Multicriteria characterization of coding domains with optimal Fourier spatial light modulator filters

V. Laude and Ph. Réfrégier

A multicriteria optimization method is introduced in order to find optimal filters for implementation on arbitrary spatial light modulators in the Fourier plane of an optical correlator. This method is applied to the trade-offs between noise robustness, sharpness of the correlation peak, and optical efficiency. A fast and simple algorithm is given in this case, which is independent of the particular form of the spatial light modulator coding constraint. It is used to characterize and to compare typical coding domains through the performances of their associated optimal filters.

Key words: Correlation, optimal filters, spatial light modulator filters.

1. Introduction

For more than 10 years many authors have studied the performances of correlation filters for optical pattern recognition. For comparison of filters, quantitative criteria have been proposed.[1,2] Optimal filters, with respect to one criterion or a set of criteria, have been derived and studied (see Ref. 3 for a review). But it is only recently that the optimal implementation of these filters in an optical correlator started to be analyzed.[4] Indeed, many optical correlators now use spatial light modulators (SLM's) to display both input scene and filter. But a drawback is that they cannot achieve every real or complex value that is required. Thus the mathematical formulation of the optimization of a filter with respect to given criteria has to include the constraint imposed by the SLM, which is that the filter values must belong to a given coding domain. It is naturally expected that the more restrained the coding domain, the less efficient the displayable filters. But this intuitive conjecture needs to be quantified, and the performances of filters displayable on typical SLM's should be compared to enable choice between SLM's for a particular application.

The aim of this paper is to present the derivation and the calculation of optimal filters for any given Fourier SLM coding domain. For the solution of this problem, different methods have been proposed.

Juday[4] introduced a general approach consisting of optimizing a metric, constructed in view of obtaining desired properties of the filter, under the constraint that the filter values belong to the coding domain. Based on this method, algorithms have been derived and filter design examples have been presented.[5–8] The differences and new results obtained with our approach are detailed at the end of Section 2.

Mait *et al.*[9] proposed an iterative algorithm with the objective, starting from a given unconstrained filter, of using design freedoms to modify the distribution of Fourier values yet maintaining its performance as a filter. They illustrated this method with binary-amplitude phase-only, ternary, and phase-only coding. Future research should make clear the relations and the distinctions of the different possible approaches to the optimization of SLM filters.

The approach we follow in this paper is to consider a set of criteria, each measuring a desired property of the filter, and seeking optimal trade-offs between them. The most useful criteria are generally thought to be signal-to-noise ratio (SNR) for noise robustness, peak-to-correlation energy (PCE) for sharpness of the correlation peak, and optical efficiency (η_H) for detection convenience.[1,2] With the multicriteria optimization method, optimal filters were obtained previously considering both unit-disk coding[10] and binary-amplitude phase-only and ternary coding.[11]

In the present paper it is generalized to any SLM coding domain. In particular, a geometric interpretation of trade-offs between criteria is given. A simple and fast algorithm is introduced for obtaining filters achieving optimal trade-offs among SNR, PCE,

The authors are with the Laboratoire Central de Recherches, Thomson-CSF, Domaine de Corbeville, Orsay Cedex F-91404, France.

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and η_H, independent of the coding-domain shape. This algorithm is used to characterize and to compare typical coding domains in terms of the performances of their associated filters. The approach we propose could also be applied to discrimination capability as introduced by Yarolavsky,[12] and it could take into account detector noise as discussed in Ref. 10 for unit-disk coding. These straightforward generalizations are not discussed in detail in the following for the sake of clarity and brevity.

2. Background: Multicriteria Optimization Method

In this section, the optimal trade-off optimization method is presented on a geometrical basis. The coherent correlator considered is sketched in Fig. 1. The filter is displayed on a SLM, in the Fourier domain. See, for example, Ref. 13 for an implementation of this architecture. All images have N pixels and are represented as one-dimensional vectors for clarity. $\mathbf{x}$ denotes the pattern or reference to be recognized, and $\hat{\mathbf{x}}$ denotes its discrete Fourier transform. The pattern element at frequency k is then denoted $\hat{x}_k$. The filter to be determined is denoted $\mathbf{h}$, which is $\hat{\mathbf{h}}$ in the Fourier domain. The correlation-peak central values is $c_0 = \Sigma_k \hat{h}_k \hat{x}_k$. The use of a SLM to display the filter imposes that the filter values $\hat{h}_k$ belong to a certain domain $\mathscr{D}$ of the complex plane. As SLM's are passive optical elements, $\mathscr{D}$ is contained in the unit disk. In Fig. 2 are represented some examples of such ideal and realistic domains. Unit-disk (full complex modulation),[14] phase-only[15] and amplitude-only[16] codings are idealized versions of what currently available SLM's can achieve. Binary phase-only and ternary codings can be achieved, respectively, with ferroelectric liquid-crystal[17] and magneto-optic SLM's.[18] Spiral coding, parameterized by maximum phase shift K, was observed with twisted-nematic liquid-crystal SLM's.[19] This list is of course not exhaustive.

