathbf{M}}^0\right)\left(-\mathbf{Q}_\mathcal{M}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right) \tag{95}$$
$$+ \left({}^t\mathbf{T}_\mathcal{M} - \left(\mathbf{Id} + {}^t\mathbf{T}_\mathcal{M}\right)\mathcal{R}_G\left(\Pi\mathcal{R}_G\Pi\right)^{-1}\right)\mathcal{P}_{\mathbb{L}^{-1}}h\, dx,$$

for every $\tilde{\mathbf{M}}^0 \in \mathbb{M}^0$ where $\mathbf{v}$ is the vector associated with $\tilde{\mathbf{M}}^0$. Using (88) this proves (i). Point (ii) follows as in Theorem 4.2. ∎

REMARK. In the above analysis, the operator $\Pi\mathcal{R}_G\Pi$ is invertible. In the context of evolution problems, G is an integro-differential operator in time, then $\Pi\mathcal{R}_G\Pi$ is not invertible. The elimination of $\mathcal{M}(L_3)$ is not possible.

Subsequently, the model is derived without eliminating $\mathcal{M}(L_3)$. The assumptions on the positivity of G and invertibility of $\Pi\mathcal{R}\Pi + \Pi_1 G \Pi_1$ are not released in order to preserve the *a priori* estimates and the convergence properties. For evolution problems, these preliminary results may be obtained by specific methods.

We use the same notations as in the preceding model, except that

$$\begin{aligned}
&\mathbf{T}_{\mathcal{N}} = -\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R} \quad \text{and} \quad \mathbf{T}_{\mathcal{M}} = -\left(\Pi_2\mathcal{R}\Pi_2\right)^{-1}\Pi_2\mathcal{R},\\
&\mathbf{Q}_{\mathcal{M}} = {}^t\left(\mathbf{Id} + \mathbf{T}_{\mathcal{M}}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}_{\mathcal{M}}\right) \quad \text{and} \quad \mathbf{Q}_{\mathcal{N}} = {}^t\left(\mathbf{Id} + \mathbf{T}_{\mathcal{N}}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}_{\mathcal{N}}\right),\\
&\mathbf{F} = 2\mathbf{Q}_{\mathcal{M}}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2},\\
&\mathcal{L} = \begin{cases} L^2(\omega_1) \text{ if faces are not metallized,}\\ \left\{E_3 \in L^2(\omega_1);\ E_3 \text{ is constant on each inclusion}\right\},\\ \text{if faces are metallized,}\end{cases}\\
&\qquad F_{\alpha\beta} \text{ are the components of } \mathbf{F} \text{ on } \mathbb{K}^0.
\end{aligned} \tag{96}$$

The matrices $\mathbf{Q}_{\mathcal{M}}$ and $\mathbf{F}$ are decomposed by blocs, $\mathbf{Q}_{\mathcal{M}} = \begin{pmatrix} \mathbf{Q}^{11}_{\mathcal{M}} & \mathbf{Q}^{12}_{\mathcal{M}} \\ \mathbf{Q}^{21}_{\mathcal{M}} & \mathbf{Q}^{22}_{\mathcal{M}}\end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix}\mathbf{F}^1\\ \mathbf{F}^2\end{pmatrix}$ corresponding to the bloc decomposition of $\mathbb{K}^0 \oplus \mathbb{L}^{-1}$; $Q^{\rho\mu}_{\mathcal{M}\alpha\beta\gamma\delta}$ and $Q_{\mathcal{N}\alpha\beta\gamma\delta}$ are the components of $\mathbf{Q}^{\rho\mu}_{\mathcal{M}}$ and $\mathbf{Q}_{\mathcal{N}}$ on $\mathbb{K}^0$.

THEOREM 4.4. *Under the same assumptions as in Theorem 4.3, the vector* $(\mathbf{u}, L_3) = (\overline{u}_1 - x_3\partial_1 u_3,\ \overline{u}_2 - x_3\partial_2 u_3,\ u_3, L_3) \in V_{KL} \times \mathcal{L}$ *is the solution of*

