Acoustic waves
- the case of fluids -

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1 Unidimensional model (1D)

1.1 Wave equation

A wave is generally speaking a perturbation of the state of equilibrium of a medium, that propagates in space and in time.

Let us consider a function $u(t,x)$, a wave equation is of the form:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

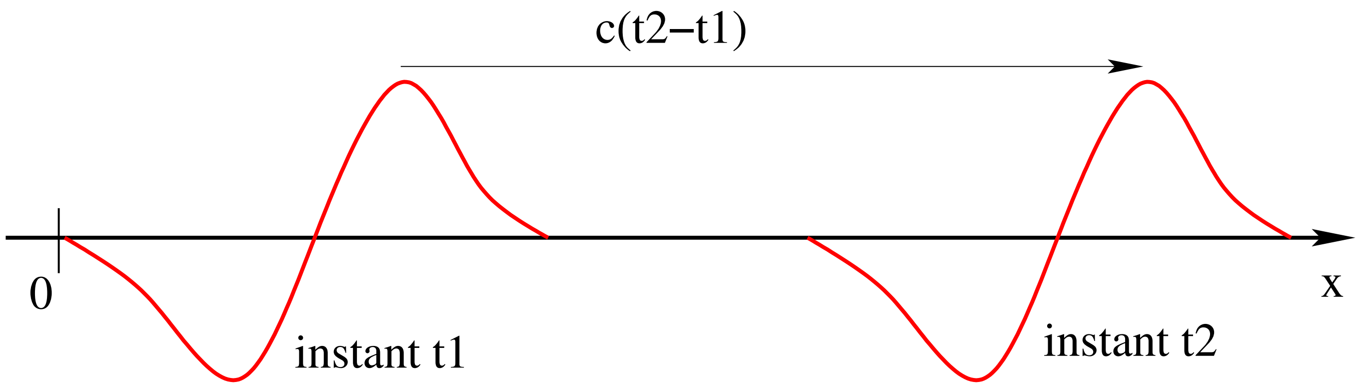
c is homogeneous to a velocity (the celerity), in m/s.

1.2 General solution?

It is easily checked that the general solution is:

$$u(t,x) = F(t - x/c) + G(t + x/c) \tag{2}$$

with F et G arbitrary functions (twice differentiable) representing a wave travelling to the right and a wave travelling to the left, independently.



Example: The vibration $F(t) = \cos(\omega t)$ yields $u(t,x) = \cos(\omega t - kx)$

$\omega = 2\pi f$ is the angular frequency; f is the frequency (in Hz).

$k = \omega/c = 2\pi/\lambda$ is the wavenumber; λ is the wavelength.

1.3 Plane wave spectrum

Any (sufficiently regular) function has a Fourier transform and reciprocally:

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(\omega)\exp(\mathrm{i}\omega t)\,\mathrm{d}\omega\,;\ \tilde{F}(\omega) = \int_{-\infty}^{\infty} F(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t \tag{3}$$

Hence the plane wave spectrum of a solution of the wave equation:

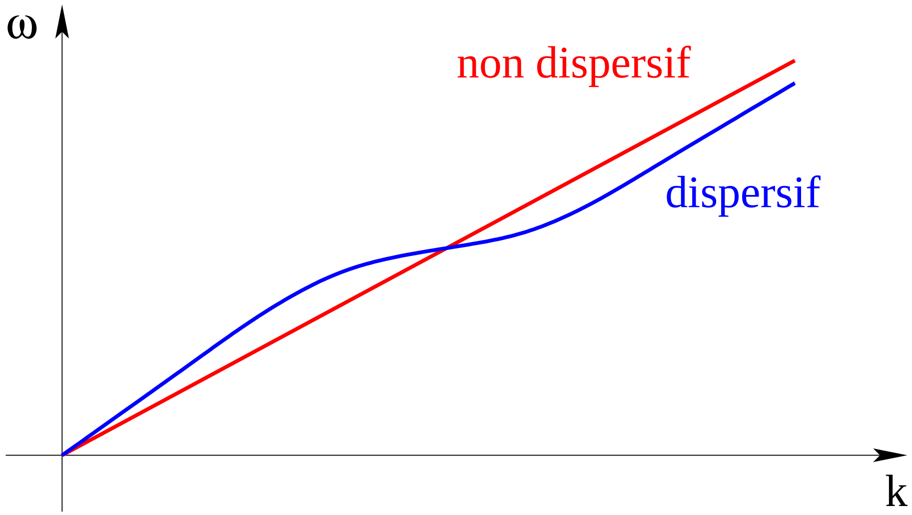
$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(\omega)\exp(\mathrm{i}\,(\omega t - kx))\,\mathrm{d}\omega \text{ avec } k(\omega) = \omega/c \tag{4}$$

(with a similar term with $\tilde{G}(\omega)$ and $k(\omega) = -\omega/c$).

$k^2(\omega) = (\omega/c)^2$ is a dispersion relation.

1.4 Dispersion and group velocity

If wave velocity is dispersive (i.e. if it depends on frequency), $c(\omega)$, then the dispersion relation $k(\omega) = \pm\omega / c(\omega)$ does not define straight lines any more.



For a wave packet: $u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega)\exp(\mathrm{i}\,(\omega t - k(\omega)\,x))\,\mathrm{d}\omega$

The phase velocity is $v(\omega) = \omega / k(\omega)$. The slowness is $s(\omega) = 1 / v(\omega)$.