Suppose we consider p criteria $E_i(\hat{\mathbf{h}})$ with $i = 1 \ldots p$ and $p \geq 2$. The criterion $E_i(\hat{\mathbf{h}})$ is a continuous real function of the filter $\hat{\mathbf{h}}$ such that the smaller its value, the more the filter $\hat{\mathbf{h}}$ displays the desired behavior. Let us denote $\{E_i(\hat{\mathbf{h}})\}$ as the point at which coordinates are the criteria value for a given filter $\hat{\mathbf{h}}$. As we consider the SLM coding domain $\mathscr{D}$, $\hat{\mathbf{h}}$ is an element of $\mathscr{D}^N$. We can plot, for every possible filter, the point $\{E_i(\hat{\mathbf{h}})\}$ on a graph on which axes correspond to the criteria values. We thus obtain a cloud of

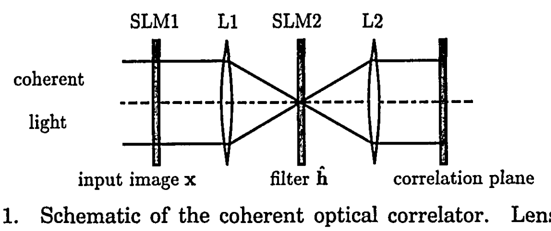


Fig. 1. Schematic of the coherent optical correlator. Lens L1 forms the Fourier transform of the input image, displayed on SLM1, in its back focal plane, where the filter is displayed on SLM2. The correlation plane is observed in the back focal plane of lens L2.

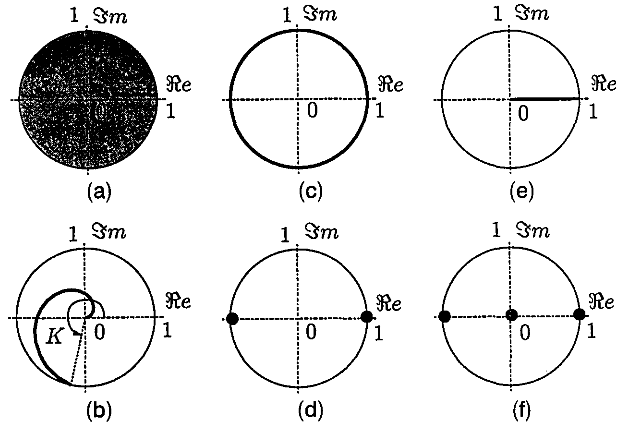


Fig. 2. Examples of coding domains: (a) unit disk; (b) phase only; (c) amplitude only; (d) spiral, parametrized by maximum phase shift K; (e) binary phase only; (f) ternary.

points $\mathscr{C}$, sketched in Fig. 3 for $p = 3$. This cloud represents all the combinations of the criteria considered that can exist for a particular coding domain $\mathscr{D}$ and permits a quantitative comparison of filters. Obviously, $\mathscr{C}$ contains in practice so many points that they cannot all be calculated. Fortunately this is not needed, as we need to consider only the best points.

This intuition can be formalized with the use of the set $\mathscr{S}_{\text{OT}}$ of optimal trade-off filters (OT filters), which

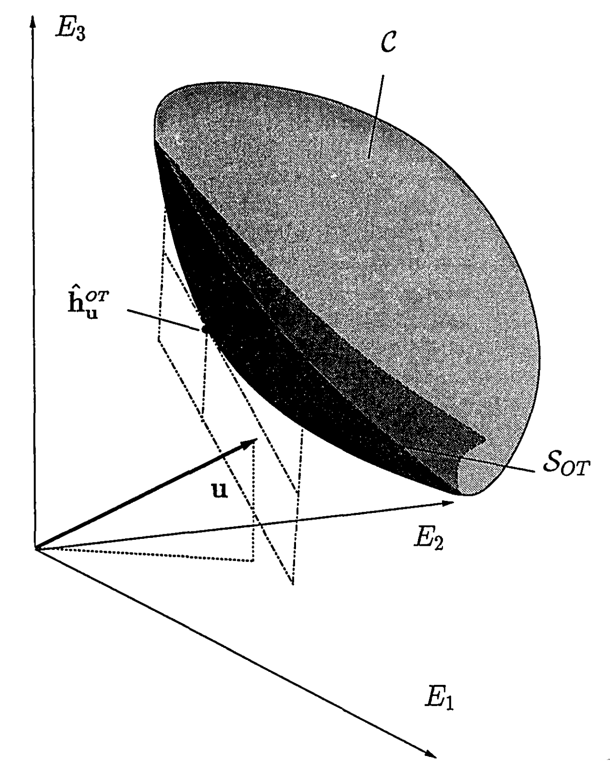


Fig. 3. Comparison of filters through their criteria values, with an example of three criteria. Set $\mathscr{C}$ is composed of all combinations of criteria values generated by filters satisfying the coding-domain constraint. $\mathscr{S}_{\text{OT}}$ is the set of OT filters that are obtained by a search for planes tangent to $\mathscr{C}$ and that are referenced by their normal vector $\hat{u}$.