$$\begin{aligned}
&\int_\omega \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \widetilde{L}_3, \partial^2_{\alpha\beta}v_3\right)\begin{pmatrix} 2Q^{11}_{\mathcal{M}\alpha\beta\gamma\delta} & 2Q^{12}_{\mathcal{M}\alpha\beta} & 0\\ 2Q^{21}_{\mathcal{M}\gamma\delta} & 2Q^{22}_{\mathcal{M}} + 4G & 0\\ 0 & 0 & \dfrac{2}{3}Q_{\mathcal{N}\alpha\beta\gamma\delta}\end{pmatrix}\begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}})\\ L_3\\ \partial^2_{\gamma\delta}u_3\end{pmatrix} dx\\
&\quad = \int_{\omega_1} 2h\widetilde{L}_3 + \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \widetilde{L}_3\right)\begin{pmatrix}F_{\alpha\beta}\\ H\end{pmatrix} dx + l_u(\mathbf{v}),\\
&\quad \textit{for every } \mathbf{v} = \left(\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3, \widetilde{L}_3\right) \in V_{KL} \times \mathcal{L}.
\end{aligned} \tag{97}$$

PROOF. We start from (93). Taking $\tilde{\mathbf{M}} \in \mathbb{M}$ such that $\mathcal{N}(\tilde{\mathbf{M}}) \in \mathcal{N}(\mathbb{M}^{-2} \oplus \mathbb{M}^{-1})$ and $\mathcal{M}(\tilde{\mathbf{M}}) \in \mathcal{M}(\Pi_2\mathbb{M})$, then $\mathcal{N}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}\Pi)^{-1}\mathcal{N}(\mathbf{M}^0)$ and $\mathcal{M}(\Pi_2\mathbf{M}) = -(\Pi_2\mathcal{R}\Pi_2)^{-1}$ $(\mathcal{M}(\mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^{-1}}L_3) + \mathcal{P}_{\mathbb{L}^0}(e_\alpha(\varphi^0))_{\alpha=1,2})$. Then

$$\begin{aligned}
&\mathcal{N}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{N}})\mathcal{N}\left(\mathbf{M}^0\right) \qquad \text{and}\\
&\mathcal{M}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{M}})\mathcal{M}\left(\mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^{-1}}L_3 + \mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right).
\end{aligned} \tag{98}$$

We choose $\tilde{\mathbf{M}}$ in an analogous way $\mathcal{N}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{N}})\mathcal{N}(\tilde{\mathbf{M}}^0)$ and $\mathcal{M}(\tilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_{\mathcal{M}})$ $\mathcal{M}(\tilde{\mathbf{M}}^0 + \mathcal{P}_{\mathbb{L}^{-1}}\tilde{L}_3)$. Then

$$\begin{aligned}
&\int_\Omega \left(\mathcal{M}\left(\tilde{\mathbf{M}}^0 + \mathcal{P}_{\mathbb{L}^{-1}}\tilde{L}_3\right), \mathcal{N}\left(\tilde{\mathbf{M}}^0\right)\right)\begin{pmatrix}\mathbf{Q}_{\mathcal{M}} + 2\Pi_1 G\Pi_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}_{\mathcal{N}}\end{pmatrix}\begin{pmatrix}\mathcal{M}\left(\mathbf{M}^0 + \mathcal{P}_{\mathbb{L}^{-1}}L_3\right)\\ \mathcal{N}\left(\mathbf{M}^0\right)\end{pmatrix} dx\\
&\quad = \int_{\Omega_1} \mathcal{M}\left(\tilde{\mathbf{M}}^0 + \mathcal{P}_{\mathbb{L}^{-1}}\tilde{L}_3\right)\mathbf{Q}_{\mathcal{M}}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2} dx + l_u(\mathbf{v}) + l_\varphi\left(\mathcal{M}\left(\tilde{L}_3\right)\right).
\end{aligned} \tag{99}$$

Letting $\mathcal{M}(\tilde{L}_3) = \widetilde{E}_3$ and $\mathcal{M}(L_3) = E_3$ leads to the theorem. ∎

4.4. Multilayered Plates

The preceding notations, statements, and proofs have been conceived in order to be easily generalized for multilayered plates. Under a few precisions concerning notations, the proofs are close to the single layer case. These precisions are stated above. For the sake of brevity, we state the plate models for each boundary condition on $\Gamma^+_{\mathrm{inc}} \cup \Gamma^-_{\mathrm{inc}}$ without giving any proof and convergence statement.