The group velocity is by definition $v_g\,(\omega) = \frac{\mathrm{d}\omega}{\mathrm{d}k} = (\frac{\mathrm{d}k(\omega)}{\mathrm{d}\omega})^{-1}$.

Property: the group velocity is the propagation velocity of te energy of the wave as a function of frequency, or

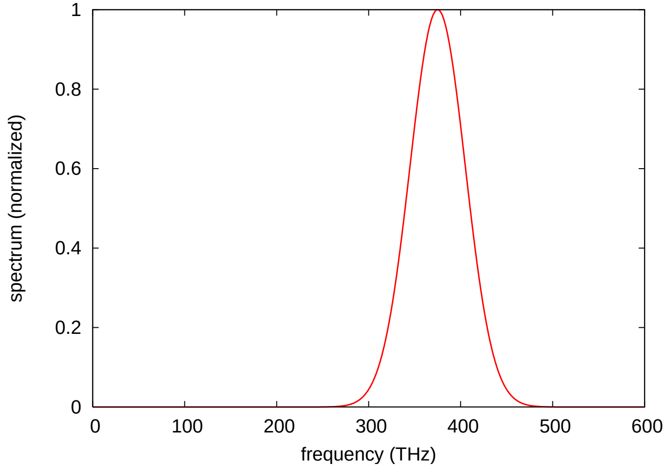
$$\int_{-\infty}^{\infty} t\,|u(t,x)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x}{v_{g(\omega)}}|\tilde{F}(\omega)|^2\,\mathrm{d}\omega \qquad (5)$$

1.5 Examples of dispersion

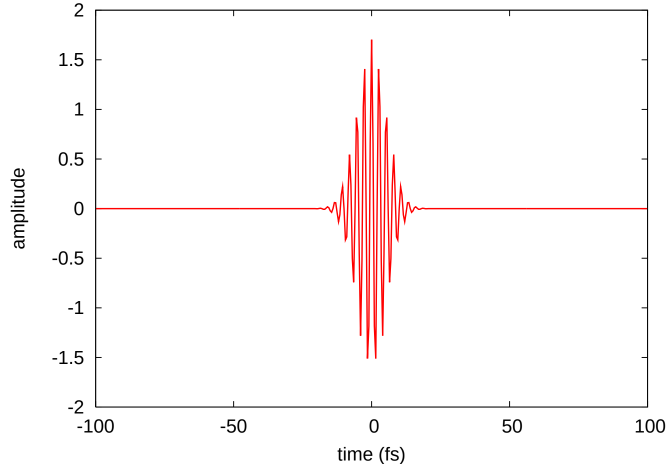
The propagation phase at point $x = L$ is $\varphi(\omega) = k(\omega)L$.

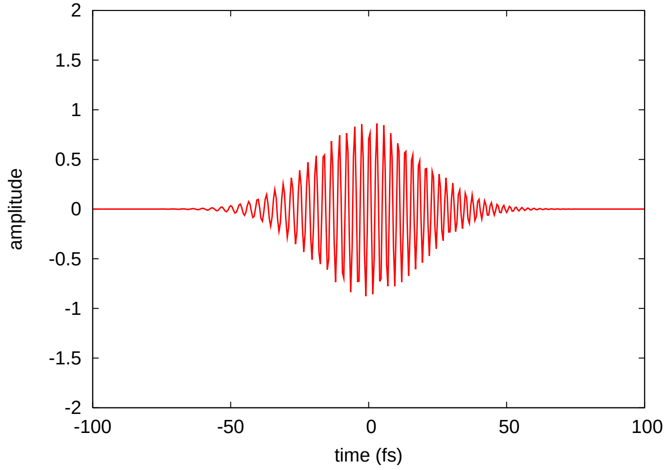
$t_g(\omega) = \mathrm{d}\varphi(\omega)\,/\,\mathrm{d}\omega = L\,/\,v_g(\omega)$ is the group velocity (time to travel distance L).

Polynomial phase $\varphi(\omega) = \varphi_0 + \varphi_0'(\omega - \omega_0) + \frac{1}{2!}\varphi_0''(\omega - \omega_0)^2 + \frac{1}{3!}\varphi_0'''(\omega - \omega_0)^3 + \ldots$

