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Semi-decentralized approximation of optimal control of distributed systems based on a functional calculus

Y. Yakoubi, M. Lenczner*,† and N. Ratier

Department Time-Frequency, FEMTO-ST Institute, 26, Rue de l'Epitaphe, 25030 Besançon, France

SUMMARY

This paper discusses a new approximation method for operators that are solution to an operational Riccati equation. The latter is derived from the theory of optimal control of linear problems posed in Hilbert spaces. The approximation is based on the functional calculus of self-adjoint operators and the Cauchy formula. Under a number of assumptions, the approximation is suitable for implementation on a semi-decentralized computing architecture in view of real-time control. Our method is particularly applicable to problems in optimal control of systems governed by partial differential equations with distributed observation and control. Some relatively academic applications are presented for illustration. More realistic examples relating to microsystem arrays have already been published. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

This work is a contribution to the area of semi-decentralized optimal control of large linear distributed systems for real-time applications. It applies to systems modeled by linear partial differential equations with observation and control distributed over the whole domain. This is a strong assumption, but it does not mean that actuators and sensors are actually continuously distributed. The models satisfying such assumption may be derived by homogenization of systems with periodic distribution of actuators and sensors.

In this paper, we consider two classes of systems: those with bounded control and bounded observation operators as in [1] and those with unbounded control but bounded observation operators as in [2]. In an example, we show how the method may also be applied to a particular boundary control problem. We view possible applications in the field of systems including a network of actuators and sensors; see, for instance, [3] dedicated to arrays of atomic force microscopes.

We consider four linear operators A, B, C, and S, and the linear quadratic regulator (LQR) problem stated classically as a minimization problem,

$$\mathcal{J}(z_0, \mathbf{u}) = \min_{u \in U} \mathcal{J}(z_0, u), \tag{1}$$

$$\text{with } \mathcal{J}(z_0, u) = \int_0^{+\infty} \|Cz\|_Y^2 + (Su, u)_U \, dt, \tag{2}$$

*Correspondence to: M. Lenczner, Department Time-Frequency, FEMTO-ST Institute, 26, Rue de l'Epitaphe, 25030 Besançon, France.
†E-mail: michel.lenczner@utbm.fr

constrained by a state equation,

$$\frac{dz}{dt}(t) = Az(t) + Bu(t) \quad \text{for } t > 0 \quad \text{and } z(0) = z_0 \tag{3}$$

for some linear spaces U and Y. Under usual assumptions, there exists a unique solution $\mathbf{u} = -S^{-1}B^*Pz$, where P is a solution of the operational Riccati equation (ORE),

$$A^*P + PA - PBS^{-1}B^*P + C^*C = 0. \tag{4}$$

For another linear space Z, we use the framework of [1] where $A : D(A) \subset Z \mapsto Z, B : U \mapsto Z, C : Z \mapsto Y, S : U \mapsto U$ and consequently $P : Z \mapsto Z$. To derive our semi-decentralized realization of Pz, we further assume that there exists a linear space X, a linear self-adjoint operator $\Lambda : X \mapsto X$, three one-to-one mappings

$$\Phi_Z : X^{n_Z} \mapsto Z, \quad \Phi_U : X^{n_U} \mapsto U \quad \text{and } \Phi_Y : X^{n_Y} \mapsto Y, \tag{5}$$

with appropriate integers n_Z, n_U and n_Y, and four continuous matrix-valued functions $\lambda \mapsto a(\lambda), \lambda \mapsto b(\lambda), \lambda \mapsto c(\lambda)$ and $\lambda \mapsto s(\lambda)$ such that

$$A = \Phi_Z a(\Lambda)\Phi_Z^{-1}, \quad B = \Phi_Z b(\Lambda)\Phi_U^{-1}, \quad C = \Phi_Y c(\Lambda)\Phi_Z^{-1} \quad \text{and } S = \Phi_U s(\Lambda)\Phi_U^{-1}. \tag{6}$$

We notice that the functions of the self-adjoint operator Λ used in the aforementioned formulae are defined using spectral theory of self-adjoint operators (having a real spectrum) with compact or not compact resolvent so as to encompass bounded and unbounded domains. From (6), it follows that the Riccati operator P is factorized as

$$P = \Phi_Z p(\Lambda)\Phi_Z^{-1}, \tag{7}$$

where $\lambda \mapsto p(\lambda)$ is a continuous function, solution of the algebraic Riccati equation (ARE)

$$a^T(\lambda)p + pa(\lambda) - pb(\lambda)s^{-1}(\lambda)b^T(\lambda)p + c^T(\lambda)c(\lambda) = 0. \tag{8}$$

Our goal is reached once separate efficient semi-decentralized approximations of Φ_Z, $p(\Lambda)$ and Φ_Z^{-1} are provided for the realization of P through (7). This is generally not an issue for Φ_Z and for Φ_Z^{-1}, and then the point is the semi-decentralized approximation of $p(\Lambda)$. It might be build by a polynomial approximation,

$$p_N(\Lambda) = \sum_{k=0}^{N} d_k \Lambda^k, \tag{9}$$

or a rational approximation,

$$p_N(\Lambda) = \frac{\sum_{k=0}^{N^N} d_k \Lambda^k}{\sum_{k'=0}^{N^D} d'_{k'} \Lambda^{k'}}. \tag{10}$$

Then, for practical implementations, the operator Λ could be replaced by a discretizations Λ_h, with parameter h. We emphasize that the formula (9) or (10) yields large approximation errors, with respect to h, because of the high powers of Λ_h. To overcome this defect, we use an approximation