The models are derived from the equations (65),(98), and one of the four boundary conditions (27)–(29), or (31). The definitions of the $\mathbb{M}^i$ are those of (68) (or (79),(80) in the case of Dirichlet conditions) except that $\mathbb{L}^0$ is defined by

$$\mathbb{L}^0 = \Big\{\mathbf{M} = (\mathbf{0}_7, \mathbf{L}) \in \left(L^2(\Omega)\right)^7 \\ \times \Big\{\left((L_\alpha)_{\alpha=1,2}, 0\right);\ L_{\alpha|\Omega_{1\xi}} = e_\alpha(\psi_\xi),\ \text{where } \psi_\xi \in H^1(\Omega_{1\xi}),\ \partial_3\psi_\xi = 0\Big\}.$$

4.4.1. Neumann boundary conditions

When Neumann condition (27) is applied on $\Gamma^+_{\mathrm{inc}} \cup \Gamma^-_{\mathrm{inc}}$, the limit $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3) \in V_{KL}$ is the unique solution of

$$\int_\omega \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \partial^2_{\alpha\beta}v_3\right) \begin{pmatrix} Q^{11}_{\alpha\beta\gamma\delta} & Q^{12}_{\alpha\beta\gamma\delta} \\ Q^{21}_{\alpha\beta\gamma\delta} & Q^{22}_{\alpha\beta\gamma\delta} \end{pmatrix} \begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}}) \\ \partial^2_{\gamma\delta}u_3 \end{pmatrix} d\mathbf{x} \\ = l_u(\mathbf{v}) + \int_{\omega_1} \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \partial^2_{\alpha\beta}v_3\right) \begin{pmatrix} F^0_{\alpha\beta} \\ F^1_{\alpha\beta} \end{pmatrix} d\mathbf{x}, \quad (100)$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL}$, where

$$\begin{aligned}
&\Pi \text{ is the projector from } \mathbb{E} \text{ onto } \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}, \\
&\mathbf{T} = -\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R}, \\
&\mathbf{Q} = \left(\mathbf{Id} + {}^t\mathbf{T}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}\right), \\
&\mathbf{H} = \left(\mathbf{T} - \left(\mathbf{Id} + {}^t\mathbf{T}\right)\mathcal{R}\left(\Pi\mathcal{R}\Pi\right)^{-1}\right)\mathcal{P}_{\mathbb{L}^{-1}}h, \\
&\mathbf{F}^0 = \int_{-1}^{1} \left(\mathbf{H} - \mathbf{Q}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right) dx_3, \\
&\mathbf{F}^1 = \int_{-1}^{1} \left(-x_3\left(\mathbf{H} - \mathbf{Q}\mathcal{P}_{\mathbb{L}^0}\left(e_\alpha(\varphi_m)\right)_{\alpha=1,2}\right)\right) dx_3, \\
&\begin{pmatrix} \mathbf{Q}^{11} & \mathbf{Q}^{12} \\ \mathbf{Q}^{21} & \mathbf{Q}^{22} \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} \mathbf{Q} & -x_3\mathbf{Q} \\ -x_3\mathbf{Q} & x_3^2\mathbf{Q} \end{pmatrix} dx_3.
\end{aligned}$$

$Q^{\rho\mu}_{\alpha\beta\gamma\delta}$ and $F^\rho_{\alpha\beta}$ are the components of $\mathbf{Q}^{\rho\mu}$ and $\mathbf{F}^\rho$ on $\mathbb{K}^0$.

REMARKS.

(i) This model is in the same form as an elastic multilayered thin plate model. The electrical field is not an unknown of the problem. The stiffness tensor $(\mathbf{Q}^{\alpha\beta})_{\alpha,\beta=1,2}$ is affected by the piezoelectricity and permitivity coefficients. The forces $(\mathbf{F}^0, \mathbf{F}^1)$ result from the mechanical forces and from the electrical sources as well.

(ii) In practice, this sort of boundary condition seems difficult to realize. In general, the following Dirichlet boundary condition is preferred.

4.4.2. Dirichlet boundary conditions

When Dirichlet condition (28) is applied to $\Gamma^+_{\text{inc}} \cup \Gamma^-_{\text{inc}}$, the limit $\mathbf{u} = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3) \in V_{KL}$ is the unique solution of