are defined[20] by the condition that, whatever filter $\hat{\mathbf{h}}$ is represented by a point of cloud $\mathscr{C}$, on OT filter $\hat{\mathbf{h}}^{OT}$, itself represented by a point of $\mathscr{C}$, performs better than $\hat{\mathbf{h}}$ for at least one criterion. $\mathscr{S}_{OT}$ is represented in Fig. 3 for $p = 3$. An important property is that, given any filter $\hat{\mathbf{h}}$, there always exists an OT filter $\hat{\mathbf{h}}^{OT}$ that performs better than $\hat{\mathbf{h}}$ for all criteria.[21] In that sense, OT filters provide a useful figure of merit for filter design. The problem now amounts to finding these optimal filters.

To expose the optimization method, let us consider the case in which $p = 3$. The following results are easily generalized to other values of p. For clarity we suppose that the criteria are positive functions, but the following construction is generalized easily to negative criteria. Let (α, β, γ) be the coordinates of a unit vector $\hat{u}$ of the first quadrant. They are positive, and $\alpha^2 + \beta^2 + \gamma^2 = 1$. Let us then consider all the planes that are orthogonal to unit vector $\hat{u}$, as shown in Fig. 3. The particular plane that is tangent to $\mathscr{C}$ defines on OT filter $\hat{\mathbf{h}}_{\hat{u}}^{OT}$ at the intersection. Mathematically this can be written as

$$\hat{\mathbf{h}}_{\hat{u}}^{OT} = \underset{\hat{\mathbf{h}} \in \mathscr{D}^N}{\operatorname{argmin}} \{\alpha E_1(\hat{\mathbf{h}}) + \beta E_2(\hat{\mathbf{h}}) + \gamma E_3(\hat{\mathbf{h}})\}. \quad (1)$$

This is the minimization of an energy function that is the scalar product of unit vector $\hat{u}$ with vector $\{E_i(\hat{\mathbf{h}})\}$. Thus by varying the (α, β, γ) parameters, we can generate OT filters. Geometrically this amounts to determining a part of the convex envelop of set $\mathscr{C}$, called the optimal characteristics curve for a two-criteria optimization problem[20] and the optimal characteristics surface (OCS) for more criteria. These optimal characteristics curves and OCS's permit a direct comparison of coding domains.

As is said above, this optimization method is applied in this paper with consideration of the SNR, PCE, and η_H criteria. These are defined as

$$\eta_H = |c_0|^2, \quad (2a)$$

$$\mathrm{SNR} = \frac{\eta_H}{\mathrm{MSE}}, \quad (2b)$$

$$\mathrm{PCE} = \frac{\eta_H}{\mathrm{CPE}}. \quad (2c)$$

The optical efficiency η_H of Eq. (2a) is generally[1] divided by the reference-image total energy $\Sigma_k |\hat{x}_k|^2$, but we omit this, as it does not affect the filter optimization. Mean-square-error (MSE) and correlation-peak energy (CPE) are expressed as

$$\mathrm{MSE} = \sum_k \hat{C}_{k,k} |\hat{h}_k|^2, \quad (3a)$$

$$\mathrm{CPE} = \sum_k \hat{D}_{k,k} |\hat{h}_k|^2. \quad (3b)$$

In these last expressions, $\hat{C}$ is the noise power spectral density and $\hat{D}$ is the reference power spectral density ($\hat{D}_{k,k} = |\hat{x}_k|^2$). $\hat{C}$ and $\hat{D}$ are diagonal matrices of dimension $N \times N$, with positive real elements. MSE and CPE are quadratic forms of the filter, an interesting property that is used in the following. It was proved[10] that the optimal trade-off filters for the SNR, PCE, and η_H criteria can be found easily from those for the MSE, CPE, and $|c_0|$ criteria. This last problem is computationally much less difficult.

An important difference with previous studies[4–8] is the search for optimal trade-offs instead of the optimization of a single metric. In particular, Juday[8] proposed optimizing the peak-to-total-energy metric, which is expressed as $\eta_H/(\mathrm{MSE} + \mathrm{CPE})$. It can be shown that this is equivalent to finding an OT filter for the MSE, CPE, and $|c_0|$ criteria, with $\alpha = \beta$, and in which γ has to be optimized. Thus the filter maximizing the peak-to-total energy yields a single point on the OCS, which is not enough to compare between coding domains. Moreover, it can be remarked that the position of this point on the OCS depends, for example, on the assumed total noise power $\Sigma_k \hat{C}_{k,k}$. It might not be clear *a priori* what the value of this constant should be in a particular application.

Thus in addition to providing a rigorous method of comparison between different codings, finding the optimal trade-offs lets one obtain all the information at hand to implement a given application optimally. In other words it lets one separate the problem of evaluating the performances obtainable with a given coding domain from the problem of the selection of a filter with given criteria values, which can be done directly on the graphical representation of the OCS.

