General solution of the coupled-wave equations of acousto-optics

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The basic problem of the diffraction of an optical plane wave by an acoustic plane wave in an anisotropic homogeneous medium is considered. The acousto-optical interaction is considered indifferently of the isotropic or of the birefringent type. Coupled-wave equations are obtained rigorously and cast into an eigenvalue value problem. A general solution is obtained for the diffraction efficiency of diffracted orders, for any interaction length and diffraction regime. The theory includes the Bragg regime, the Raman–Nath regime, and all intermediate situations in the same formulation. The method of solution is both exact and computationally efficient. It is similar in character to the rigorous coupled-wave analysis of Moharam and Gaylord but differs by the choice of basis functions adapted to propagating rather than static gratings. Examples are given for acousto-optical interaction in paratellurite, TeO_2.

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1. INTRODUCTION

The study of the diffraction of an optical plane wave by an acoustic plane wave in a general anisotropic homogeneous medium is a basic problem of acousto-optics. The determination of the diffraction efficiencies of the various orders of diffraction relies on its solution. This study has generally been divided according to the diffraction regime, e.g., the Raman–Nath and the Bragg regimes,[1–3] and to the interaction type, i.e., isotropic or birefringent interaction.[4] The Raman–Nath and the Bragg regimes are special cases for which analytic solutions have long been known because of the simplifications that can be made in the equations. The coupled-wave equations for isotropic media were established in the 1930s and have in general no analytic solution. However, a complete numerical solution was given by Klein and Cook in 1967.[1] Their solution relies on a finite-difference approximation of the coupled-wave equations, which can be solved with a Runge–Kutta-type algorithm. Korpel[5] summarized various formal and numerical approaches to the problem of light diffraction by a sound column. For birefringent acousto-optical interaction, most studies focus on the Bragg regime only, as it is widely used in applications.

The purpose of this paper is to obtain a general solution of the coupled-wave equations of acousto-optical interaction that is valid whatever the diffraction regime and the crystal anisotropy and for both isotropic and birefringent interactions. Furthermore, this general solution does not rely on a numerical approximation of the coupled-wave equations, as in the method of Klein and Cook,[1] but on the analytical solution of a vectorial first-order differential equation with constant coefficients. The solution is obtained at once in the whole depth of the crystal by solving a linear eigenvalue problem.

The coupled-wave equations are derived rigorously in Section 2. This is necessary because previous derivations assumed isotropic media with the coupled-wave equations expressed for the electric field. This is not consistent in a general anisotropic medium, in which the electric displacement field should be used instead. In Section 3 it is shown that the coupled-wave equations can be given the form of a vectorial first-order differential equation with constant coefficients and that the general solution can be obtained by solving a linear eigenvalue problem. As will be seen, the theory is similar in character to the rigorous coupled-wave analysis of Moharam and Gaylord.[6,7] However, as discussed in Section 3, a closer look shows that it differs mainly by the choice of basis functions that are adapted to propagating rather than static gratings. In Section 4 some examples are given of acousto-optical interactions in paratellurite, TeO_2.

2. COUPLED-WAVE EQUATIONS

Tensor notations are used with the usual convention of summation over repeated indices. The reference frame of Euclidian space is denoted (x_1, x_2, x_3) and can be taken to be the natural crystallographic reference frame of the crystal in which acousto-optical interaction takes place or any other rotated reference frame best suited to the interaction geometry. The optical wave equation expressed with respect to the electric displacement vector $\mathbf{D}$ in an homogeneous anisotropic medium assumes the form

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = \nabla^2(\eta : \mathbf{D}) - \nabla[\nabla \cdot (\eta : \mathbf{D})], \qquad (1)$$

where η is the impermeability tensor, the inverse of the dielectric tensor ϵ according to the relation

$$\epsilon_{ij}\eta_{jk} = \delta_{ik}, \qquad (2)$$

where δ_{ik} is the Kronecker symbol. η_{ij} and ϵ_{ij} are rank-2 symmetric tensors, with indices i and j running from 1 to

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3. The notation $\eta:\mathbf{D}$ represents the right-hand-side contraction of tensor η by vector $\mathbf{D}$, i.e., the vector with components $\eta_{ij}D_j$. Similarly, $\mathbf{D}:\eta$ would represent the left-hand-side contraction, i.e., the transposed vector with components $D_i\eta_{ij}$.

The electric displacement of an optical plane wave can be written as

$$D_i = D_0 d_i \exp[j(\omega t - k_j x_j)] \tag{3}$$

for $i, j = 1 \dots 3$, where ω is the angular frequency, D_0 is a complex amplitude, d_i are the coordinates of a unit polarization vector, and k_j are the coordinates of the wave vector. These are also written as $k_i = kl_i$ where the l_i are director cosines ($l_il_i = 1$). The electric displacement of a plane wave is always transverse, that is, $d_il_i = 0$. By inserting Eq. (3) into Eq. (1), a dispersion relation is readily found that serves as a definition of the index of refraction seen by the plane wave, i.e.,

$$\frac{1}{n^2} = \frac{\omega^2}{k^2c^2} = \mathbf{d}:\eta:\mathbf{d} = d_i\eta_{ij}d_j. \tag{4}$$

There are in general at most two different independent polarization states for propagation in a given direction.