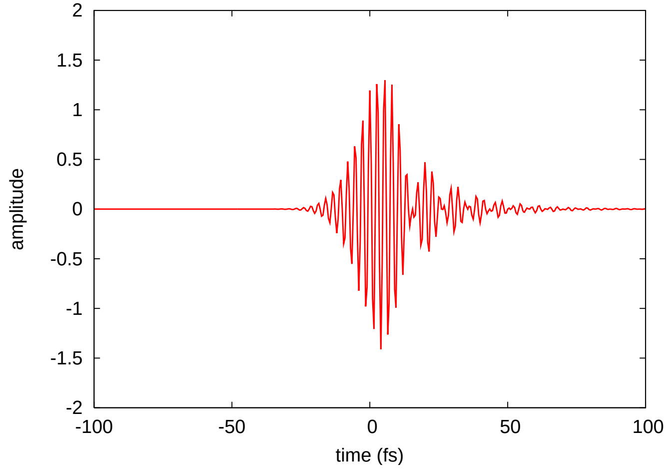


$\longleftarrow$ spectrum ; pulse $\longrightarrow$





$\longleftarrow \varphi_0''\ ;\ \varphi_0''' \longrightarrow$

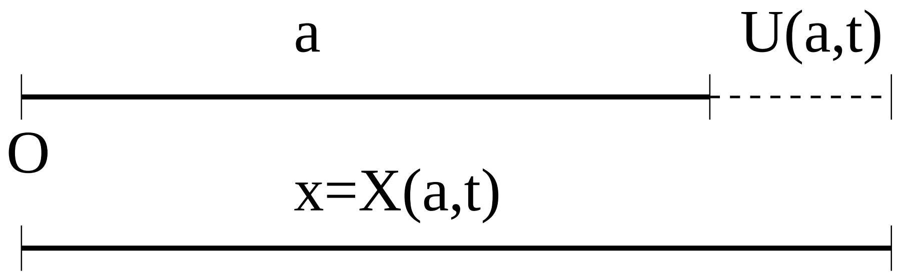


2 1D acoustic waves

2.1 Lagrangian and Eulerian descriptions

Consider a continuous, isotropic, homogeneous fluid, perfectly compressible.

- Lagrange variables, for a material point: equilibrium position a and time t. Physical quantity: $G(a,t)$.
- Euler variables, for a geometrical point of a referential: coordinate x and time t. The same physical quantity: $g(x,t)$.

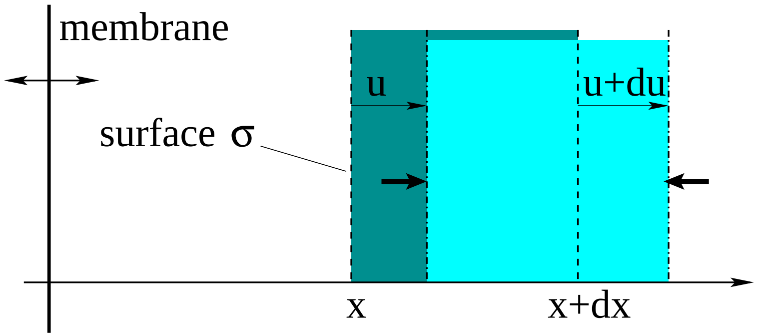


Position of the material point: $x = X(a,t)$, hence $G(a,t) = g(X(a,t),t)$
Displacement: $U(a,t) = X(a,t) - a = u(X(a,t),t)$
Particle velocity $V_p = \partial U / \partial t = \partial X / \partial t$ and local velocity $v = \partial u / \partial t$

$$V_p = v + V_p \frac{\partial u}{\partial x} \tag{6}$$

Approximation of linear acoustics: $\partial u / \partial x \ll 1$ and then $V_p \simeq v$

2.2 Relations between pressure and displacement



$$\mathrm{d}u = \frac{\partial u(t,x)}{\partial x}\,\mathrm{d}x \ll \mathrm{d}x$$

Total pressure force acting on a slice of width $\mathrm{d}x$ and surface σ :

$$\mathrm{d}F = \sigma\, p(t, x+u) - \sigma\, p(t, x+u+\mathrm{d}x) \simeq -\sigma\,\frac{\partial p}{\partial x}\,\mathrm{d}x$$

By application of the dynamical (Newton) principle:

$$-\frac{\partial p}{\partial x} = \rho_0\,\frac{\partial^2 u}{\partial t^2} \tag{7}$$

with ρ_0 the (static) density of the fluid.

2.3 Relations between pressure and displacement (cont.)

Pressure is the sum of the static pressure and of the dynamic pressure δp:

$$p(t,x) = p_0 + \delta p(t,x) \tag{8}$$

For a compressible fluid, we have ($\mathrm{d}V = \sigma\,\mathrm{d}x$) :

$$\delta p = -\frac{1}{\chi}\frac{\delta(\mathrm{d}V)}{\mathrm{d}V} = -\frac{1}{\chi}\frac{\partial u}{\partial x} \tag{9}$$

with χ the compressibility coefficient. By definition, $S(t,x) = \partial u/\partial x$ is the local dilatation (strain).

Gathering (7) and (9), a wave equation is obtained:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \text{ ou } \frac{\partial^2(\delta p)}{\partial t^2} - c^2\frac{\partial^2(\delta p)}{\partial x^2} = 0 \text{ with } c = (\rho_0\,\chi)^{-1/2} \tag{10}$$

The velocity v and the strain S satisfy exactly the same wave equation.

2.4 Sound velocity

How can we estimate the celerity c in air, supposed a perfect gas?

- The state equation for a perfect gas, with molar mass M, for n moles is $pV = nRT$ or $p = \rho RT/M$, (T temperature, $R = 8.314$ J/mole.K)
- Compressions and dilatations caused by the acoustic wave are adiabatic (but not isothermal) and follow the law $pV^\gamma = \text{Cst}$. From which $\chi = (\gamma p_0)^{-1}$. $\gamma = 1.67$ for a monoatomic gas and 1.4 for a diatomic gas (approximately the case of air).

$$\frac{\mathrm{d}p}{p} + \gamma \frac{\mathrm{d}V}{V} = 0 \text{ so that } \chi = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{1}{\gamma p_0}$$

$$\text{and then } c = \sqrt{\gamma \frac{RT}{M}}$$

You should better trust experiment!
$c \simeq 343$ m/s for air at $T = 293$ K.
And what about water?
$c \simeq 1480$ m/s for water at $T = 293$ K.