$$\sum_{\xi=1}^{N} \int_{\omega} \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \partial^2_{\alpha\beta} v_3\right) \begin{pmatrix} Q^{11\xi}_{\alpha\beta\gamma\delta} & Q^{12\xi}_{\alpha\beta\gamma\delta} \\ Q^{21\xi}_{\alpha\beta\gamma\delta} & Q^{22\xi}_{\alpha\beta\gamma\delta} \end{pmatrix} \begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}}) \\ \partial^2_{\gamma\delta} u_3 \end{pmatrix} dx$$
$$= l_u(\mathbf{v}) + \sum_{\xi=1}^{N} \int_{\omega_{1\xi}} \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \partial^2_{\alpha\beta} v_3\right) \begin{pmatrix} F^{1\xi}_{\alpha\beta} \\ F^{2\xi}_{\alpha\beta} \end{pmatrix} dx, \quad (101)$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3) \in V_{KL}$, where

$$\begin{pmatrix} \mathbf{Q}^{11\xi} & \mathbf{Q}^{12\xi} \\ \mathbf{Q}^{21\xi} & \mathbf{Q}^{22\xi} \end{pmatrix} = \begin{pmatrix} (b_\xi - a_\xi)\,\mathbf{Q}^\xi_{\mathcal{M}} & \dfrac{a_\xi^2 - b_\xi^2}{2}\,\mathbf{Q}^\xi_{\mathcal{M}} \\ \dfrac{a_\xi^2 - b_\xi^2}{2}\,\mathbf{Q}^\xi_{\mathcal{M}} & \dfrac{b_\xi^3 - a_\xi^3}{3}\,\mathbf{Q}^\xi_{\mathcal{M}} + \dfrac{(b_\xi - a_\xi)^3}{12}\,\mathbf{Q}^\xi_{\mathcal{N}} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{F}^{1\xi} \\ \mathbf{F}^{2\xi} \end{pmatrix} = \begin{pmatrix} (b_\xi - a_\xi) \\ \dfrac{a_\xi^2 - b_\xi^2}{2} \end{pmatrix} \mathbf{Q}^\xi_{\mathcal{M}} \left(\mathcal{P}_{\mathcal{M}(\mathbf{L}^{-1})}\varphi_{c\xi} + \mathcal{P}_{\mathbf{L}^0}\left(e_\alpha(\varphi_{m\xi})\right)_{\alpha=1,2}\right),$$
$$\text{for every } \xi = 1, \dots, N.$$

$Q^{\rho\mu\xi}_{\alpha\beta\gamma\delta}$ and $F^{\rho\xi}_{\alpha\beta}$ are the components of $\mathbf{Q}^{\rho\mu\xi}$ and $\mathbf{F}^{\rho\xi}$ on $\mathbb{K}^0$. The tensors $\mathbf{Q}^\xi_{\mathcal{M}}$ and $\mathbf{Q}^\xi_{\mathcal{N}}$ are the restriction of the tensors $\mathbf{Q}_{\mathcal{M}}$ and $\mathbf{Q}_{\mathcal{N}}$ to the layer number ξ defined by

$$\mathcal{M}_{|\Omega_\xi}(\mathbf{M}) = \frac{1}{b_\xi - a_\xi}\int_{a_\xi}^{b_\xi} \mathbf{M}\, dx_3 \quad \text{and} \quad \mathcal{N}_{|\Omega_\xi}(\mathbf{M}) = \mathbf{M} - \mathcal{M}_{|\Omega_\xi}(\mathbf{M}), \qquad \text{for any } \mathbf{M} \in \mathbb{E},$$
$$\Pi_1 \text{ is the projector from } \mathbb{E} \text{ onto } \mathbb{L}^{-1},$$
$$\Pi_2 \text{ the projector from } \mathbb{E} \text{ onto } \mathbb{M}^{-2} \oplus \left(\mathbb{K}^{-1} \times \{0_3\}\right),$$
$$\mathbf{T}_{\mathcal{N}} = -\left(\Pi\mathcal{R}\Pi\right)^{-1}\Pi\mathcal{R} \quad \text{and} \quad \mathbf{T}_{\mathcal{M}} = -\left(\Pi_2\mathcal{R}\Pi_2\right)^{-1}\Pi_2\mathcal{R},$$
$$\mathbf{Q}_{\mathcal{N}} = {}^t\left(\mathbf{Id} + \mathbf{T}_{\mathcal{N}}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}_{\mathcal{N}}\right) \quad \text{and} \quad \mathbf{Q}_{\mathcal{M}} = {}^t\left(\mathbf{Id} + \mathbf{T}_{\mathcal{M}}\right)\mathcal{R}\left(\mathbf{Id} + \mathbf{T}_{\mathcal{M}}\right).$$

REMARKS.