The optical instantaneous Poynting vector is defined by $\mathcal{S} = \mathbf{E}\times\mathbf{H}$, where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic vectors, respectively, and represent the transport of energy by the plane wave. The norm of the optical Poynting vector can be shown to be, in the plane wave case,

$$|\mathcal{S}| = \frac{1}{2}\frac{c}{\epsilon_0 n^3}D_0^2, \tag{5}$$

where c is the speed of light in a vacuum and ϵ_0 is the vacuum permittivity.

The polarization of the acoustic plane wave is represented by the rank-2 symmetric strain tensor, S_{ij}, according to

$$S_{ij}(t, \mathbf{r}) = S\hat{s}_{ij}\sin(\Omega t - \mathbf{K}\cdot\mathbf{r}), \tag{6}$$

where S is a complex amplitude, Ω is the acoustic pulsation, $\mathbf{K}$ is the acoustic wave vector, and $\hat{s}$ is a unit polarization rank-2 tensor. The acoustic transported power is given by[3]

$$P_{\text{ac}} = I_{\text{ac}}S_t, \tag{7}$$

where S_t is the transducer's effective surface and where

$$I_{\text{ac}} = \tfrac{1}{2}\rho V_a^3|S|^2 \tag{8}$$

is the acoustic intensity, i.e., the acoustic power-flux density.

The elasto-optic effect is represented in the Pockels theory by the deformation of the impermeability tensor caused by the acoustic wave, according to

$$\begin{aligned}\Delta\eta_{ij}(t, \mathbf{r}) &= p_{ijkl}S_{kl}(t, \mathbf{r})\\ &= S(p_{ijkl}\hat{s}_{kl})\sin(\Omega t - \mathbf{K}\cdot\mathbf{r})\\ &= \Delta\eta_{ij}\sin(\Omega t - \mathbf{K}\cdot\mathbf{r}).\end{aligned} \tag{9}$$

To solve for the optical propagation problem, it is necessary to insert such a spatial dependence of the impermeability tensor into the wave equation, Eq. (1). The usual procedure[3] is to represent the optical field by the superposition

$$\mathbf{D}(\mathbf{r}, t) = \sum_{m=-\infty}^{\infty} D_m(z_m)\mathbf{d}_m \exp[j(\omega_m t - \mathbf{k}_m\cdot\mathbf{r})], \tag{10}$$

where

$$\omega_m = \omega + m\Omega, \tag{11}$$

$$\mathbf{k}_m = \mathbf{k} + m\mathbf{K}. \tag{12}$$

It is worthwhile noting that the elementary plane waves in the expansion of Eq. (10) are not necessarily propagation modes, since they will not, in general, satisfy the dispersion relation of Eq. (4). However, this is not a problem as long as their superposition is itself a solution of the wave equation. They will be referred to as the diffraction orders. The unit polarization vectors $\mathbf{d}_m$ are as yet unspecified. The functions $D_m(z_m)$ are assumed to be slowly varying envelope functions that depend on some coordinate z_m, which is in most derivations chosen arbitrarily and identical for all diffraction orders, e.g., along the acoustic wave vector.[3] We next show that, in fact, the direction of variation of the envelopes $D_m(z_m)$ cannot be chosen arbitrarily. Let us write this direction as

$$z_m = \mathbf{u}_m\cdot\mathbf{r}, \tag{13}$$

where $\mathbf{u}_m$ is a unit vector, so that the partial derivatives with respect to the space coordinates of a function $f(z_m)$ are

$$\frac{\partial f(z_m)}{\partial x_i} = (u_m)_i\frac{\mathrm{d}f(z_m)}{\mathrm{d}z_m} = (u_m)_i f'(z_m). \tag{14}$$

The divergence of Eq. (10) must vanish in accordance with Maxwell's equations. Expressing this condition for all space and time positions, we arrive at

$$\mathbf{u}_m\cdot\mathbf{d}_m D'_m(z_m) - j\mathbf{k}_m\cdot\mathbf{d}_m D_m(z_m) = 0, \tag{15}$$

which holds for every diffraction order and every spatial position z_m. Then we have necessarily

$$\mathbf{u}_m\cdot\mathbf{d}_m = 0, \tag{16}$$

$$\mathbf{k}_m\cdot\mathbf{d}_m = 0. \tag{17}$$

As Eq. (17) shows, the polarization direction must be chosen transverse. As is well known, the intersection of the index ellipsoid with a plane orthogonal to a given wavevector direction is an ellipse, the two main axes of which give the eigenmodes of propagation. Then the diffraction order can be decomposed along the ordinary and extraordinary polarizations, and only the polarization that is best compatible with the dispersion relation, Eq. (4), can be retained in the analysis in practice. The refractive index for order m is, in any case, defined by

$$\frac{1}{n_m^2} = \mathbf{d}_m:\eta:\mathbf{d}_m. \tag{18}$$

Furthermore, as Eq. (16) shows, it is not possible, in general, to consider a single z axis for all diffraction orders. Although there remains some degree of freedom in the choice of $\mathbf{u}_m$, the most obvious choice is to take it as the unit vector in the direction of the wave vector $\mathbf{k}_m$, i.e., $\mathbf{u}_m = \mathbf{l}_m$.