Moreover, it is shown in Section 3 that the determination of the optimal trade-offs leads to a very simple and efficient algorithm. In particular there is no need to use differential calculus to prove the minimum-distance principle introduced by Juday.[8]

3. Simple Algorithm for Quadratic Criteria

In this section we now consider the particular case of quadratic criteria and derive a simple solution for this case for any given SLM coding domain $\mathscr{D}$. For clarity the method is exposed with the example of the MSE, CPE, and $|c_0|$ criteria, and the necessary proofs are given in Appendix A.

The energy function we wish to minimize can be written as follows:

$$E(\hat{\mathbf{h}}) = \alpha \mathrm{MSE}(\hat{\mathbf{h}}) + \beta \mathrm{CPE}(\hat{\mathbf{h}}) - 2\gamma |c_0(\hat{\mathbf{h}})|, \quad (4)$$

where $(\alpha, \beta, 2\gamma)$ are the three positive coordinates of a unit vector $\hat{u}$, implying that $\alpha^2 + \beta^2 + 4\gamma^2 = 1$. The minus sign before $|c_0|$ replaces maximization by minimization, and the factor 2 is only used for convenience. Equation (4) can be written as

$$E(\hat{\mathbf{h}}) = \sum_k [\hat{B}_{\hat{u}}]_{k,k} |\hat{h}_k|^2 - 2\gamma \left| \sum_k \hat{h}_k \hat{x}_k \right|, \quad (5)$$

where the diagonal matrix $\hat{B}_{\hat{u}}$ is defined by $[\hat{B}_{\hat{u}}]_{k,k} = \alpha \hat{C}_{k,k} + \beta \hat{D}_{k,k}$. Minimization of this energy function without consideration of the coding-domain constraint leads to the classical OT filter[10]:

$$(\hat{h}_{\hat{u}}^0)_k = \gamma \frac{\hat{x}_k^*}{[\hat{B}_{\hat{u}}]_{k,k}} \cdot \qquad (6)$$

In the graph of Fig. 4(b) is plotted a typical complex histogram of an OT filter $\hat{\mathbf{h}}_{\hat{u}}^0$. Each point of this histogram corresponds to a complex value $(\hat{\mathbf{h}}_{\hat{u}}^0)_k$. With this representation, γ appears as a scaling parameter, measuring the saturation of the filter in the sense that it determines the number of points contained in the unit disk. $\hat{x}_k^*/[\hat{B}_{\hat{u}}]_{k,k}$ determines the shape of the histogram. It can be seen that some points can be very far from the coding domain, illustrated in the example of Fig. 4(b) with a spiral coding.

As demonstrated in the appendix, minimizing the energy function of Eq. (5) is equivalent to minimizing

$$E_\varphi(\hat{\mathbf{h}}) = \sum_k |\hat{h}_k - (\hat{h}_{\hat{u}}^0)_k \exp(i\varphi)|^2, \qquad (7)$$

with $\hat{h}_k \in \mathscr{D}$ and where $\varphi \in [0, 2\pi]$ is an angular parameter to be optimized. This expression is the Euclidean distance between filters $\hat{\mathbf{h}}$ and $\hat{\mathbf{h}}_{\hat{u}}^0[\exp(i\varphi)]$. If displayed in Fig. 4(b), the complex histogram of filter $\hat{\mathbf{h}}_{\hat{u}}^0[\exp(i\varphi)]$ would simply be that of $\hat{\mathbf{h}}_{\hat{u}}^0$ rotated by an angle φ. For a fixed φ the sum in Eq. (7) is minimized if all of its terms are minimized. This is obtained when point $\hat{h}_k$ is the Euclidean projection of point $(\hat{h}_{\hat{u}}^0)_k[\exp(i\varphi)]$ on the coding domain $\mathscr{D}$ for each spatial frequency k. We write this projected filter as

$$\hat{\mathbf{h}}_{\hat{u}}^\varphi = \mathscr{P}_{\mathscr{D}}[\hat{\mathbf{h}}_{\hat{u}}^0 \exp(i\varphi)]. \qquad (8)$$

Let us then define the function

$$f_{\hat{u}}(\varphi) = E(\hat{\mathbf{h}}_{\hat{u}}^\varphi) = E\{\mathscr{P}_{\mathscr{D}}[\hat{\mathbf{h}}_{\hat{u}}^0 \exp(i\varphi)]\}, \qquad (9)$$

obtained by inserting $\hat{\mathbf{h}}_{\hat{u}}^\varphi$ in Eq. (5). A typical function $f_u(\varphi)$ is plotted in Fig. 5, which was obtained with the spiral coding and the reference image of Fig. 4. Once obtained, the angle φ^S minimizing this function yields the optimum SLM filter minimizing $E(\hat{\mathbf{h}})$ in $\mathscr{D}^N$ simply by replacement of φ by φ^S in Eq. (8). Thus the optimization problem of Eq. (1), a search over all the filters that are elements of $\mathscr{D}^N$, is replaced by a simple search over a continuous real parameter φ. It can be shown, although the demonstration is beyond the scope of this paper, that $f_{\hat{u}}(\varphi)$ is always a continuous function, whatever the shape of the coding domain $\mathscr{D}$.