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based on the Cauchy integral, which requires to know the poles of p. In practice, we first approximate the function $\lambda \mapsto p(\lambda)$ by a polynomial approximation or a rational approximation $p_N(\lambda)$ with degrees N or (N^N, N^D) sufficiently high to insure a very small error. When p_N is known, so as its poles, so we can state the Cauchy formula for $p_N(\Lambda)$. This yields to introduce the equations of the complex function $v = v_1 + i\,v_2$ for each input $z \in Z$,

$$(\xi - \Lambda)v = -i\,\xi' p_N(\xi) z, \tag{11}$$

where $\xi : (0, 2\pi) \to \mathbb{C}$ is the contour of the Cauchy formula. Denoting by v^ℓ the solution corresponding to a quadrature point ξ_ℓ of the contour and ω_ℓ some quadrature weights, the final approximation of $p(\Lambda)z$ is

$$p_{N,M}(\Lambda)z = \frac{1}{2\pi} \sum_{\ell=1}^{M} \omega_\ell v_1^\ell. \tag{12}$$

Remark that the number M of quadrature points is the only important parameter governing the approximation error. For real-time computation, the expression of p_N is pre-computed, so the approximation cost is also governed by M only. With this method, we do not observe a lack of precision when Λ is replaced by its discretizations Λ_h and M is large. In the sequel, we show that the same derivation can be carried out directly for $Qz = -S^{-1}B^*Pz$ provided that the isomorphisms Φ_Z and Φ_U are also some functions of Λ.

This approach based on functional calculus is relatively simple, but in each case, it requires to determine the isomorphisms (5). The theory has already been applied in [4] to an LQR control problem with a bounded operator B that is not a function of Λ. It has been shown how the control approximation can be implemented through a distributed electronic circuit. In [5] and [3], it has also been applied to a one-dimensional array of cantilevers with regularly spaced actuators and sensors for which the operator C is not a function of Λ. The underlying model was derived with a multiscale method, an implementation of the semi-decentralized control was provided in the form of a periodic network of resistors, and the numerical validations of the complete strategy was carried out. In the present paper, we illustrate the theory with four simpler examples, namely a simple heat equation with internal bounded control and observation operators, a heat equation with an unbounded control operator, a vibrating Euler–Bernoulli beam, and a heat equation with a boundary controls.

We notice that our method together improves and generalizes a previous paper [6]. It was related to a specific application, namely vibration control problem for a plate with a periodic distribution of piezoelectric actuators and sensors. There, the general isomorphisms (5) and the general factorization (6) were not introduced, and $p(\Lambda)$ was approximated by a polynomial as in (9), which were severely limiting the accuracy of the approximation. In both papers, the control method is an LQR, but the theory is applicable to Riccati equations that may arise in a number of other control problems, for instance, for H_2 or H_∞ dynamic compensators. Other extensions are also possible; for instance, we may want to deal with functions of a non-self-adjoint operator Λ. In such a case, another functional calculus, such as those in [7] or in [8], could be used instead of the spectral theory. Other frameworks for control problems of infinite dimensional systems could also be used, for instance, that of [9] for optimal control with unbounded observations and unbounded controls.

Other techniques have already been established; see [10–14] and the references therein. But they are mostly focused on the infinite length systems; see [10–12, 14] for systems governed by partial differential equations and [13] for discrete systems. Other points of view exist as for instance in [15] for controlling large networks of autonomous vehicles. Finally, in [16], we developed another theoretical framework based on the *diffusive realization* applicable to a broad range of linear operators on bounded or unbounded domains. In principle, this approach allows to cover general distributed control problems with internal or boundary control. However, in this first paper in the subject, only one-dimensional domains and linear operational equations (e.g. Lyapunov equations) are covered.

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The paper is organized as follows. Notations and basic definitions are recalled in Section 2. In Section 3, the abstract approximation method is stated in the framework of bounded control and observation operators. The framework of unbounded control operators is treated in Section 4. Some extensions are outlined in Section 5. Most proofs are concentrated in Section 6. The illustrative examples are detailed in Section 7, and finally, the paper is concluded by Section 8.

2. PRELIMINARY RESULTS AND NOTATIONS

The norm and the inner product of a Hilbert space E are denoted by $||.||_E$ and $(.,.)_E$. For a second Hilbert space F, $\mathcal{L}(E,F)$ denotes the space of continuous linear operators from E to F. In addition, $\mathcal{L}(E,E)$ is denoted by $\mathcal{L}(E)$. One says that $\Phi \in \mathcal{L}(E,F)$ is an isomorphism from E to F if Φ is one-to-one and if its inverse is continuous.

Because the approximation method of P is based on the concept of matrices of functions of a self-adjoint operator, this section is devoted to their definition. Let Λ be a self-adjoint operator on a separable Hilbert space X with domain $D(\Lambda)$; we denote by $\sigma(\Lambda)$ its spectrum and by $I_\sigma = (\sigma_{\min}, \sigma_{\max}) \subset \mathbb{R}$ an open interval that includes $\sigma(\Lambda)$. We recall that if Λ is compact, then $\sigma(\Lambda)$ is bounded and is only composed of eigenvalues λ_k. They are the solutions to the eigenvalue problem $\Lambda\phi_k = \lambda_k\phi_k$, where ϕ_k is an eigenvector associated to λ_k chosen normed in X, such that $||\phi_k||_X = 1$. For a given real valued function f, continuous on I_σ, $f(\Lambda)$ is the linear self-adjoint operator on X defined by

$$f(\Lambda)z = \sum_{k=1}^{\infty} f(\lambda_k) z_k \phi_k \quad \text{where } z_k = (z, \phi_k)_X,$$

with domain $D(f(\Lambda)) = \left\{ z \in X \;\middle|\; \sum_{k=1}^{\infty} |f(\lambda_k) z_k \right|^2 < \infty \right\}$. Then, if f is a $n_1 \times n_2$ matrix of real valued functions f_{ij}, continuous on I_σ, $f(\Lambda)$ is a matrix of linear operators $f_{ij}(\Lambda)$ with domain

$$D(f(\Lambda)) = \left\{ z \in X^{n_2} \;\middle|\; \sum_{k=1}^{\infty} \sum_{j=1}^{n_2} |f_{ij}(\lambda_k)(z_j)_k \right|^2 < \infty \quad \forall i = 1 \dots n_1 \right\}.$$