Coupled-wave equations are obtained by inserting Eq. (10) into Eq. (1). This procedure is quite straightforward but long and is summarized in Appendix A. We give only the final results here. The problem considered, depicted in Fig. 1, is to find the diffraction efficiencies of all diffracted orders when the interaction region is restricted to a region of space limited by two parallel planes a distance L apart. All the wave vectors of the diffracted orders lie in the plane of incidence defined by $\mathbf{k}_0$ and $\mathbf{K}$. A distance L_m can be associated with each diffraction order, measured along the direction of its wave vector. The square moduli of the envelopes are made proportional to the diffraction efficiencies by use of the transformed functions

$$\hat{D}_m(Z) = \frac{1}{\sqrt{n_m^3 L_m}} D_m(z_m). \tag{19}$$

The division by the factor $\sqrt{n_m^3}$ ensures that the envelopes are proportional to the square root of the modulus of the optical Poynting vector, according to Eq. (5). The division by $\sqrt{L_m}$ is a correction term compensating for the variation with the diffraction order of the apparent surface crossed by the optical flux. We have also set $Z = z_m/L_m$, where L_m is the effective interaction length for the mth diffraction order, so that all envelopes depend on a single reduced coordinate that equals 0 at the entrance of the crystal and 1 at the output. Z is termed the normalized depth. With these notations, the coupled-wave equations assume the simple form

$$\hat{\mathbf{D}}'(Z) = jM\hat{\mathbf{D}}(Z), \tag{20}$$

where $\hat{\mathbf{D}}(Z)$ is the vector whose components are the amplitudes $\hat{D}_m(Z)$ and M is a tridiagonal square matrix whose only nonzero elements are

$$M_{m,m} = \frac{1}{2} k_m L_m \left(1 - \frac{\omega_m^2 n_m^2}{c^2 k_m^2}\right), \tag{21}$$

$$M_{m,m-1} = -\frac{j\omega}{4c} (L_m L_{m-1} n_m^3 n_{m-1}^3)^{1/2} \Delta n|_m^{m-1}, \tag{22}$$

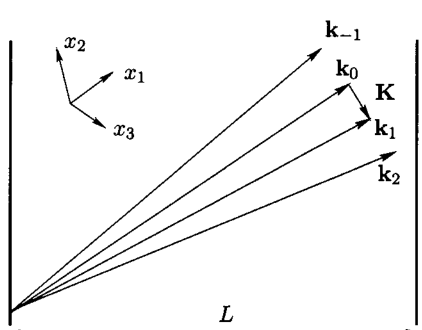

Fig. 1. Schematic of the interaction region. All diffraction orders lie in the plane of incidence defined by wave vectors $\mathbf{k}_0$ and $\mathbf{K}$.

$$M_{m,m+1} = \frac{j\omega}{4c} (L_m L_{m+1} n_m^3 n_{m+1}^3)^{1/2} \Delta n|_m^{m+1}, \tag{23}$$

with the notation

$$\Delta n|_m^{m\pm 1} = \mathbf{d}_m : \Delta\eta : \mathbf{d}_{m\pm 1}. \tag{24}$$

By direct inspection, it can easily be seen that matrix M is Hermitian symmetric, i.e., $M_{m-1,m} = M_{m,m-1}^*$ and $M_{m+1,m} = M_{m,m+1}^*$, whereas $M_{m,m}$ is purely real.

3. METHOD OF SOLUTION

In this section we use matrix notations with explicit summations. The coupled-wave equations, Eq. (20), assume the form of a vectorial first-order differential equation with constant coefficients. As is well known, such equations can be simply solved by obtaining the eigenvalues v_m and the eigenvectors $\mathbf{x}_m$ of matrix M. Such a solution is the basis of the nth-order order approximation method[8] of solving the Raman–Nath equations. With the eigenvectors arranged vertically in matrix X, the solution of Eq. (20) is

$$\hat{\mathbf{D}}(Z) = X\Delta(Z)\mathbf{a}, \tag{25}$$

where $\Delta(Z)$ is a diagonal matrix with elements

$$\Delta_{m,m}(Z) = \exp(jv_m Z) \tag{26}$$

and where $\mathbf{a}$ is a vector of unknown parameters, which is obtained by expressing initial conditions. Since at the entrance of the crystal, $Z = 0$, only the diffracted order $m = 0$ is incident, then

$$\hat{D}(0) = X_{m,n} a_n = \delta_m, \tag{27}$$

where δ_m equals 1 if $m = 0$ and 0 otherwise, and vector $\mathbf{a}$ is obtained by solution of a linear system. A few observations can be made regarding this matrix solution. First, matrices and vectors have to be truncated in practice. The usual procedure would then be to increase the number of selected orders of diffraction until the computation result is no longer dependent on this number. Second, the solution is obtained at once in the whole depth of the crystal. The computation time is mostly dictated by the number of diffraction orders that are retained for analysis and is mostly consumed by the eigenvalue and eigenvectors determination. Third, as matrix M is Hermitian symmetric, its eigenvalues are necessarily real, and, by proper normalization of the eigenvectors, matrix X can be made unitary, i.e.,