(i) This model is the more classical one (see [1,2]). In general, $\varphi_{m\xi}$ is taken equal to zero. The forces $\mathbf{F}^\xi_{\mathcal{M}}$ and $\mathbf{F}^\xi_{\mathcal{N}}$ are only affected by $\varphi_{c\xi}$, the tension between the upper and lower inclusion faces. Electronic devices based on operational amplifiers permit us to impose a such condition.

(ii) For the open loop control problem, the Dirichlet boundary condition is the more natural one. For the design of closed loop control with numerically computed feedback, this condition is also the more usual one.

4.4.3. Mixed boundary conditions

When mixed condition (29) is applied to $\Gamma^+_{\text{inc}} \cup \Gamma^-_{\text{inc}}$, the limit $(\mathbf{u}, L_3) = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3, L_3) \in V_{KL} \times \mathcal{L}$ is the unique solution of

$$\sum_{\xi=1}^{N} \int_{\omega_\xi} \left(s_{\alpha\beta}(\overline{\mathbf{v}}), \widetilde{L}_{3\xi}, \partial^2_{\alpha\beta} v_3\right) \begin{pmatrix} Q^{11\xi}_{\alpha\beta\gamma\delta} & Q^{11\xi}_{\alpha\beta3} & Q^{12\xi}_{\alpha\beta\gamma\delta} \\ Q^{11\xi}_{3\gamma\delta} & Q^{11\xi}_{33} + G^\xi & Q^{12\xi}_{3\gamma\delta} \\ Q^{21\xi}_{\alpha\beta\gamma\delta} & Q^{21\xi}_{\alpha\beta3} & Q^{22\xi}_{\alpha\beta\gamma\delta} \end{pmatrix} \begin{pmatrix} s_{\gamma\delta}(\overline{\mathbf{u}}) \\ L_{3\xi} \\ \partial^2_{\gamma\delta} u_3 \end{pmatrix} dx$$
$$= l_u(\mathbf{v}) + \sum_{\xi=1}^{N} \int_{\omega_{1\xi}} (b_\xi - a_\xi) h\, \widetilde{L}_{3\xi} + \left(s_{\alpha\beta}(\overline{\mathbf{v}})\, \widetilde{L}_{3\xi},\, \partial^2_{\alpha\beta} v_3\right) \begin{pmatrix} F^{1\xi}_{\alpha\beta} \\ F^{1\xi}_{3} \\ F^{2\xi}_{\alpha\beta} \end{pmatrix} dx, \quad (102)$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3, \widetilde{L}_3) \in V_{KL} \times \mathcal{L}$, where

$$\begin{pmatrix} \mathbf{F}^{1\xi} \\ \mathbf{F}^{2\xi} \end{pmatrix} = \begin{pmatrix} (b_\xi - a_\xi) \\ \dfrac{a_\xi^2 - b_\xi^2}{2} \end{pmatrix} \mathbf{Q}_{\mathcal{M}}^{\xi} \mathcal{P}_{\mathbb{L}^0} \left(e_\alpha \left(\varphi_{m\xi}\right)\right)_{\alpha=1,2},$$

$$\begin{pmatrix} \mathbf{Q}^{11\xi} & \mathbf{Q}^{12\xi} \\ \mathbf{Q}^{21\xi} & \mathbf{Q}^{22\xi} \end{pmatrix} = \begin{pmatrix} (b_\xi - a_\xi)\mathbf{Q}_{\mathcal{M}}^{\xi} & \dfrac{a_\xi^2 - b_\xi^2}{2}\mathbf{Q}_{\mathcal{M}}^{\xi} \\ \dfrac{a_\xi^2 - b_\xi^2}{2}\mathbf{Q}_{\mathcal{M}}^{\xi} & \dfrac{b_\xi^3 - a_\xi^3}{3}\mathbf{Q}_{\mathcal{M}}^{\xi} + \dfrac{(b_\xi - a_\xi)^3}{12}\mathbf{Q}_{\mathcal{N}}^{\xi} \end{pmatrix}, \tag{103}$$

$$G^\xi = (b_\xi - a_\xi)^2 G.$$

$\mathbf{Q}_{\mathcal{M}}^{\xi}$ and $\mathbf{Q}_{\mathcal{N}}^{\xi}$ being the restriction of the tensors $\mathbf{Q}_{\mathcal{M}}$ and $\mathbf{Q}_{\mathcal{N}}$ to the layer number ξ defined by (96).