We now can give a simple algorithm for obtaining the OCS for any coding domain $\mathscr{D}$. For a fixed unit vector $\hat{u}$, do the following:

- Calculate the OT filter $\hat{\mathbf{h}}_{\hat{u}}^0$, without coding-domain constraint, by Eq. (6).
- For every φ, do the following:
  Calculate the projected filter $\hat{\mathbf{h}}_{\hat{u}}^\varphi$ by Eq. (8).
  Calculate the corresponding value $f_{\hat{u}}(\varphi)$ by Eq. (9).
- Select angle φ^S, which minimizes $F_{\hat{u}}(\varphi)$. The optimum SLM constrained filter is then the projected filter $\hat{\mathbf{h}}_{\hat{u}}^{\varphi S}$.
- Calculate the values of SNR, PCE, and η_H, as given by Eqs. (2), for $\hat{\mathbf{h}}_{\hat{u}}^{\varphi S}$. A point on the OCS is thus obtained.
- Repeat the preceding steps for another unit vector $\hat{u}$.

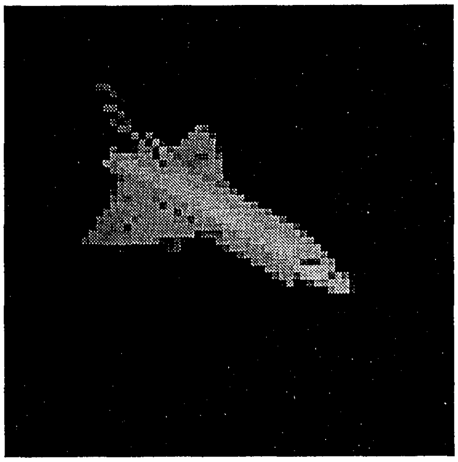
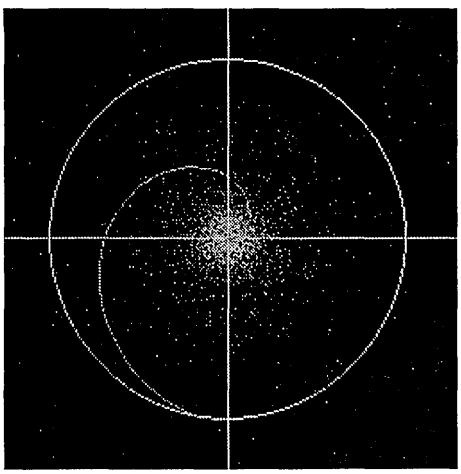

(a)

(b)

Fig. 4. (a) 64 × 64 reference image used for simulations; (b) typical complex histogram of an OT filter, generated with the reference in (a) without the coding contraint. Each point of the histogram has to be projected on the coding constraint (here the spiral), which is rotated to find the global optimum.

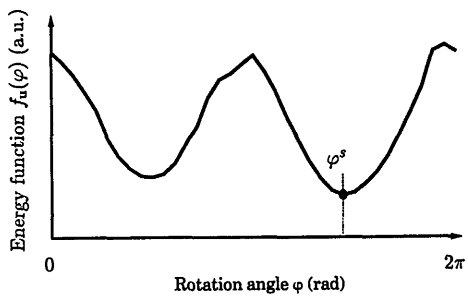


Fig. 5. Typical function $f_u(\varphi)$, measuring the quality of projected filters versus rotation angle φ. Optimal angle φ^S yields the OT filter satisfying the coding-domain contraint.

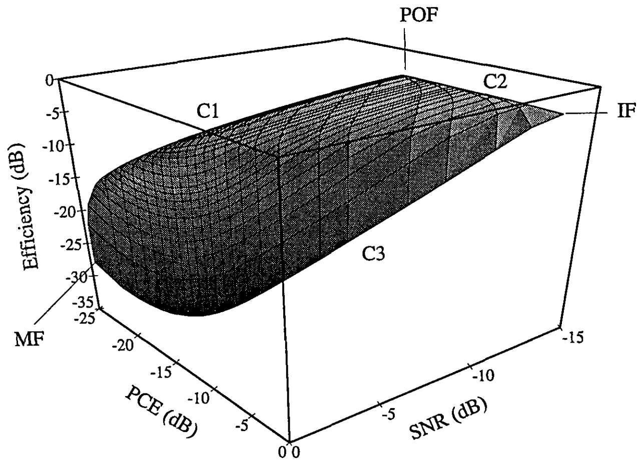


Fig. 6. Optimal characteristics surface (OCS) for unit-disk coding. POF, MF, and IF stand for phase-only, matched, and inverse filters, respectively. Boundary curves C1, C2, and C3 joining these points represent, respectively, the optimal trade-offs between SNR and optical efficiency, between PCE and optical efficiency, and between SNR and PCE.

Practically, for the numerical simulations described in Section 4 we calculated $f_{\hat{u}}(\varphi)$ every 10° and kept the minimum. For the coding domains considered, this rather loose sampling always led sufficiently close to the actual global minimum. The most time-consuming step in the algorithm is the projection, depending directly on the number of points defining the coding domain.