$$X^\dagger X = I, \tag{28}$$

where I is the identity matrix and (†) represents the transpose conjugate of a matrix or vector. It then follows, by one's taking the norm of Eq. (27), that

$$\sum_m |a_m|^2 = 1 \tag{29}$$

and then, from Eq. (25) that

$$\sum_m |\hat{D}_m(Z)|^2 = 1. \tag{30}$$

As this last identity proves, the diffraction efficiencies always sum to unity, whatever the particular material properties, diffraction regime, and interaction length. This is obviously a compelling physical property. Furthermore, left multiplying Eq. (27) by $X^\dagger$ yields

$$a_n = X^*_{0,n}, \tag{31}$$

where (*) stands for the complex conjugate. Equation (25) can then be written explicitly as

$$\hat{D}_m(Z) = \sum_n X_{m,n} X^*_{0,n} \exp(j v_n Z), \tag{32}$$

which emphasizes that the amplitude of each diffraction order is the result of the interference of harmonic eigenmodes.

The solution that has been obtained in this section bears a strong similarity to the rigorous coupled-wave analysis solution of plane-wave diffraction by planar-diffraction gratings of Moharam and Gaylord.[6,7] Indeed, the coupled-wave equations obtained by these authors were cast in a vectorial first-order differential equation with constant coefficients and solved by an eigenvector superposition. However, an important difference between planar gratings and acousto-optical diffraction is the static nature of the former. When one tries to apply the rigorous coupled-wave analysis solution of Ref. 7 to acousto-optical diffraction, a problem arises in connection with the choice of basis functions. Indeed, as can be seen from Eqs. (10)–(12), the expansion used is not over true propagation modes, since the basis functions do not satisfy the dispersion relation. When one uses true propagation modes, e.g., by using phase matching for two components of the wave vectors and the dispersion relation to find the third, the coupled-wave equations will still be in the form of a first-order differential equation but with nonconstant coefficients. This arises because phase matching in the case of acousto-optical diffraction involves both the spatial and the temporal dependences of plane waves. The change in frequency of diffracted orders is independent of the change in wave vector, and hence either temporal or spatial modulation of the coupling strength results.

4. EXAMPLES

The examples in this section are mostly intended to illustrate the application of the general solution of the coupled-wave equations but not especially to be interesting in practice. The interactions considered are in the intermediate regime; i.e., they are neither in the Raman–Nath nor in the Bragg regime. The interaction geometry is depicted in Fig. 2. A pure-shear acoustic wave is propagating along the [110] direction in a TeO_2 crystal, polarized along the $[1\bar{1}0]$ direction, with a velocity of 613 m/s. This wave is often used in practical devices because the value of the figure of merit M_2 is high.[3,9–11] It can be used for isotropic interaction, i.e., without any change in the optical polarization of the diffraction orders, and for birefringent interaction. We assume that the incident light, in the zeroth order, is ordinary polarized. In the case of birefringent interaction, odd diffraction orders have an extraordinary polarization, whereas even diffraction orders have an ordinary polarization. The material constants for TeO_2, i.e., elasto-optic and dielectric coefficients, are taken from Ref. 12. Seventeen diffraction orders are taken into account in all computations, with the index m running from -8 to 8. It was verified that the diffraction efficiencies of higher diffraction orders are negligible in the cases considered. The crystal is assumed to be cut normally to the [001] axis of Fig. 2. L is the interaction length measured along the [001] axis.

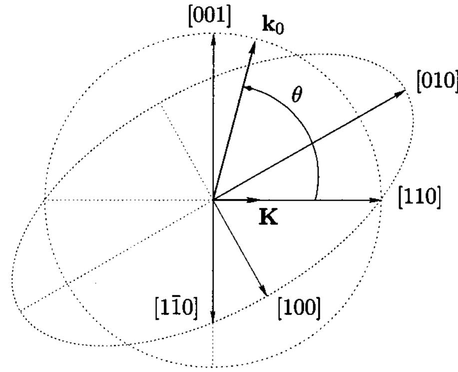

Fig. 2. Definitions for the acousto-optical interaction in TeO_2 considered in Figs. 3–5. Acoustic waves are propagating along the [110] direction and are shear polarized along the $[1\bar{1}0]$ direction. The plane of incidence is defined by axes [001] and [110]. θ is the angle of incidence for incoming light.