In the general case, where Λ is not compact and where f is still a continuous function, the self-adjoint operator $f(\Lambda)$ is defined on X by the Stieltjes integral

$$f(\Lambda) = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda,$$

and its domain is $D(f(\Lambda)) = \left\{ z \in X \mid \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, d||E_\lambda z||_X^2 < \infty \right\}$ where E_λ is the spectral family associated to Λ; see [17]. When f is a matrix, $f(\Lambda)$ is a matrix of linear operators with entries defined by the aforementioned formula and with domain

$$D(f(\Lambda)) = \left\{ z \in X^{n_2} \mid \int_{-\infty}^{+\infty} \sum_{j=1}^{n_2} |f_{ij}(\lambda)|^2 \, d||E_\lambda z_j||_X^2 < \infty \quad \forall i = 1 \dots n_1 \right\}.$$

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3. BOUNDED CONTROL OPERATORS

In this section, we state the approximation result in the framework of bounded input operators. We follow the mathematical setting [1] of the LQR problem (1)–(3). So, A is the infinitesimal generator of a continuous semigroup on a separable Hilbert space Z with dense domain $D(A)$, $B \in \mathcal{L}(U, Z)$, $C \in \mathcal{L}(Z, Y)$, and $S \in \mathcal{L}(U, U)$, where U and Y are two Hilbert spaces. We assume that (A, B) is stabilizable and that (A, C) is detectable, in the sense that there exist $Q \in \mathcal{L}(Z, U)$ and $F \in \mathcal{L}(Y, Z)$ such that $A - BQ$ and $A - FC$ are the infinitesimal generators of two uniformly exponentially stable continuous semigroups. For each $z_0 \in Z$, the LQR problem (1)–(3) admits a unique solution $\mathbf{u} = -S^{-1}B^*Pz$, where $P \in \mathcal{L}(Z)$ is the unique self-adjoint nonnegative solution of the ORE

$$(A^*P + PA - PBS^{-1}B^*P + C^*C)z = 0 \tag{13}$$

for all $z \in D(A)$. The adjoint A^* of the unbounded operator A is defined from $D(A^*) \subset Z$ to Z by the equality $(A^*z, z')_Z = (z, Az')_Z$ for all $z \in D(A^*)$ and $z' \in D(A)$. The adjoint $B^* \in \mathcal{L}(Z, U)$ of the bounded operator B is defined by $(B^*z, u)_U = (z, Bu)_Z$, the adjoint $C^* \in \mathcal{L}(Y, Z)$ being defined similarly.

Now, we state specific assumptions for the approximation method. Here, Λ is a given self-adjoint operator on a separable Hilbert space X, which is chosen to be easily approximable on a semi-decentralized architecture. Generally, Λ is chosen with regard to A, and then Φ_Z and Φ_U can be chosen so as to have also a natural semi-decentralized approximation.

Assumption (H1)
There exist three integers n_Z, n_U, and $n_Y \in \mathbb{N}^*$; three isomorphisms $\Phi_Z \in \mathcal{L}(X^{n_Z}, Z)$, $\Phi_U \in \mathcal{L}(X^{n_U}, U)$, and $\Phi_Y \in \mathcal{L}(X^{n_Y}, Y)$; and four matrices of functions $a(\lambda) \in \mathbb{R}^{n_Z \times n_Z}$, $b(\lambda) \in \mathbb{R}^{n_Z \times n_U}$, $c(\lambda) \in \mathbb{R}^{n_Y \times n_Z}$, and $s(\lambda) \in \mathbb{R}^{n_U \times n_U}$ continuous on I_σ such that

$$A = \Phi_Z a(\Lambda)\Phi_Z^{-1}, \quad B = \Phi_Z b(\Lambda)\Phi_U^{-1}, \quad C = \Phi_Y c(\Lambda)\Phi_Z^{-1}, \quad \text{and} \quad S = \Phi_U s(\Lambda)\Phi_U^{-1}.$$

One of the consequences of this assumption, for a system governed by a partial differential equation posed in a domain Ω, is that both the control and the observation must be distributed throughout the domain, in conformity with what has been stated from the beginning.

Remark 3.1

1. In case all operators are functions of Λ, then the isomorphisms Φ are or not useful or can be chosen as a function of Λ. In both cases, P is also a function p of Λ.
2. Introducing the isomorphisms Φ_Z, Φ_Y, and Φ_U allows to deal with problems where operators A, B, and C are not functions of Λ.
3. When control is distributed over the entire domain, Assumption (H1) is generally satisfied. In Section 7.3, there is an example of observation operator C that is not a function of Λ, while in the paper [5], it is the case for B the control operator.
4. For boundary control or observation problems, it is impossible to find such isomorphisms. Nevertheless, in Section 7.4, we show how to proceed to address some boundary control problems.
5. Multiscale models with controls at the microscale, as in [5] and [3], are also possible applications.

We introduce the ARE

$$a^T(\lambda)p + pa(\lambda) - pb(\lambda)s^{-1}(\lambda)b^T(\lambda)p + c^T(\lambda)c(\lambda) = 0. \tag{14}$$

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Assumption (H2)
For all $\lambda \in I_\sigma$, the ARE (14) admits a unique nonnegative symmetric solution denoted by $p(\lambda)$.