4. Characterization of Typical Coding Domains

In this section the previous results are used to characterize typical coding domains, through their optimal filters for the SNR, PCE, and η_H criteria. For each of the six coding domains depicted in Fig. 2 the OCS was generated with the 64 × 64 reference image of Fig. 4(a). For the spiral coding two values of maximal phase shift are considered, $K = 180°$ and $K = 360°$, and both are quantized on 32 levels.

As all the coding domains considered are subsets of the unit disk, it is clear that this last coding domain must yield the best performances of all. Consequently all criteria are normalized by their maximum values attained with the unit disck and expressed in decibel units, i.e., $10 \log(E/E_{\max})$ for criterion E.

It is well known that the optical efficiency η_H is maximized by the phase-only filter $\hat{h}_k = \hat{x}_k^*/|\hat{x}_k|$, yielding $(\eta_H)_{\max} = (\Sigma_k |\hat{x}_k|)^2$ from Eq. (2a). This filter is obtained with unit-disk coding as γ tends to 1/2 (fully saturated filter).

Similarly, signal-to-noise ratio (SNR) is maximized by the matched filter $\hat{h}_k = \gamma \hat{x}_k^*/\hat{C}_{k,k}$, yielding $(\mathrm{SNR})_{\max} = \Sigma_k \hat{D}_{k,k}/\hat{C}_{k,k}$ from Eq. (2b). The matched filter is obtained with unit-disk coding for $\beta = 0$ and for γ sufficiently small to avoid saturation.

Finally, peak-to-correlation energy (PCE) is maximized by the inverse filter $\hat{h}_k = \gamma \hat{x}_k^*/\hat{D}_{k,k}$, yielding $(\mathrm{PCE})_{\max} = N$ from Eq. (2c). The inverse filter is obtained with unit-disk coding for $\alpha = 0$ and for γ sufficiently small to avoid saturation.

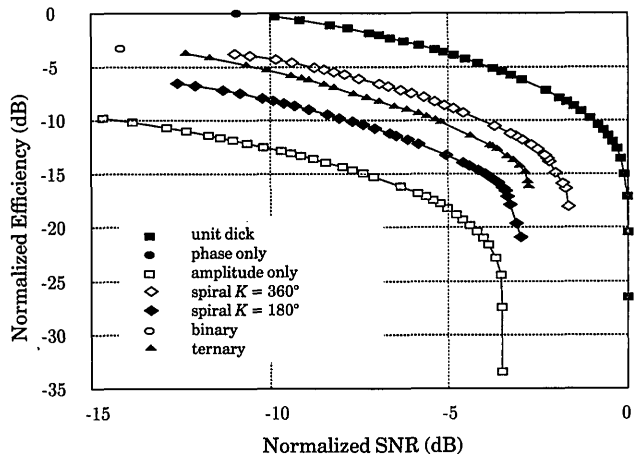


Fig. 7. Optimal trade-offs between SNR and optical efficiency for the coding domains of Fig. 2.

The OCS for unit-disk coding is shown in Fig. 6. The phase-only, matched, and inverse filters appear as limiting points on this surface. The boundary curves that join these points, denoted C1, C2, and C3 in Fig. 6, are used to compare the different coding domains of Fig. 2. Figure 7, corresponding to C1, presents the optimal trade-offs between SNR and optical efficiency. It is obtained by the setting of $\beta = 0$; i.e., we do not take into account the PCE and we vary γ, for example. Similarly, Fig. 8, corresponding to C2, presents the optimal trade-offs between PCE and optical efficiency. It is obtained by the setting of $\alpha = 0$; i.e., we do not take into account the SNR and we vary γ. Figure 9, corresponding to C3, presents the optimal trade-offs between SNR and PCE. It is obtained by variation of α, for example, and optimization of γ for each value of α. Although the present analysis is restricted to these three sections, others might be used as well; for example, constant optical efficiency sections may be used.

From Figs. 7–9 the following observations can be made. As expected, unit-disk coding always yields the best performances. It is the only coding that can

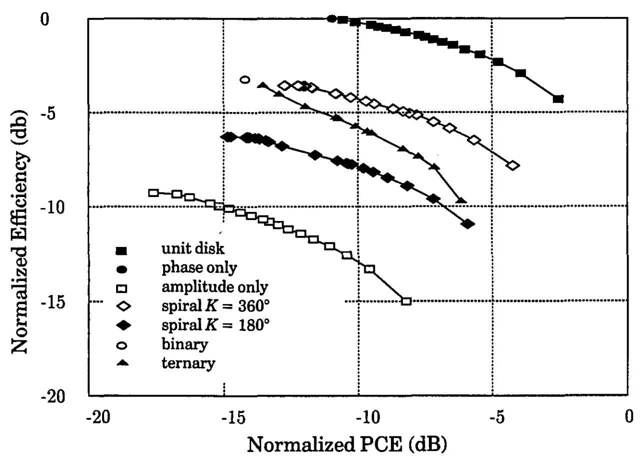


Fig. 8. Optimal trade-offs between PCE and optical efficiency for the coding domains of Fig. 2.