Figure 3 shows the diffraction efficiencies obtained around normal incidence in a case close to the Raman–Nath diffraction regime, as a function of the optical wavelength. Diffraction regimes are customarily classified since Klein and Cook[1] according to the value of the parameter Q defined by

$$Q = (K^2 L)/k. \tag{33}$$

The conditions $Q \ll 1$ and $Q \gg 1$ define the Raman–Nath and the Bragg interaction regimes, respectively. For the computations shown in Fig. 3, the acoustic frequency is $\Omega = 20\,\mathrm{MHz}$ and the interaction length is $L = 1\,\mathrm{mm}$. Then Q equals approximately 1.1, 1.7, and 2.9 for an optical wavelength of 0.4, 0.6, and 1 μm, respectively. These values are higher than what is required for the Raman–Nath diffraction regime but still close to that regime. The acoustic intensity is chosen to give maximum diffraction efficiency for the first diffraction order at an optical wavelength of 0.63 μm. For normal incidence, Fig. 3(a), the diffraction efficiencies are rather adequately described by the Bessel functions' solution specific to the Raman–Nath regime. In particular, a maximum diffraction efficiency of 39% is obtained for the first order, slightly higher than the 33.9% upper limit in the Raman–

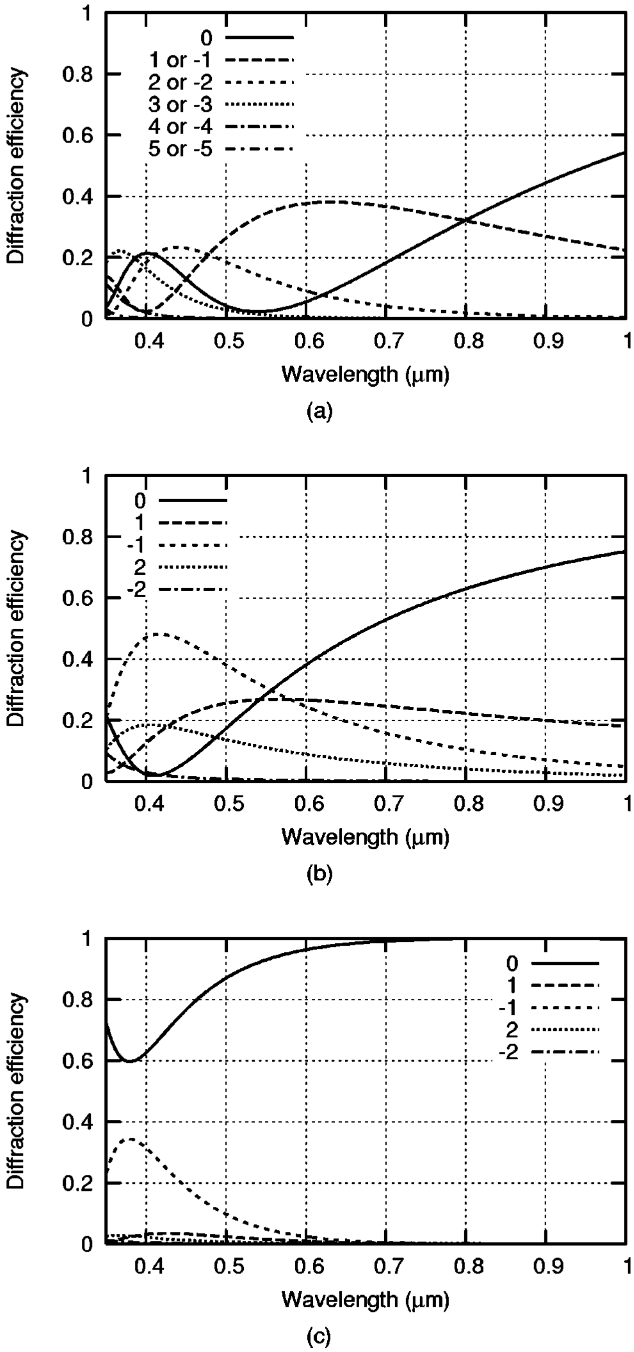

Fig. 3. Diffraction efficiency for isotropic (ordinary light) acousto-optical interaction in TeO_2 in the intermediate regime. The interaction length is $L = 1$ mm, the acoustic intensity is $I_{ac} = 10^5$ kg/s^3, and the acoustic frequency is $\Omega = 20$ MHz. The angles of incidence are (a) 90°, (b) 89°, and (c) 88°. Figure labels are for diffraction orders.

Nath regime, the maximum of the squared J_1 Bessel function. However, as the angle of incidence becomes different from 90°, the behavior of the diffraction efficiencies changes rapidly, as depicted in Figs. 3(b) and 3(c). The negative first diffraction order becomes dominant over other nonzero diffraction orders.

Figure 4 shows the buildup of the Bragg interaction regime for birefringent interaction and ordinary incident light. With the acoustic frequency set to $\Omega = 60$ MHz, the Bragg condition is achieved in the first order of diffraction for $\lambda = 0.82\ \mu$m approximately at an angle of incidence of 89°. In Figs. 4(a), 4(b), and 4(c) the interaction length L is 2, 6, and 10 mm, respectively; the value of the parameter Q is then approximately 36, 108, and 180, respectively. The acoustic intensity is chosen so that maximum diffraction efficiency is achieved in the first order at $\lambda = 0.82\ \mu$m. The first-order diffraction efficiency as a function of the optical wavelength assumes the shape typical of the Bragg regime, with only the zeroth and first diffraction orders having nonnegligible diffraction efficiency in the vicinity of $\lambda = 0.82\ \mu$m. However, the second order of diffraction also satisfies the Bragg condition at $\lambda = 0.42\ \mu$m approximately, and it is seen that its diffraction efficiency can become quite important if Q is not too high, i.e., 57% in Fig. 4(a). To emphasize this phenomenon, we show in Fig. 5 the variation of the diffraction efficiencies as a function of the position inside the crystal at $\lambda = 0.42\ \mu$m, in the same conditions as in Fig. 4(a). The normalized depth Z is allowed to vary from 0 to 3; i.e., positions measured along the [001] axis run from 0 to 6 mm. It can be seen that the first-order diffraction efficiency oscillates, whereas the second-order diffraction efficiency progressively builds up until $Z = 1.3$, at which point the diffraction efficiency reaches a maximum of 94% and then decreases.