2.5 Acoustic impedance

Displacement u is a solution to the wave equation (10), hence

$$u(t,x) = F(t - x/c) + G(t + x/c)$$

with F and G two arbitrary functions. Then

$$v(t,x) = \frac{\partial u}{\partial t} = F'(t - x/c) + G'(t + x/c)$$

$$\delta p(t,x) = -\frac{1}{\chi}\frac{\partial u}{\partial x} = Z\,(F'(t - x/c) - G'(t + x/c))$$

with the acoustic impedance $Z = \rho_0 c = \frac{1}{c\chi} = \sqrt{\rho_0/\chi}$.

Pressure and velocity are proportional for waves propagating to the right, $\delta p_+ = Z v_+$, and for waves propagating to the left, $\delta p_- = -Z v_-$.

This relation is analogous to the electrical impedance: $U = ZI$

2.6 Representation of propagation loss?

A fluid can not react instantly to an excitation. Phenomenologically, (9) is modified as:

$$\delta p = -\frac{1}{\chi}\left(S + \tau\frac{\partial S}{\partial t}\right) \tag{11}$$

with τ a time constant.

Illustration - For $\delta p = H(t)$, it can be shown that $S = -\chi(1 - \exp(-t/\tau))H(t)$. The propagation equation becomes $\partial^2 u/\partial t^2 - c^2\,\partial^2/\partial x^2(u + \tau\partial u/\partial t) = 0$ (this is no more a wave equation!). For a monochromatic plane wave, $F(\omega t - kx)$, the complex dispersion relation $\omega^2 = c^2(1 + \mathrm{i}\omega\tau)\,k^2$ is obtained.

Exercise - Write $k = \beta - \mathrm{i}\alpha$ so that the harmonic plane wave is

$$u(t,x) = \exp(\mathrm{i}(\omega t - kx)) = \exp(-\alpha x)\exp(\mathrm{i}(\omega t - \beta x)) \tag{12}$$

Show that $\alpha \simeq \frac{\omega^2\tau}{2c}$ and $\beta \simeq \frac{\omega}{c}(1 - \frac{3}{8}\omega^2\tau^2)$ for $\omega\tau \ll 1$. α is expressed in dB/m.

Property - In practice, the compressibility coefficient can be complexified $\chi \to \chi/(1 + \mathrm{i}\omega\tau)$ and the plane wave spectrum (4) can be formed with damped harmonic plane waves (12).

3 3D scalar wave model

3.1 3D wave equation

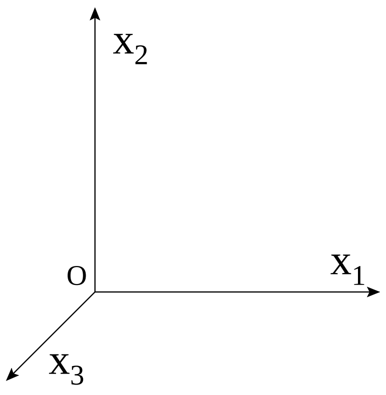
For a function $u(t,\boldsymbol{r})$, an isotropic wave equation is of the form:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0 \tag{13}$$

with the Laplacian $\triangle = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Isotropy: the medium properties are invariant under any rotation in space. Equivalently, propagation is the same in any direction.

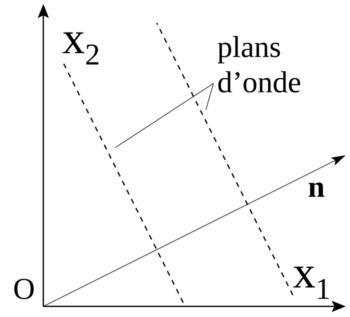
An anisotropic wave equation is of the form:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{3} c_{ij}^2 \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = 0 \tag{14}$$



Wave propagation depends on the direction.

3.2 Plane wave an harmonic plane wave



A 3D plane wave is of the form

$$u(t,\boldsymbol{r}) = F(t - \boldsymbol{n}.\boldsymbol{r}/c) = F\left(t - \frac{n_1 x_1 + n_2 x_2 + n_3 x_3}{c}\right) \tag{15}$$

with $\boldsymbol{n}$ a unit vector representing the direction of propagation. The decomposition (2) is not anymore the general solution to the wave equation.

A harmonic plane wave is of the form

$$u(t,\boldsymbol{r}) = \exp(\mathrm{i}(\omega t - \boldsymbol{k}.\boldsymbol{r})) \tag{16}$$

For the isotropic wave equation (13), we have the dispersion relation $\omega^2 = c^2\,\boldsymbol{k}.\boldsymbol{k} = c^2\,k^2$, with $\boldsymbol{k} = k\boldsymbol{n}$.