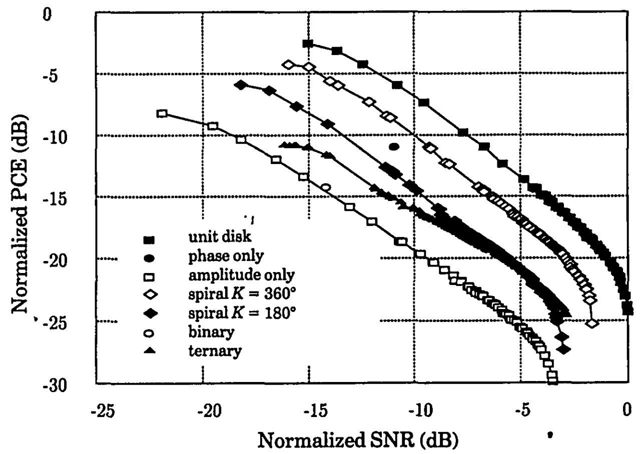


Fig. 9. Optimal trade-offs between SNR and PCE for the coding domains of Fig. 2.

be used to display any given filter, provided the saturation parameter γ is sufficiently small, that accordingly lowers the optical efficiency.

For phase-only and binary phase-only codings the following holds: there is no way to trade off between the three chosen criteria, variation of unit vector $\hat{u}$ always yields the same filter, and the OCS reduces to a single point. Both codings are overspecialized for optical efficiency.

Amplitude-only coding is always associated with the poorest performance of all. This makes it unsuited for practical applications.

For spiral coding, associated with twisted-nematic liquid-crystal SLM's, the larger the maximum phase shift K, the better the performance. This is an interesting property that might be used, as maximum phase shifts larger than 400° were observed recently with such modulators.[22]

Ternary coding turns out to perform surprisingly well although it uses only three levels, and it almost always does better than the 180° spiral coding, which uses 32 levels. These observations demonstrate the key importance of choosing a SLM for use in the Fourier plane of an optical correlator.

5. Conclusion

We have introduced a multicriteria optimization method, based on a geometrical interpretation of trade-offs between criteria, for any Fourier SLM coding-domain constraint. We have applied this method to the SNR, PCE, and η_H criteria. In this case we have given a fast and simple algorithm for designing OT filters for any coding domain. We have characterized unit-disk, phase-only, amplitude-only, spiral, binary phase-only, and ternary codings through their optimal characteristics surfaces and curves, and we have compared them. Unit-disk coding yields the best performances. According to the criteria chosen, phase-only and binary phase-only codings provide very few freedom degrees for filter design, and amplitude coding is not suited for Fourier filtering. But, remarkably, spiral and ternary coding have turned out to show interesting performances.

Appendix A: Some Results for Quadratic Criteria

In this appendix the properties used in Section 3 are demonstrated. For brevity we use vectorial notations for images. Superscripts T and $\dagger$ denote, respectively, transposition and complex-conjugate transposition of a vector.

With these notations the energy function of Eq. (5) becomes

$$E(\hat{\mathbf{h}}) = \hat{\mathbf{h}}^{\dagger}\hat{B}_{\hat{u}}\hat{\mathbf{h}} - 2\gamma|\hat{\mathbf{h}}^{T}\cdot\hat{\mathbf{x}}|.$$

Let us introduce the modified energy function,

$$E_{\varphi}(\hat{\mathbf{h}}) = \hat{\mathbf{h}}^{\dagger}\hat{B}_{\hat{u}}\hat{\mathbf{h}} - 2\gamma\Re[\hat{\mathbf{h}}^{T}\cdot\hat{\mathbf{x}}\exp(-i\varphi)], \quad \text{(A1)}$$

where $\Re$ stands for the real part of a complex number and φ is a real parameter in the range $[0, 2\pi]$. Defining $\phi(\hat{\mathbf{h}})$ as the phase of the correlation peak $\hat{\mathbf{h}}^{T}\cdot\hat{\mathbf{x}}$, we can write

$$E_{\varphi}(\hat{\mathbf{h}}) = \hat{\mathbf{h}}^{\dagger}\hat{B}_{\hat{u}}\hat{\mathbf{h}} - 2\gamma|\hat{\mathbf{h}}^{T}\cdot\hat{\mathbf{x}}|\cos[\phi(\hat{\mathbf{h}}) - \varphi], \quad \text{(A2)}$$

from which it can be seen that

$$E_{\varphi}(\hat{\mathbf{h}}) \geq E(\hat{\mathbf{h}}), \quad \text{(A3)}$$

with equality occurring if and only if $\phi(\hat{\mathbf{h}}) = \varphi$.