5. CONCLUSION

A general solution of the classic problem of the diffraction of an optical plane wave by an acoustic plane wave in an anisotropic homogeneous medium has been presented. The acousto-optical interaction can be indifferently of the isotropic or of the birefringent type. Coupled-wave equations have been obtained rigorously and cast in an eigenvalue value problem. General formulas have been given for the diffraction efficiency of all diffraction orders, for any interaction length and diffraction regime. The theory includes the Bragg regime, the Raman–Nath regime, and all intermediate situations in the same formulation. The method of solution is both exact and computationally efficient and can be used to design a particular acousto-optical device or as the basis of a realistic model including the finite sizes of the optical and acoustic beams. Examples were given for acousto-optical diffraction in paratellurite, TeO_2. The theory will be further developed by considering the case of multifrequency acousto-optical diffraction.[13–15]

APPENDIX A: DERIVATION OF THE COUPLED-WAVE EQUATIONS

In this appendix we outline the derivation of the coupled-wave equations (20)–(23). From Eq. (10), we wish to determine all terms in the wave equation, Eq. (1). The time derivative is

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = -\sum_m \frac{\omega_m^2}{c^2} D_m(z_m)\mathbf{d}_m \exp(j\phi_m), \qquad \text{(A1)}$$

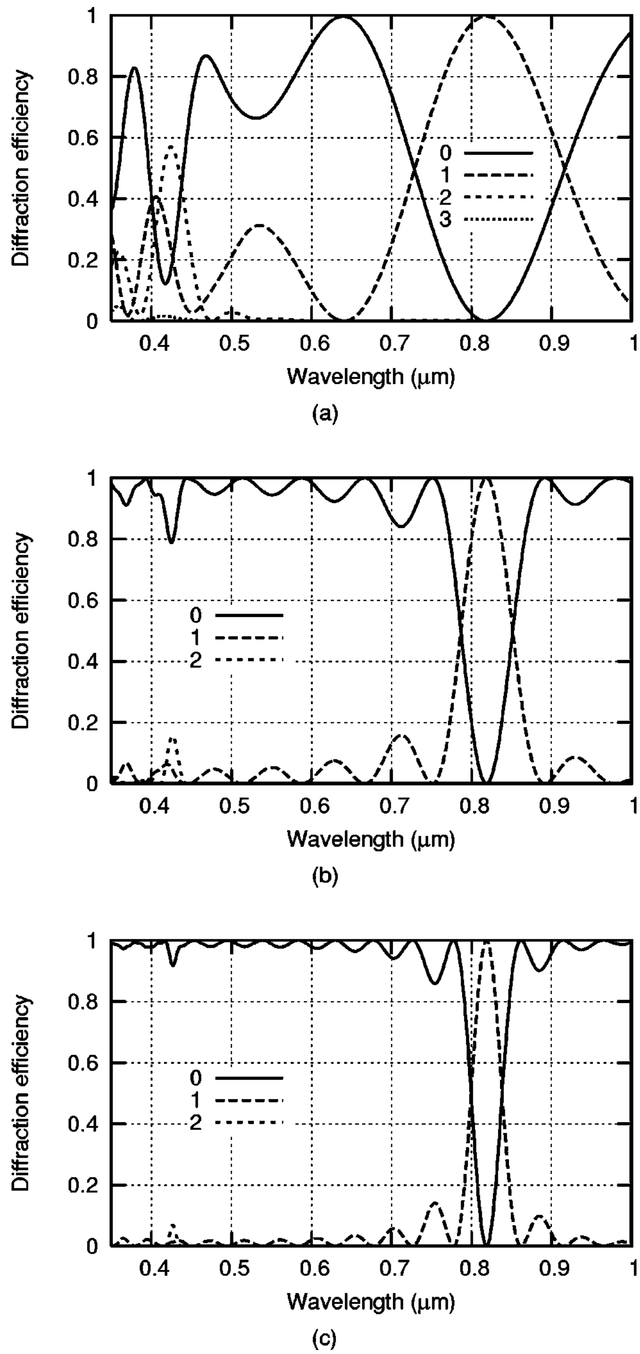

Fig. 4. Diffraction efficiency for birefringent acousto-optical interaction in TeO_2 in the intermediate regime, for ordinary incident light. The acoustic frequency is $\Omega = 60\,\mathrm{MHz}$, and the angle of incidence is 89°; the Bragg condition is achieved in the first order of diffraction for $\lambda = \sim 0.82\,\mu\mathrm{m}$. The interaction length and the acoustic intensity are, respectively, (a) $L = 2\,\mathrm{mm}$ and $I_{ac} = 119{,}200\,\mathrm{kg/s^3}$, (b) $L = 6\,\mathrm{mm}$ and $I_{ac} = 13{,}000\,\mathrm{kg/s^3}$, and (c) $L = 10\,\mathrm{mm}$ and $I_{ac} = 4768\,\mathrm{kg/s^3}$. The acoustic intensity is set to yield maximum diffraction efficiency in the first diffraction order at $\lambda = 0.82\,\mu\mathrm{m}$. Figure labels are for diffraction orders.