$$\begin{pmatrix} Q_{\alpha\beta\gamma\delta}^{11\xi} & Q_{\alpha\beta3}^{11\xi} \\ Q_{3\gamma\delta}^{11\xi} & Q_{33}^{11\xi} \end{pmatrix}, \quad \begin{pmatrix} Q_{\alpha\beta\gamma\delta}^{12\xi} \\ Q_{3\gamma\delta}^{12\xi} \end{pmatrix}, \quad \left(Q_{\alpha\beta\gamma\delta}^{21\xi} \quad Q_{\alpha\beta3}^{21\xi}\right), \quad \text{and} \quad Q_{\alpha\beta\gamma\delta}^{22\xi} \quad \text{and} \quad F_{\alpha\beta}^{\rho\xi}$$

are the components of $\mathbf{Q}^{11\xi}$, $\mathbf{Q}^{12\xi}$, $\mathbf{Q}^{21\xi}$, $\mathbf{Q}^{22\xi}$, and $\mathbf{F}^{\rho\xi}$ on $\mathbb{K}^0$; $F_3^{1\xi}$ is the component of $\mathbf{F}^{1\xi}$ on $\mathbb{L}^{-1}$.

REMARKS.

(i) In this model, the electrical tensions $L_{3\xi}$ are unknown. In addition to the electrical sources, the effect of the electrical circuit results from the admittance G.

(ii) This model is well suited for the design of dynamic feedback (see [21]). The control variable is L_3.

(iii) General electrical circuits include also active components based on operational amplifiers. They do not have an admittance (see [19], for example). Then, the electrical circuits considered in this model lead to a particular model of coupling between piezoelectric plate and an electrical circuit.

4.4.4. Boundary conditions coupling the layers

When mixed condition coupling the layers (31) is applied to $\Gamma_{\rm inc}^{+} \cup \Gamma_{\rm inc}^{-}$, the limit $(\mathbf{u}, L_3) = (\overline{u}_1 - x_3\partial_1 u_3, \overline{u}_2 - x_3\partial_2 u_3, u_3, L_3) \in V_{KL} \times \mathcal{L}$ is the unique solution of

$$\begin{aligned} &\sum_{\xi=1}^{N} \int_{\omega_\xi} \left(s_{\alpha\beta}(\overline{v}), \widetilde{L}_{3\xi}, \partial_{\alpha\beta}^2 v_3\right) \begin{pmatrix} Q_{\alpha\beta\gamma\delta}^{11\xi} & Q_{\alpha\beta3}^{11\xi} & Q_{\alpha\beta\gamma\delta}^{12\xi} \\ Q_{3\gamma\delta}^{11\xi} & Q_{33}^{11\xi} & Q_{3\gamma\delta}^{12\xi} \\ Q_{\alpha\beta\gamma\delta}^{21\xi} & Q_{\alpha\beta3}^{21\xi} & Q_{\alpha\beta\gamma\delta}^{22\xi} \end{pmatrix} \begin{pmatrix} s_{\gamma\delta}(\overline{u}) \\ L_{3\xi} \\ \partial_{\gamma\delta}^2 u_3 \end{pmatrix} dx \\ &\quad + \sum_{\xi=1}^{N} \sum_{\eta=1}^{N} r_\xi \widehat{A}_{\xi\eta} \widetilde{L}_{3\xi} L_{3\eta} = l_u(v) + \sum_{\xi=1}^{N} \int_{\omega_{1\xi}} (b_\xi - a_\xi) h \, \widetilde{L}_3 \\ &\quad + \left(s_{\alpha\beta}(\overline{v}), \widetilde{L}_3, \partial_{\alpha\beta}^2 v_3\right) \begin{pmatrix} F_{\alpha\beta}^{1\xi} \\ F_3^{1\xi} \\ F_{\alpha\beta}^{2\xi} \end{pmatrix} dx, \end{aligned} \tag{104}$$

for every $\mathbf{v} = (\overline{v}_1 - x_3\partial_1 v_3, \overline{v}_2 - x_3\partial_2 v_3, v_3, \widetilde{L}_3) \in V_{KL} \times \mathcal{L}$, where the notations are defined in Section 4.4.3 and $\widehat{\mathbf{A}}$ is defined in (58).

REMARK. The same remarks as for mixed boundary conditions hold. The introduction of a coupling between the layers shows that it may be possible (using more general electronical circuits) to design a dynamic feedback using the finite difference approximation of $\partial_3 L_3$.

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