Remark 3.2
This assumption is stronger than the typical sufficient condition for the mere existence of a solution to the Riccati equation [1].

We make the following choices for the inner products of Z, U, and Y:

$$(z,z')_Z = \left(\Phi_Z^{-1}z, \Phi_Z^{-1}z'\right)_{X^{n_Z}},\ (u,u')_U = (\Phi_U^{-1}u, \Phi_U^{-1}u')_{X^{n_U}}, \text{ and } (y,y')_Y = \left(\Phi_Y^{-1}y, \Phi_Y^{-1}y'\right)_{X^{n_Y}}.$$

Thus, P, Q and p, q are related as follows.

Theorem 3.3
If (H1) and (H2) are fulfilled, then

$$P = \Phi_Z\ p(\Lambda)\ \Phi_Z^{-1} \quad \text{and} \quad \mathbf{u} = -Qz,$$

where the controller Q admits the factorization $Q = \Phi_U q(\Lambda)\Phi_Z^{-1}$ with $q(\Lambda) = s^{-1}(\Lambda)b^T(\Lambda)p(\Lambda)$.

Now, we focus on a semi-decentralized approximation of Q, which reduces to provide such an approximation for $q(\Lambda)$. We restrict the presentation to the case of bounded operators Λ because they have a bounded spectra. This is sufficient for applications to systems governed by partial differential equations in bounded domains.

Assumption (H3)
The operator Λ is bounded, and its spectrum $\sigma(\Lambda)$ is bounded, so there exists $R > 0$ with $\sigma(\Lambda) \subset (-R, R)$.

This assumption can be relaxed; see Section 5.

Assumption (H4)
The operators Φ_Z, Φ_Z^{-1}, Λ, and $(\xi I - \Lambda)^{-1}$ admit semi-decentralized approximations for all $\xi \in \mathbb{C}$ with $|\xi| = R$.

Now, we introduce two successive approximations $q_N(\Lambda)$ and $q_{N,M}(\Lambda)$ of $q(\Lambda)$ that play a key role in our method.

▷ *The rational approximation* $q_N(\Lambda)$: Because the interval I_σ is bounded, each entry q_{ij} of the matrix q admits a rational approximation on I_σ. This defines a matrix of rational approximations of $q(\lambda)$,

$$q_N(\lambda) = \frac{\sum_{k=0}^{N^N} d_k \lambda^k}{\sum_{k'=0}^{N^D} d'_{k'} \lambda^{k'}}, \tag{15}$$

to be understood componentwise, so each d_k, $d'_{k'}$ is a matrix and $N = (N^N, N^D)$ is a pair of matrices of polynomial degrees. The particular case $N^D = 0$ corresponds to a classical polynomial approximation. For any $\eta > 0$, the degrees of approximations can be chosen so that the uniform estimate

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$$\sup_{\lambda \in I_\sigma} |q(\lambda) - q_N(\lambda)| \leqslant C_1(q)\eta \tag{16}$$

holds.

▷ *Approximation $q_{N,M}(\Lambda)$ by quadrature of the Cauchy integral*: For any complex valued function $g(\theta)$ continuous on $[0, 2\pi]$, we introduce $I_M(g) = \sum_{\ell=1}^{M} \omega_\ell g(\theta_\ell)$ a quadrature rule for the integral $I(g) = \int_0^{2\pi} g(\theta)\ d\theta$, $(\theta_\ell)_\ell$ denoting the nodes of a regular subdivision of $[0, 2\pi]$ and ω_ℓ the associated quadrature weights. The quadrature rule is assumed to satisfy an error estimate as

$$|I(g) - I_M(g)| \leqslant C_2(g)\eta. \tag{17}$$

$z \in X^{nz}$ and $\xi = \xi_1 + i\xi_2$, a sufficiently regular complex contour enlacing $\sigma(\Lambda)$ and not surrounding any pole of q_N. We parameterize it by a parameter varying in $[0, 2\pi]$. We further introduce the solution $(v_i)_{i=1,2}$ of the system

$$\begin{cases} \xi_1 v_1 - \xi_2 v_2 - \Lambda v_1 = \Re e\,(-i\xi' q_N(\xi))\, z, \\ \xi_2 v_1 + \xi_1 v_2 - \Lambda v_2 = \Im m\,(-i\xi' q_N(\xi))\, z, \end{cases} \tag{18}$$

and the second approximation of $q(\Lambda)$ through its realizations

$$q_{N,M}(\Lambda) z = \frac{1}{2\pi} \sum_{\ell=1}^{M} \omega_\ell v_1^\ell. \tag{19}$$

We notice that two approximations p_N and $p_{N,M}$ of the function p can be constructed by following the same steps. The next theorem states the approximations of the operators P and Q.

Theorem 3.4

Under the Assumptions (H1)–(H4), P and Q can be approximated by one of the two semi-decentralized approximations

$$P_N = \Phi_Z p_N(\Lambda)\Phi_Z^{-1} \quad \text{and} \quad Q_N = \Phi_U q_N(\Lambda)\Phi_Z^{-1}$$
$$\text{or } P_{N,M} = \Phi_Z p_{N,M}(\Lambda)\Phi_Z^{-1} \quad \text{and} \quad Q_{N,M} = \Phi_U q_{N,M}(\Lambda)\Phi_Z^{-1}.$$

Moreover, for any $\eta > 0$, there exist N and M such that

$$\|P - P_N\|_{\mathcal{L}(Z)} \leqslant C_3\eta, \|Q - Q_N\|_{\mathcal{L}(Z,U)} \leqslant C_3'\eta$$
$$\text{and } \|P - P_{N,M}\|_{\mathcal{L}(Z)} \leqslant C_4\eta, \|Q - Q_{N,M}\|_{\mathcal{L}(Z,U)} \leqslant C_4'\eta,$$

C_3, C_3' and C_4, C_4' being independent of η, N and M.