where we have introduced the notation

$$\phi_m = \omega_m t - \mathbf{k}_m \cdot \mathbf{r}. \tag{A2}$$

The time- and space-dependent impermeability tensor uses Eq. (9):

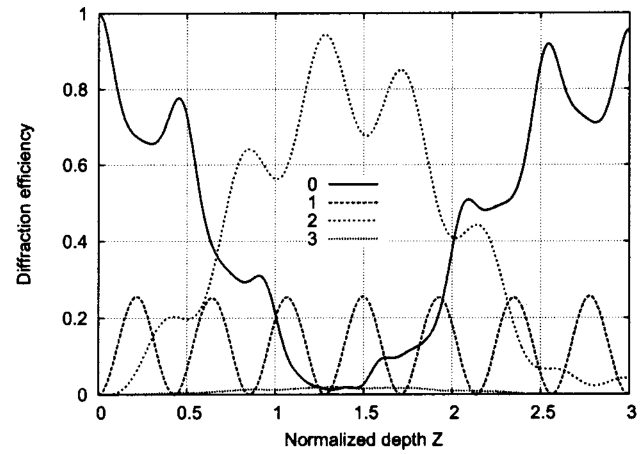

Fig. 5. Evolution of the diffraction efficiency for anisotropic acousto-optical interaction in TeO_2 as a function of the position inside the crystal, for ordinary incident light. The acoustic frequency is $\Omega = 60\,\mathrm{MHz}$, and the angle of incidence is 89°. $L = 2\,\mathrm{mm}$ and $I_{ac} = 119{,}200\,\mathrm{kg/s^3}$ as in Fig. 4(a). Figure labels are for diffraction orders.

$$\eta(t, \mathbf{r}) = \eta + \frac{1}{2j}\Delta\eta\{\exp[j(\Omega t - \mathbf{K}\cdot\mathbf{r})] - \exp[-j(\Omega t - \mathbf{K}\cdot\mathbf{r})]\}. \tag{A3}$$

We next obtain

$$\eta : \mathbf{D}(\mathbf{r}, t) = \sum_m D_m(z_m)\,\eta : \mathbf{d}_m \exp(j\phi_m) + \frac{1}{2j}\sum_m D_m(z_m)\Delta\eta : \mathbf{d}_m \exp(j\phi_{m+1}) - \frac{1}{2j}\sum_m D_m(z_m)\Delta\eta : \mathbf{d}_m \exp(j\phi_{m-1}). \tag{A4}$$

The Laplacian of this expression reads as

$$\nabla^2[\eta : \mathbf{D}(\mathbf{r}, t)] = -\sum_m [k_m^2 D_m(z_m) + 2j(\mathbf{k}_m \cdot \mathbf{u}_m)D'_m(z_m)]\,\eta : \mathbf{d}_m \exp(j\phi_m) - \frac{1}{2j}\sum_m D_m(z_m)k_{m+1}^2\Delta\eta : \mathbf{d}_m \times \exp(j\phi_{m+1}) + \frac{1}{2j}\sum_m D_m(z_m)k_{m-1}^2 \times \Delta\eta : \mathbf{d}_m \exp(j\phi_{m-1}). \tag{A5}$$

In this equation, all terms that are of the second order or higher with respect to $\Delta\eta$ and the derivatives of envelope functions $D_m(z_m)$ have been neglected. This amounts to assuming $\Delta\eta \ll \eta$ and the envelope functions are slowly varying. On rearrangement of the summation order, it comes to

$$\nabla^2[\eta:\mathbf{D}(\mathbf{r},t)] = -\sum_m [k_m^2 D_m(z_m) + 2jk_m D'_m(z_m)]\eta:\mathbf{d}_m \exp(j\phi_m) - \frac{1}{2j}\sum_m k_m^2 D_{m-1}(z_{m-1})\Delta\eta:\mathbf{d}_{m-1} \times \exp(j\phi_m) + \frac{1}{2j}\sum_m k_m^2 D_{m+1}(z_{m+1}) \times \Delta\eta:\mathbf{d}_{m+1}\exp(j\phi_m). \tag{A6}$$