In order to find the minimum of $E_{\varphi}(\hat{\mathbf{h}})$ without consideration of the coding domain, let us consider a small variation $\delta\hat{\mathbf{h}}$ of filter $\hat{\mathbf{h}}$. The corresponding variation $\delta E_{\varphi}(\hat{\mathbf{h}})$ of the modified energy function $E_{\varphi}(\hat{\mathbf{h}})$ is

$$\delta E_{\varphi}(\hat{\mathbf{h}}) = E_{\varphi}(\hat{\mathbf{h}} + \delta\hat{\mathbf{h}}) - E_{\varphi}(\hat{\mathbf{h}}), \quad \text{(A4)}$$

whose first order is

$$\delta E_{\varphi}(\hat{\mathbf{h}}) = 2\Re[\delta\hat{\mathbf{h}}^{\dagger}\hat{B}_{\hat{u}}\hat{\mathbf{h}} - \gamma\delta\hat{\mathbf{h}}^{T}\cdot\hat{\mathbf{x}}\exp(-i\varphi)]. \quad \text{(A5)}$$

It is easily verified that equating this first-order variation to zero yields the extremum filter $\hat{\mathbf{h}}_{\hat{u}}^{0}[\exp(i\varphi)]$, where $\hat{\mathbf{h}}_{\hat{u}}^{0}$ is given by Eq. (6). Furthermore, it is immediately proven that

$$E_{\varphi}[\hat{\mathbf{h}}_{\hat{u}}^{0}\exp(i\varphi)] = E[\hat{\mathbf{h}}_{\hat{u}}^{0}\exp(i\varphi)] = E(\hat{\mathbf{h}}_{\hat{u}}^{0}) \quad \text{(A6)}$$

and that the modified energy function takes the following form:

$$E_{\varphi}(\hat{\mathbf{h}}) = [\hat{\mathbf{h}} - \hat{\mathbf{h}}_{\hat{u}}^{0}\exp(i\varphi)]^{\dagger}\hat{B}_{\hat{u}}[\hat{\mathbf{h}} - \hat{\mathbf{h}}_{\hat{u}}^{0}\exp(i\varphi)] + E(\hat{\mathbf{h}}_{\hat{u}}^{0}). \quad \text{(A7)}$$

The right-hand side of this last equation is composed of two terms. The first term is a positive quadratic form, reaching zero when $\hat{\mathbf{h}} = \hat{\mathbf{h}}_{\hat{u}}^{0}[\exp(i\varphi)]$, and the second term is a constant independent of parameter φ. With the help of these properties it is seen that the extremum filter $\hat{\mathbf{h}}_{\hat{u}}^{0}[\exp(i\varphi)]$ indeed minimizes $E_{\varphi}(\hat{\mathbf{h}})$ without consideration of the coding constraint. And in addition, because $\hat{B}_{\hat{u}}$ is a diagonal matrix with positive real elements, minimizing $E_\varphi(\hat{\mathbf{h}})$ in $\mathscr{D}^N$ is equivalent to minimizing the Euclidean distance of Eq. (7). For a given parameter φ the filter that minimizes $E_\varphi(\hat{\mathbf{h}})$ considering the coding constraint is then the filter $\hat{\mathbf{h}}_{\hat{u}}^{\varphi} = \mathscr{P}_{\mathscr{D}}[\hat{\mathbf{h}}_{\hat{u}}^{0} \exp(i\varphi)]$ of inequality (8), in which the projection operator on the coding constraint $\mathscr{P}_{\mathscr{D}}$ is defined in Section 3.

Let us then consider a filter $\hat{\mathbf{h}}_{\hat{u}}^{S}$ that minimizes $E(\hat{\mathbf{h}})$ in $\mathscr{D}^N$. From inequality (A3) we can write for any value of parameter φ the following:

$$E_\varphi(\hat{\mathbf{h}}_{\hat{u}}^{\varphi}) \geq E(\hat{\mathbf{h}}_{\hat{u}}^{\varphi}) \geq E(\hat{\mathbf{h}}_{\hat{u}}^{S}), \qquad \text{(A8)}$$

from which comes, with definition (9) of $f_{\hat{u}}(\varphi)$,

$$\min_{\varphi\in[0,2\pi]} \{f_{\hat{u}}(\varphi)\} \geq E(\hat{\mathbf{h}}_{\hat{u}}^{S}). \qquad \text{(A9)}$$

Furthermore, defining the phase $\varphi^S = \phi(\hat{\mathbf{h}}_{\hat{u}}^{S})$, from inequality (A3) we then have $E_{\varphi_S}(\hat{\mathbf{h}}_{\hat{u}}^{S})$. But then $\hat{\mathbf{h}}_{\hat{u}}^{S}$ necessarily minimizes $E_{\varphi S}(\hat{\mathbf{h}})$ in $\mathscr{D}^N$, as is seen from inequality (A8). This last property can be written $f_{\hat{u}}(\varphi^S) = E(\hat{\mathbf{h}}_{\hat{u}}^{S})$, which shows that inequality (A9) is indeed an equality. This proves that in order to find the OT filter that minimizes $E(\hat{\mathbf{h}})$ in $\mathscr{D}^N$, we can seek the phase φ^S that minimizes $f_{\hat{u}}(\varphi)$.

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