Remark 3.5

In the case of a polynomial approximation, that is, $N^D = 0$, we can set a circle as contour $\xi(\theta) = Re^{i\theta}$. For actual rational approximations, the contour must leave the poles outside, so we choose an ellipse centered at $\frac{R_1}{2}$ parameterized by $\xi(\theta) = \frac{R_1}{2}(1 + \cos(\theta)) + iR_2 \sin(\theta)$ where R_1 and R_2 are for the major and minor radii and R_2 is small enough.

Remark 3.6

The approximation of p used in [6] is based on Taylor series, so it is applicable only if the interval I_σ is sufficiently small. The approximation proposed in our paper does not suffer from this drawback.

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Remark 3.7
In case where the solution P of a Riccati equation is a kernel operator (see [18] for optimal control of systems governed by partial differential equations), that is, $Pz(x) = \int_\Omega \overline{p}(x,x')z(x')dx'$, and if Λ is a compact operator, then the kernel may be decomposed on the basis of eigenvectors of Λ,

$$\overline{p}(x,x') = \sum_{k=1}^{\infty} p(\lambda_k)\phi_k(x)\phi_k(x').$$

The truncation technique used in [10] can be applied to build a semi-decentralized approximation of P. However, when the decay of $\overline{p}$ is not very fast, this technique is not efficient; see, for example, the case $p(\lambda) = \lambda$ that may yield from an LQR problem.

For concrete real-time computations, one can use either of the two formulae (15) or (19) given that both are semi-decentralized, but we prefer the second because it does not make use of powers of λ. The reason will become clearer when discretizing. In a real-time computation, the realization $q_{N,M}(\Lambda)z$ requires solving M systems (18) corresponding to M complex values $\xi(\theta_\ell)$. So, the parameter M is essential to evaluate the cost of our algorithm. The matrix q_N is pre-computed off-line once and for all, and we choose N sufficiently large that q_N is a very good approximation of q. Consequently, M is the only parameter that influences the accuracy of the method, except the parameter space discretization that is discussed now.

The end of the section is devoted to spatial discretization. For the sake of simplicity, the interval is meshed with regularly spaced nodes separated by a distance h.

▷ *Spatial discretization with polynomial approximation*: First, we introduce Λ_h^k the finite differences discretizations of Λ^k, with $k = 1 \cdots N$. For $N^D = 0$, the discretization $q_{N,h}$ of q_N in (15) can be written as

$$q_{N,h}z_h = \sum_{k=0}^{N} d_k \Lambda_h^k z_h,$$

where z_h is the vector of nodal values of z. Their discretization yields very high errors because of the powers of Λ. This can be avoided by using the Cauchy formula, that is, Equation (18).

▷ *Spatial discretization with Cauchy formula approximation*: For each quadrature point $\xi := \xi_{1,\ell} + i\xi_{2,\ell}$, the discrete approximation $\left(v_{1,h}^\ell, v_{2,h}^\ell\right)$ of $\left(v_1^\ell, v_2^\ell\right)$ is the solution of the discrete set of equations

$$\begin{cases} \xi_{1,\ell} v_{1,h}^\ell - \xi_2 v_{2,h}^\ell - \Lambda_h v_{1,h} = \Re e(-i\xi_\ell q_N\left(\xi_{\ell,\ell}\right))\, z_h, \\ \xi_{2,\ell} v_{1,h}^\ell + \xi_{1,\ell} v_{2,h}^\ell - \Lambda_h v_{2,h} = \Im m(-i\xi_\ell q_N\left(\xi_\ell\right))\, z_h. \end{cases} \tag{20}$$

Thus, we deduce the discretization $q_{N,M,h}$ of the approximation $q_{N,M}$ in (19),

$$q_{N,M,h}z_h = \frac{1}{2\pi}\sum_{\ell=1}^{M} \omega_\ell v_{1,h}^\ell. \tag{21}$$

Under the Assumption (H4), we introduce $\Phi_{U,h}$ and $\Phi_{Z,h}$ the semi-decentralized approximations of Φ_U and Φ_Z. So, the approximations of $\mathbf{u}_N$ and $\mathbf{u}_{N,M}$ by a spatial discretization are

$$\mathbf{u}_{N,h} = -\Phi_{U,h} q_{N,h} \Phi_{Z,h}^{-1} z_h \quad \text{and} \quad \mathbf{u}_{N,M,h} = -\Phi_{U,h} q_{N,M,h} \Phi_{Z,h}^{-1} z_h. \tag{22}$$

This constitutes two different final semi-decentralized approximations of $\mathbf{u}$.