Proceeding to the second term of the right-hand side of Eq. (10), we have

$$\nabla\cdot[\eta:\mathbf{D}(\mathbf{r},t)] = \sum_m [-jD_m(z_m)(\mathbf{k}_m:\eta:\mathbf{d}_m) + D'_m(z_m)(\mathbf{u}_m:\eta:\mathbf{d}_m)]\exp(j\phi_m) - \frac{1}{2}\sum_m D_m(z_m)(\mathbf{k}_{m+1}:\Delta\eta:\mathbf{d}_m)\exp(j\phi_{m+1}) + \frac{1}{2}\sum_m D_m(z_m)(\mathbf{k}_{m-1}:\Delta\eta:\mathbf{d}_m)\exp(j\phi_{m-1}) \tag{A7}$$

and then

$$\nabla\{\nabla\cdot[\eta:\mathbf{D}(\mathbf{r},t)]\} = -\sum_m \mathbf{k}_m[D_m(z_m)(\mathbf{k}_m:\eta:\mathbf{d}_m) + 2jD'_m(z_m)(\mathbf{l}_m:\eta:\mathbf{d}_m)] \times \exp(j\phi_m) + \frac{j}{2}\sum_m D_m(z_m) \times \mathbf{k}_{m+1}(\mathbf{k}_{m+1}:\Delta\eta:\mathbf{d}_m) \times \exp(j\phi_{m+1}) - \frac{j}{2}\sum_m D_m(z_m) \times \mathbf{k}_{m-1}(\mathbf{k}_{m-1}:\Delta\eta:\mathbf{d}_m) \times \exp(j\phi_{m-1}) \tag{A8}$$

or, on rearrangement,

$$\nabla\{\nabla\cdot[\eta:\mathbf{D}(\mathbf{r},t)]\} = -\sum_m \mathbf{k}_m[D_m(z_m)(\mathbf{k}_m:\eta:\mathbf{d}_m) + 2jD'_m(z_m)(\mathbf{l}_m:\eta:\mathbf{d}_m)]\exp(j\phi_m) + \frac{j}{2}\sum_m \mathbf{k}_m D_{m-1}(z_{m-1})(\mathbf{k}_m:\Delta\eta:\mathbf{d}_{m-1})\exp(j\phi_m) - \frac{j}{2}\sum_m \mathbf{k}_m D_{m+1}(z_{m+1})(\mathbf{k}_m:\Delta\eta:\mathbf{d}_{m+1})\exp(j\phi_m). \tag{A9}$$

All terms have now been expressed, and Eqs. (A1), (A6), and (A9) can be inserted into Eq. (1). In the resulting equation, terms with the same ϕ_m dependence must be grouped and equated to zero, by virtue of the phase-matching principle. Left multiplying with $\mathbf{d}_m$, we see it further comes to

$$\left(\frac{\omega_m^2}{c^2} - \frac{k_m^2}{n_m^2}\right)D_m(z_m) - 2j\frac{k_m}{n_m^2}D'_m(z_m) = \frac{k_m^2}{2j}[\Delta n|_m^{m-1}D_{m-1}(z_{m-1}) - \Delta n|_m^{m+1}D_{m+1}(z_{m+1})]. \tag{A10}$$

On rearranging, the coupled-wave equations are of the form

$$D'_m(z_m) = \frac{j}{2}k_m\left(1 - \frac{\omega_m^2 n_m^2}{c^2 k_m^2}\right)D_m(z_m) + \frac{1}{4}n_m^2 k_m \times [\Delta n|_m^{m-1}D_{m-1}(z_{m-1}) - \Delta n|_m^{m+1}D_{m+1}(z_{m+1})]. \tag{A11}$$

By use of definition (19), it comes to

$$\hat{D}'_m(Z) = \sqrt{L_m/n_m^3}D'_m(z_m) \tag{A12}$$

and then

$$\hat{D}'_m(Z) = \frac{j}{2}k_m L_m\left(1 - \frac{\omega_m^2 n_m^2}{c^2 k_m^2}\right)\hat{D}_m(Z) + \frac{1}{4}\sqrt{n_m L_m}k_m\sqrt{n_{m-1}^3 L_{-1}}\Delta n|_m^{-1}\hat{D}_{-1}(Z) - \frac{1}{4}\sqrt{n_m L_m}k_m\sqrt{n_{m+1}^3 L_{m+1}}\Delta n|_m^{+1}\hat{D}_{m+1}(Z). \tag{A13}$$

As discussed in Section 3, matrix M in Eqs. (20)–(23) should be Hermitian symmetric to have the physical property that the computed diffraction efficiencies sum to unity. Equation (A13) does not possess this property but can be made to have it by use of the approximation

$$k_m \approx n_m\frac{\omega}{c}, \tag{A14}$$

but only for the off-diagonal terms of M. This leads to the final Hermitian symmetric form

$$\hat{D}'_m(Z) = \frac{j}{2}k_m L_m\left(1 - \frac{\omega_m^2 n_m^2}{c^2 k_m^2}\right)\hat{D}_m(Z) + \frac{1}{4}\frac{\omega}{c}\sqrt{n_m^3 L_m}\sqrt{n_{m-1}^3 L_{m-1}}\Delta n|_m^{m-1}\hat{D}_{-1}(Z) - \frac{1}{4}\frac{\omega}{c}\sqrt{n_m^3 L_m}\sqrt{n_{m+1}^3 L_{m+1}}\Delta n|_m^{m+1}\hat{D}_{m+1}(Z). \tag{A15}$$

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