For the anisotropic wave equation (14), we have $\omega^2 = \sum_{i,j=1}^{3} c_{ij}^2 k_i k_j$

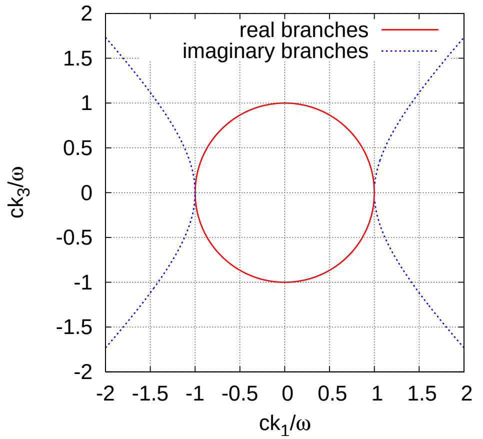
3.3 Plane wave spectrum

Is it possible to generalize to 3D the 1D plane wave spectrum (4)? Taking the Fourier transform in time and space, valid for all functions u :

$$u(t,\boldsymbol{r}) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \mathrm{d}\omega \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{k}\, \tilde{u}(\omega,\boldsymbol{k}) \exp(\mathrm{i}\,(\omega t - \boldsymbol{k}.\boldsymbol{r})) \tag{17}$$

If u is a solution of the wave equation, then ω et $\boldsymbol{k}$ are linked by a dispersion relation. Hence k_3, for instance, is a function of ω, k_1 and k_2 :

$$u(t,\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}\omega\, \mathrm{d}k_1 \mathrm{d}k_2\, \tilde{u}(\omega,\boldsymbol{k}) \exp(\mathrm{i}\,(\omega t - k_1 x_1 - k_2 x_2 - k_3(\omega,k_1,k_2) x_3)) \tag{18}$$



Example - if $k^2 = \omega^2/c^2$, then

$$k_3 = \pm\sqrt{\omega^2/c^2 - k_1^2 - k_2^2} \text{ if } \omega^2/c^2 - k_1^2 - k_2^2 \geq 0 \text{ and } k_3 = \pm\mathrm{i}\sqrt{|\omega^2/c^2 - k_1^2 - k_2^2|} \text{ if not}$$

3.4 Temporal and spatial dispersion

Assume we know the dispersion relation in the form $k(\omega, \boldsymbol{n})$. Then:

- $v(\omega, \boldsymbol{n}) = \omega / k(\omega, \boldsymbol{n})$ the phase velocity ; $s(\omega, \boldsymbol{n}) = k(\omega, \boldsymbol{n}) / \omega$ the slowness
- $v_g(\omega, \boldsymbol{n}) = (\partial k / \partial \omega)^{-1}$ the (temporal) group velocity gives the propagation velocity of a signal.
- $\boldsymbol{v}_g(\omega, \boldsymbol{n}) = \omega(\nabla_{\boldsymbol{n}} k^{-1}) = (\nabla_{\boldsymbol{n}} v)$ the (spatial) group velocity gives the velocity and the direction of propagation of the wavefront.

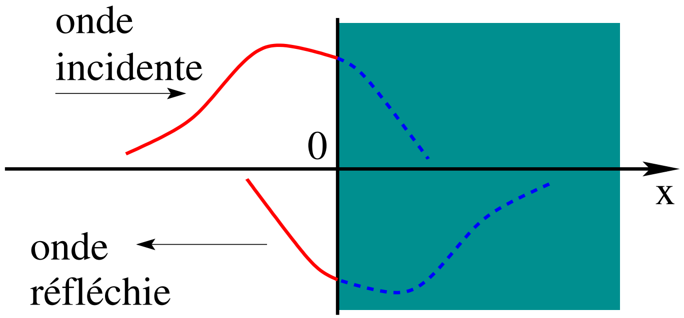
Stationary phase principle - If we can use the representation (typical of the far field):

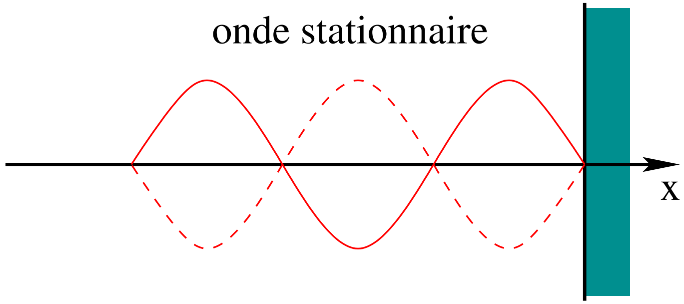
$$u(t, \boldsymbol{r}) = \frac{1}{2\pi} \int \mathrm{d}\omega \int \mathrm{d}\boldsymbol{n}\, \tilde{u}(\omega, \boldsymbol{n}) \exp(\mathrm{i}(\omega t - k(\omega, \boldsymbol{n}) \boldsymbol{n}.\boldsymbol{r})) \tag{19}$$

then energy concentrates along trajectories such that the phase in the exponential function is stationary in time and space, or

$$t = v_g^{-1}(\boldsymbol{n}.\boldsymbol{r}) \text{ and } v\boldsymbol{r} = \boldsymbol{v}_g\,(\boldsymbol{n}.\boldsymbol{r}) \tag{20}$$

3.5 Total reflection of a plane wave - normal incidence



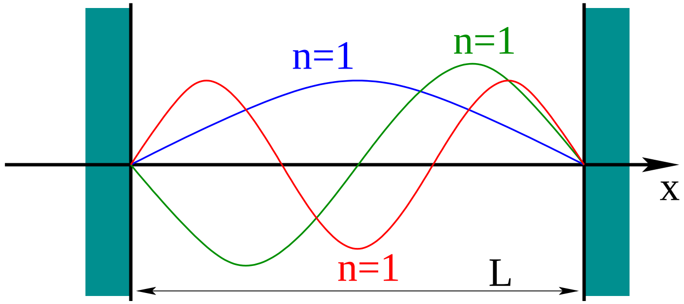


Let the incident plane wave be $F_i(t - x/c)$, the reflected wave $G_r(t + x/c)$ is also plane. The total wave is $u(t, \boldsymbol{r}) = F_i(t - x/c) + G_r(t + x/c)$.