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Remark 3.8
The approximations $\mathbf{u}_{N,h}$ and $\mathbf{u}_{N,M,h}$ are given in the general case where the isomorphisms Φ_Z and Φ_U are not functions of Λ only. Therefore, we use our approximation technique to represent $q(\Lambda)$. In some cases, Φ_Z and Φ_U are function of Λ, and so is Q, and the approximation is developed directly on it, which we denote by $k(\Lambda)$,

$$\mathbf{u}_{N,h} = -k_{N,h}(\Lambda)z_h \quad \text{and} \quad \mathbf{u}_{N,M,h} = -k_{N,M,h}(\Lambda)z_h. \tag{23}$$

In the case where Φ_Z and $\Phi_U^{-1}Q$ are functions of Λ, then the approximation is developed on $\Phi_U^{-1}Q$; we will also denote it by $k(\Lambda)$ without risk of confusion,

$$\mathbf{u}_{N,h} = -\Phi_{U,h}k_{N,h}(\Lambda)z_h \quad \text{and} \quad \mathbf{u}_{N,M,h} = -\Phi_{U,h}k_{N,M,h}(\Lambda)z_h.$$

4. UNBOUNDED CONTROL OPERATORS

When the input operator B is unbounded from U to Z and the observation operator C is bounded from Z to Y, we use the framework of [2] where V is another Hilbert space, V' is its dual space with respect to the pivot space Z, $A \in \mathcal{L}(V, V')$, $B \in \mathcal{L}(U, V')$, and $C \in \mathcal{L}(Z, Y)$. A number of other technical assumptions are not detailed here. The state equations are written in the sense of V' with $z_0 \in Z$. The optimal control is $\mathbf{u} = -B^*Pz$ where P is the unique nonnegative solution of the ORE

$$(A^*P + PA - PBB^*P + C^*C)v = 0, \tag{24}$$

for all $v \in V$. The adjoint $A^* \in \mathcal{L}(V, V')$ is defined by $\langle A^*v, v'\rangle_{V',V} = \langle v, Av'\rangle_{V,V'}$ when $B^* \in \mathcal{L}(V', U)$ is defined as the adjoint of a bounded operator. We keep the same inner products for Z, U, and Y, and those of V and V' are

$$(v, v')_V = \left(\Phi_V^{-1}v, \Phi_V^{-1}v'\right)_{X^nZ} \quad \text{and} \quad \left(v, v'\right)_{V'} = \left(\Phi_{V'}^{-1}v, \Phi_{V'}^{-1}v'\right)_{X^nZ}.$$

Moreover, we choose $J = \Phi_V\Phi_{V'}^{-1}$ as the canonical isomorphism from V' to V, and the duality product between V and V' is

$$\left\langle v, v'\right\rangle_{V,V'} = (v, Jv')_V.$$

Assumption (H1')
The same statement as (H1) except that

$$A = \Phi_{V'}a(\Lambda)\Phi_V^{-1} \quad \text{and} \quad B = \Phi_{V'}b(\Lambda)\Phi_U^{-1}$$

where $\Phi_V \in \mathcal{L}(X^nZ, V)$ and $\Phi_{V'} \in \mathcal{L}(X^nZ, V')$ are two additional isomorphisms. Moreover,

$$\Phi_V = \phi_V(\Lambda), \quad \Phi_Z = \phi_Z(\Lambda), \quad \text{and} \quad \Phi_{V'} = \phi_{V'}(\Lambda)$$

are some functions of Λ.

Here, the ARE is

$$\begin{aligned}&\phi_{V'}(\lambda)a^T(\lambda)\ p\ \phi_{V'}^{-1}(\lambda)\phi_V(\lambda) + \phi_V(\lambda)\ p\ a(\lambda)\\ &- \phi_V(\lambda)\ p\ b(\lambda)s^{-1}(\lambda)\,b^T(\lambda)\ p\ \phi_{V'}^{-1}(\lambda)\phi_V(\lambda)\\ &+ \phi_Z(\lambda)c^T(\lambda)c(\lambda)\,\phi_Z^{-1}(\lambda)\phi_V(\lambda) = 0.\end{aligned} \tag{25}$$

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Assumption (H2')
For all $\lambda \in I_\sigma$, the ARE (25) admits a unique nonnegative solution denoted by $p(\lambda)$.

Theorem 4.1
If (H1', H2') are fulfilled, then

$$P = \Phi_V \; p(\Lambda) \; \Phi_{V'}^{-1} \quad \text{and} \quad \mathbf{u} = -Qz$$

where Q admits the factorization $Q = \Phi_U q(\Lambda)$ with $q(\Lambda) = b^T(\Lambda)\phi_{V'}^{-1}\phi_V\, p(\Lambda)\phi_{V'}^{-1}$.

The following assumptions are necessary for the semi-decentralized approximation of P.

Assumption (H4')
The same statement as (H4), completed by Φ_V, $\Phi_{V'}$, and $\Phi_{V'}^{-1}$ admit a semi-decentralized approximation.

In the next statement, the approximations q_N and $q_{N,M}$ of q are built according to the formulae (15) and (19).

Theorem 4.2
Under the Assumptions (H1', H2', H3, H4'), P and Q can be approximated by one of the two semi-decentralized approximations

$$P_N = \Phi_V\, p_N(\Lambda)\Phi_{V'}^{-1} \text{ and } Q_N = \Phi_U q_N(\Lambda),$$
$$\text{or } P_{N,M} = \Phi_V\, p_{N,M}(\Lambda)\Phi_{V'}^{-1} \text{ and } Q_{N,M} = \Phi_U q_{N,M}(\Lambda).$$

Moreover, for any $\eta > 0$, there exist N and M such that

$$\|P - P_N\|_{\mathcal{L}(V',V)} \leqslant C_3\eta, \quad \|Q - Q_N\|_{\mathcal{L}(V',U)} \leqslant C_3'\eta,$$
$$\text{and } \|P - P_{N,M}\|_{\mathcal{L}(V',V)} \leqslant C_4\eta, \quad \|Q - Q_{N,M}\|_{\mathcal{L}(V',U)} \leqslant C_4'\eta,$$

C_3, C_3', C_4, and C_4' being independent of η, N, and M.

Remark 4.3
An example of unbounded control operators is given in the Section 7.2.

The approximations of $\mathbf{u}$ and $\mathbf{u}_h$ are constructed using the same method as in the case of bounded control operators.

5. EXTENSIONS

In this section, we mention possible extensions of the theoretical framework presented earlier.