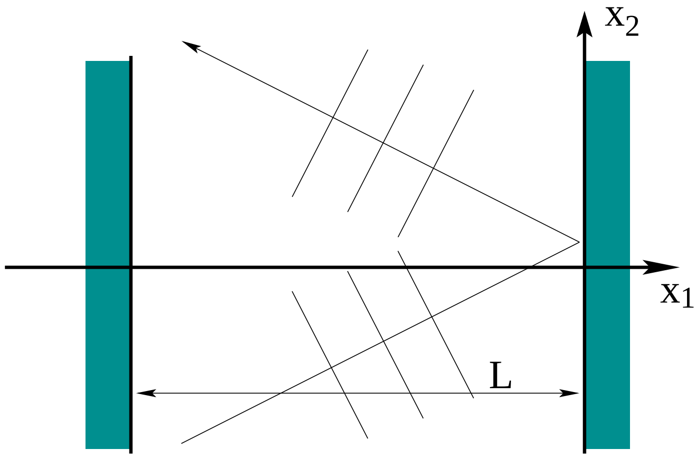
Next, we assume that the wave amplitude vanishes on the mirror (clamped condition), then $G_r(t) = -F_i(t)$ and $u(t, \boldsymbol{r}) = F_i(t - x/c) - F_i(t + x/c)$.

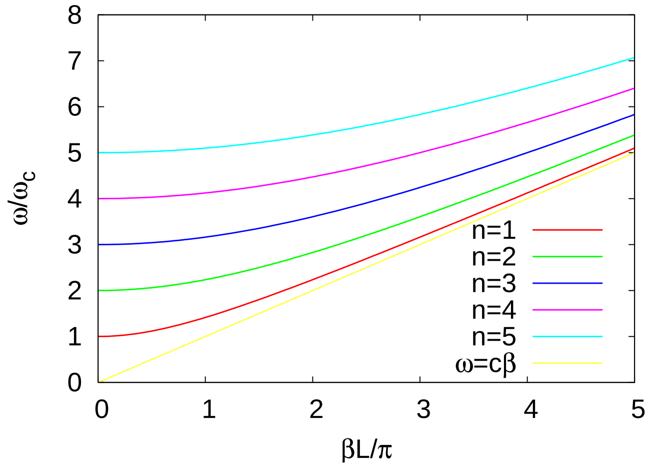
If $F_i(t) = \exp(\mathrm{i}\omega t)$, then $u(t, \boldsymbol{r}) = -2\mathrm{i}\exp(\mathrm{i}\omega t)\sin(\omega x/c)$ is a stationary wave.

In a resonator, modes are discrete: $\omega L/c = n\pi$ with $n \geqslant 1$ an integer



3.6 Guidance of waves between two plane mirrors





In order for the superposition of two harmonic plane waves to satisfy boundary conditions on the mirrors, phase matching must be observed:

- frequency is conserved ;
- the wavenumber along the mirrors is conserved.

Hence the decomposition:

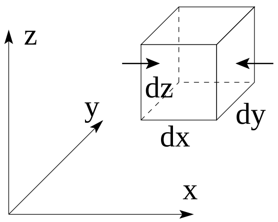
$$u(t,\boldsymbol{r}) = \exp(\mathrm{i}(\omega t - k_1 x_1 - k_2 x_2)) - \exp(\mathrm{i}(\omega t + k_1 x_1 - k_2 x_2)) = -2\mathrm{i}\exp(\mathrm{i}(\omega t - k_2 x_2))\sin(k_1 x_1)$$

representing a wave propagating along x_2 but stationary along x_1.
Dispersion relation: $k_1 L = n\pi$ and $k_2^2 = \beta^2 = \omega^2/c^2 - (n\pi/L)^2$, for $n \geqslant 1$.
There is a cut-off frequency $\omega_c = \pi c/L$ (or $f_c = c/(2L)$).

4 3D acoustic waves

4.1 Relations between pressure and displacements

Relation (8) is generalized to



$$p(t,\boldsymbol{r}) = p_0 + \delta p(t,\boldsymbol{r}) \text{ with the position vector } \boldsymbol{r} = (x,y,z)^T \tag{21}$$

The local strain becomes

$$S(t,\boldsymbol{r}) = \frac{\delta(\mathrm{d}V)}{\mathrm{d}V} = \nabla\,.\,\boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{22}$$

Fundamental dynamical relation:

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\left(\frac{\partial(\delta p)}{\partial x}, \frac{\partial(\delta p)}{\partial y}, \frac{\partial(\delta p)}{\partial z}\right)^T = -\nabla(\delta p) \tag{23}$$

Equation (23) shows that the polarization of a plane wave is longitudinal in a fluid: displacements occur only along the propagation direction.

4.2 3D acoustic wave equation

For a compressible linear fluid, we still assume $S = -\chi \delta p$.