The same strategy applies directly to dynamic estimators and compensators derived by the H_2 to the H_∞ theories. For instance, the condition $\rho(P\overline{P}) < \gamma$ on the spectral radius of the product of the solution of the two Riccati equation can be expressed under the form of a condition on the spectral radius of the product of two parameterized matrices $\rho\,(p(\lambda)\overline{p}(\lambda)) < \gamma$ for all $\lambda \in I_\sigma$; see Lemma 6.2 (*6*).

The spectral theory of self-adjoint operators has been chosen for its relative simplicity. We are aware of its limitation, so we mention possible extensions based on more general functional calculi such as those developed in [7] or [8] to cite only two.

Other frameworks for the well-posedness of the LQR problem can be used. In particular, that of [9] for optimal control with unbounded observation and control may be incorporated in this approach.

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6. PROOFS

First, we remark that for E and F two Hilbert spaces and Φ an isomorphism from E to F, if F is equipped with the inner product $(z, z')_F = (\Phi^{-1}z, \Phi^{-1}z')_E$, then $\Phi^* = \Phi^{-1}$. In the next lemma, we state few functional calculus properties.

Lemma 6.1
For Λ a self-adjoint operator on a separable Hilbert space X, and for f, g two functions continuous on I_σ,

1. $f(\Lambda)$ is self-adjoint;
2. for $\mu \in \mathbb{R}$, $(\mu f)(\Lambda) = \mu f(\Lambda)$ on $D(f(\Lambda))$;
3. $(f+g)(\Lambda) = f(\Lambda) + g(\Lambda)$ on $D(f(\Lambda)) \cap D(g(\Lambda))$;
4. $g(\Lambda)f(\Lambda) = (g\ f)(\Lambda)$ when the range of $f(\Lambda)$ is included in $D(g(\Lambda))$;
5. if $f \neq 0$ in I_σ, then $f(\Lambda)^{-1}$ exists and is equal to $\frac{1}{f}(\Lambda)$;
6. if $f(\lambda) \geqslant 0$ for all $\lambda \in I_\sigma$, then $f(\Lambda) \geqslant 0$;
7. $||f(\Lambda)x||_X^2 \leqslant \sup_{\lambda \in I_\sigma} |f(\lambda)|^2 ||x||_X^2$ for all $x \in D(f(\Lambda))$.

Proof
The proofs of the first five statements can be found in [17]. We prove (*6*), that is, that $\sum_{i,j=1}^{n} (f_{ij}(\Lambda)z_j, z_i)_X \geqslant 0$. First, assume that I_σ is bounded. We recall that for a function g continuous on I_σ and for $z \in X$, the integral $\int_{\sigma_{\min}}^{\sigma_{\max}} g(\lambda) dE_\lambda z$ is defined as the strong limit in X of the Riemann sums, see [17], $\sum_{k=1}^{p} g(\lambda'_k)(E_{\lambda_{k+1}} - E_{\lambda_k})z$ when $\max_k |\lambda_{k+1} - \lambda_k|$ vanishes, where $\lambda'_k \in [\lambda_k, \lambda_{k+1}]$ and $\sigma_{\min} = \lambda_1 < \lambda_2 \ldots < \lambda_p = \sigma_{\max}$. When I_σ is not bounded, we use a subdivision of a bounded interval $I_{\widetilde{\sigma}} = (\widetilde{\sigma}_{\min}, \widetilde{\sigma}_{\max})$, and the integral $\int_{\sigma_{\min}}^{\sigma_{\max}} g(\lambda) dE_\lambda z$ is defined by passing to the limit in the integral bounds. Let us establish that the Riemann sum $\sum_{i,j=1}^{n} \sum_{k=1}^{p} f_{ij}(\lambda'_k)((E_{\lambda_{k+1}} z_j, z_i) - (E_{\lambda_k} z_j, z_i))$ is nonnegative, so the result will follow by passing to the limit. Because $(E_{\lambda_{k+1}} z_j, z_i) - (E_{\lambda_k} z_j, z_i) = ((E_{\lambda_{k+1}} - E_{\lambda_k})z_j, z_i) = \left(y_j^k, y_i^k\right)$ where $y_j^k = (E_{\lambda_{k+1}} - E_{\lambda_k})z_j$, then the Riemann sum is the sum over k of the nonnegative terms $\sum_{i,j=1}^{n} f_{ij}\left(\lambda'_k\right)\left(y_j^k, y_i^k\right)$, which in turn is nonnegative.

Now we prove (*7*):

$$||f(\Lambda))x||_X^2 = \int_{\sigma_{\min}}^{\sigma_{\max}} |f(\lambda)|^2 \, d||E_\lambda x||_X^2 \leqslant \sup_{\lambda \in I_\sigma} |f(\lambda)|^2 \int_{\sigma_{\min}}^{\sigma_{\max}} d||E_\lambda x||_X^2$$
$$\leqslant \sup_{\lambda \in I_\sigma} |f(\lambda)|^2 ||x||_X^2.$$

□

For two integers n_E, n_F, an $n_E \times n_F$ matrix f of functions continuous on I_σ and two Hilbert spaces E, F isomorphic with X^{n_E} and X^{n_F} by Φ_E^{-1} and Φ_F^{-1}, respectively, we introduce the so-called generalized matrix of functions of Λ: $f^\phi(\Lambda) = \Phi_E f(\Lambda)\Phi_F^{-1} \in \mathcal{L}(F, E)$ with domain $D(f^\phi(\Lambda)) = \Phi_F D(f(\Lambda))$. For the sake of shortness, the spaces E and F do not appear explicitly in the notation f^ϕ, so they will be associated to each matrix at the beginning of their use. Then, no confusion will be possible. In the next lemma, we state some properties of generalized matrices of functions.