Hence the 3D scalar wave equation (for either δp or S) or vector wave equation (for $\boldsymbol{u}$ or $\boldsymbol{v}$) :

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c^2 \triangle \boldsymbol{u} = 0 \text{ or } \frac{\partial^2 (\delta p)}{\partial t^2} - c^2 \triangle (\delta p) = 0 \text{ with } c = (\rho_0 \chi)^{-1/2} \tag{24}$$

Exercise - Show (24)!

Generalization - Assume there exists a body force distribution per unit volume, $\boldsymbol{f}$, for instance due to gravity ($\boldsymbol{f} = \rho \boldsymbol{g}$) or to external sources, then (23) and (24) become

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \nabla (\delta p) = \boldsymbol{f}(t, \boldsymbol{r}) \tag{25}$$

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c^2 \triangle \boldsymbol{u} = \boldsymbol{f} / \rho_0 \; ; \; \frac{\partial^2 (\delta p)}{\partial t^2} - c^2 \triangle (\delta p) = -c^2 \nabla \boldsymbol{f} \tag{26}$$

4.3 Power flux and Poynting vector

We define the following energy quantities:

- kinetic energy $E_c = \int_V e_c \mathrm{d}V$ with $e_c = \frac{1}{2}\rho_0 \boldsymbol{v}.\boldsymbol{v}$
- potential energy $E_p = \int_V e_p \mathrm{d}V$ with $e_p = \frac{1}{2}\frac{S^2}{\chi} = \frac{1}{2}\chi(\delta p)^2$
- Poynting vector $\boldsymbol{P} = \delta p\, \boldsymbol{v}$
- work of internal forces $W = \int_V w\, \mathrm{d}V$ with $\frac{\partial w}{\partial t} = \boldsymbol{f}.\boldsymbol{v}$

From (25): (with $\nabla(\delta p\, \boldsymbol{v}) = \nabla(\delta p).\boldsymbol{v} + \delta p \nabla \boldsymbol{v}$ and $\nabla \boldsymbol{v} = \partial S / \partial t$)

$$\frac{\partial w}{\partial t} = \rho_0 \boldsymbol{v}.\frac{\partial \boldsymbol{v}}{\partial t} + \nabla(\delta p).\boldsymbol{v} = \frac{\partial e_c}{\partial t} + \frac{\partial e_p}{\partial t} + \nabla.\boldsymbol{P}$$

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}(E_c + E_p) + \int_\sigma \boldsymbol{P}.\boldsymbol{l}\, \mathrm{d}\sigma \tag{27}$$

The Poynting vector flux represents the power carried by the wave.

4.4 Energy relations for plane waves

The Poynting vector represents the instantaneous power density per unit surface carried by the wave. The acoustic intensity is by definition

$$I = <\boldsymbol{P}(t).\boldsymbol{l}> = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\,\delta p\,\boldsymbol{v}.\boldsymbol{l} \tag{28}$$

For a plane wave in direction $\boldsymbol{l}$, $u = F(t - x/c)$, $v = F'(t - x/c)$ and $\delta p = \mathrm{ZF}'(t - x/c)$, with x along axis $\boldsymbol{l}$.
Then $e_c = e_p = \frac{1}{2}\rho_0 F'^2(t - x/c)$ and $\boldsymbol{P}.\boldsymbol{l} = ZF'^2(t - x/c) = c(e_c + e_p)$.
For a harmonic plane wave in direction $\boldsymbol{l}$, $u = u_m\sin(\omega(t - x/c))$, then $v = \omega u_m\cos(\omega(t - x/c)) = v_m\cos(\omega(t - x/c))$.

- $e_c = e_p = \frac{1}{2}\rho_0\omega^2 u_m^2\cos^2(\omega(t - x/c))$ and $<e_c> = <e_p> = \frac{1}{4}\rho_0\omega^2 u_m^2 = \frac{1}{4}\rho_0 v_m^2$
- $\boldsymbol{P}.\boldsymbol{l} = Zv_m^2\cos^2(\omega(t - x/c))$
- $I = \frac{1}{2}Zv_m^2 = \frac{1}{2Z}(\delta p_m)^2$

For complex harmonic plane waves, the replacement is

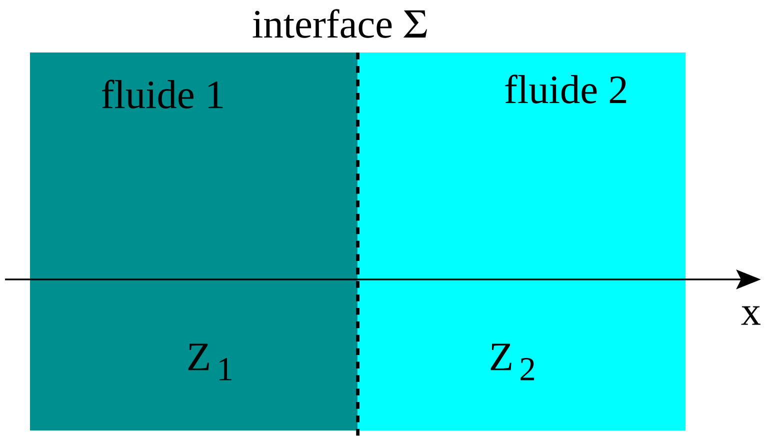
$$e_c = \frac{1}{4}\rho_0\mathrm{Re}(\boldsymbol{v}^*.\boldsymbol{v})\ ;\ e_p = \frac{1}{4}\chi\mathrm{Re}(\delta p^*\delta p)\ ;\ \boldsymbol{P} = \frac{1}{2}\mathrm{Re}(\delta p\,\boldsymbol{v}^*) \tag{29}$$

4.5 Reflection and refraction

4.5.1 Boundary conditions

The boundary conditions at the interface between two non viscous fluids (assumed separated by an infinitely thin boundary) are:

- continuity of the normal component of the displacement ;
- continuity of pressure variations δp at the interface.



If the interface is defined by $x=0$, then

$$u_{1x}(t,x=0,y,z)=u_{2x}(t,x=0,y,z) \tag{30}$$

and similarly for the normal component of the velocity, and

$$\delta p_1(t,x=0,y,z)=\delta p_2(t,x=0,y,z) \tag{31}$$

4.5.2 Normal incidence for a plane wave

A normally incident plane wave gives rise to reflected and transmitted plane waves. The normal displacements at the interface are $u_{1x}(t,\boldsymbol{r}) = F_i(t - x/c_1) + F_r(t + x/c_1)$ and $u_{2x}(t,\boldsymbol{r}) = F_t(t - x/c_2)$. At the interface $(x=0)$:

$$F_i'(t) + F_r'(t) = F_t'(t) \text{ and } Z_1(F_i'(t) - F_r'(t)) = Z_2 F_t'(t)$$

From these equations, we obtain the reflection and transmission coefficients for velocity

$$r_v = \frac{F_r'(t)}{F_i'(t)} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \text{ and } t_v = \frac{F_t'(t)}{F_i'(t)} = \frac{2Z_1}{Z_1 + Z_2} \tag{32}$$

the reflection and transmission coefficients for pressure

$$r_p = -\frac{F_r'(t)}{F_i'(t)} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \text{ and } t_p = \frac{Z_2}{Z_1}\frac{F_t'(t)}{F_i'(t)} = \frac{2Z_2}{Z_1 + Z_2} \tag{33}$$

the reflection and transmission coefficients for acoustic power

$$R = \frac{|P_r|}{|P_i|} = -r_v\, r_p = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \text{ and } T = t_v\, t_p = \frac{4\, Z_1\, Z_2}{(Z_1 + Z_2)^2} = 1 - R \tag{34}$$

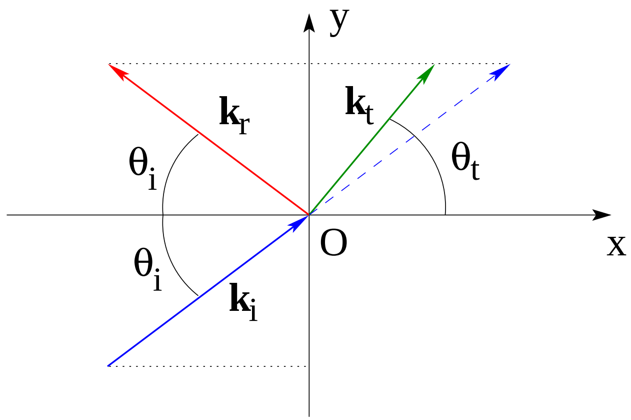
4.5.3 Oblique incidence for a harmonic plane wave

For a harmonic plane wave, equating the normal components of the displacement gives at $\boldsymbol{r} = (0, y, z)^T$:

$$A_{ix}\text{exp}(\text{i}(\omega_i t - \boldsymbol{k}_i.\boldsymbol{r})) + A_{rx}\text{exp}(\text{i}(\omega_r t - \boldsymbol{k}_r.\boldsymbol{r})) = A_{tx}\text{exp}(\text{i}(\omega_t t - \boldsymbol{k}_t.\boldsymbol{r}))$$

This relation is valid $\forall t \in \mathbb{R}$ and $\forall \boldsymbol{r} \in \Sigma$, hence

$$\omega_i = \omega_r = \omega_t \text{ and } \boldsymbol{k}_i.\boldsymbol{r} = \boldsymbol{k}_r.\boldsymbol{r} = \boldsymbol{k}_t.\boldsymbol{r}$$



The following properties apply:

- Reflexion and transmission on a static interface occur without any frequency change.
- Snell-Descartes law: the components along the interface of the wavevector are conserved: $\theta_r = \theta_i$ and $\sin\theta_t / c_2 = \sin\theta_i / c_1$.

The pressure on Σ is $\delta p(t,\boldsymbol{r}) = (A_i + A_r)\exp(\mathrm{i}(\omega t - \boldsymbol{k}.\boldsymbol{r})) = A_t\exp(\mathrm{i}(\omega t - \boldsymbol{k}.\boldsymbol{r}))$, along with the continity of the normal component of velocity we have

$$A_i + A_r = A_t \;\text{ and }\; \frac{A_i}{Z_1}\cos\theta_i - \frac{A_r}{Z_1}\cos\theta_i = \frac{A_t}{Z_2}\cos\theta_t$$

Hence the reflection and transmission coefficients for pressure

$$r_p = \frac{A_r}{A_i} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_i} \text{ and } t_p = \frac{A_t}{A_i} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} \tag{35}$$

and the reflection and transmission coefficients for acoustic power

$$R = \frac{|P_r|}{|P_i|} = |r_p|^2 \text{ and } T = 1 - R \tag{36}$$

4.5.4 Oblique incidence for a harmonic plane wave (cont.)