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Lemma 6.2
For any generalized matrices of functions of Λ, $f^\phi(\Lambda) = \Phi_E f(\Lambda)\Phi_F^{-1}$ and $g^\phi(\Lambda) = \Phi_E g(\Lambda)\Phi_F^{-1}$, and any real number μ,

1. $(f^\phi(\Lambda))^* = (f^T)^\phi(\Lambda)$;
2. $\mu f^\phi(\Lambda) = (\mu f)^\phi(\Lambda)$ on $D(f^\phi(\Lambda))$;
3. $f^\phi(\Lambda) + g^\phi(\Lambda) = (f+g)^\phi(\Lambda)$ on $D(f^\phi(\Lambda)) \cap D(g^\phi(\Lambda))$;
4. for another Hilbert space G and $g^\phi(\Lambda) = \Phi_F g(\Lambda)\Phi_G^{-1}$, $f^\phi(\Lambda)g^\phi(\Lambda) = (fg)^\phi(\Lambda) = \Phi_E(fg)(\Lambda)\Phi_G^{-1}$ when the range $R(f^\phi(\Lambda)) \subset D(g^\phi(\Lambda))$;
5. when $F = E$, if $f(\lambda) \geqslant 0$ for all $\lambda \in I_\sigma$, then $f^\phi(\Lambda) \geqslant 0$;
6. $\sigma(f^\phi(\Lambda)) = \sigma(f)$.

Proof
The properties (*1*)–(*4*) are direct consequences of Lemma 6.1. For the derivation of (*5*), we remark that for $z \in D(f^\phi(\Lambda)) \subset E$, $(f^\phi(\Lambda)z, z)_E = \left(f(\Lambda)\Phi_E^{-1}z, \Phi_E^{-1}z\right)_{X^{n_E}}$, which is nonnegative if $f(\Lambda)$ is nonnegative. The conclusion uses Lemma 6.1 (*5*). Finally, the derivation of (*6*) is a direct consequence of the definition of the spectrum of an operator. □

Proof of Theorem 3.3
From Lemma 6.2 (*1*) and (*4*),

$$A^* = \Phi_Z a^T(\Lambda)\Phi_Z^{-1}, \quad BB^* = \Phi_Z b(\Lambda)b^*(\Lambda)\Phi_Z^{-1}, \quad \text{and} \quad C^*C = \Phi_Z c^*(\Lambda)c(\Lambda)\Phi_Z^{-1}$$

are some generalized matrices of functions of Λ on Z. We write

$$e(\lambda) = a^T(\lambda)\ p(\lambda) + p(\lambda)\ a(\lambda) - p(\lambda)\ b(\lambda)b^T(\lambda)\ p(\lambda) + c^T(\lambda)c(\lambda),$$

so by construction, $e(\lambda) = 0$ and $e(\Lambda) = 0$. Multiplying the last equality by Φ_Z to the left and by Φ_Z^{-1} to the right, using Lemma 6.2 (*3*) and (*4*), and posing $\widetilde{P} = \Phi_Z\ p(\Lambda)\ \Phi_Z^{-1}$, we find that $\widetilde{P}$ satisfies the Riccati equation (13). Next, the nonnegativity and symmetry of p with Lemma 6.2 (*1*) and (*5*) yield the nonnegativity and self-adjointness of $\widetilde{P}$. Finally, we conclude that $P = \widetilde{P}$ thanks to the uniqueness of the solution, so $\mathbf{u} = -Qz$, where $Q = S^{-1}B^*\widetilde{P} = \Phi_U s^{-1}(\Lambda)b^T(\Lambda)p(\Lambda)\Phi_Z^{-1}$. □

Proof of Theorem 3.4
The estimate $||q(\Lambda) - q_N(\Lambda)||_{\mathcal{L}(X^{n_Z}, X^{n_U})}$ results from (16) and Lemma 6.1 (*7*). In the following, we derive the estimate

$$||q_N(\Lambda) - q_{N,M}(\Lambda)||_{\mathcal{L}(X^{n_Z}, X^{n_U})} \leqslant C_5\eta.$$

Because q_N is holomorphic in $\mathbb{C}$, and Λ is a bounded operator on X with a spectrum included in $(-R, R)$, $p_N(\Lambda)$ may be represented by the Cauchy formula, see [19],

$$q_N(\Lambda) = \frac{1}{2i\pi}\int_{\mathcal{C}(R)} q_N(\xi)(\xi I - \Lambda)^{-1}d\xi$$

where $\mathcal{C}(R) \subset \mathbb{C}$, provided that all its poles are out of the contour $\mathcal{C}(R)$. By choosing ξ, function of θ, with $\theta \in (0, 2\pi)$ as a parametrization of $\mathcal{C}(R)$, we find

$$q_N(\Lambda) = \frac{1}{2\pi}\int_0^{2\pi} -i\xi' q_N(\xi)(\xi I - \Lambda)^{-1}d\theta.$$

Then, we use the quadrature formula to approximate $q_N(\lambda)$ by

$$q_{N,M}(\lambda) = \frac{1}{2\pi} I_M(-i\xi' q_N(\xi)(\xi - \lambda)^{-1}).$$

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Combining the estimate (17) and Lemma 6.1 (*5*) yields the wanted estimate. The triangular inequality yields

$$\begin{aligned}\|q(\Lambda)-q_{N,M}(\Lambda)\|_{\mathcal{L}(X^{n_Z},X^{n_U})} &\leqslant \|q(\Lambda)-q_N(\Lambda)\|_{\mathcal{L}(X^{n_Z},X^{n_U})}+\|q_N(\Lambda)-q_{N,M}(\Lambda)\|_{\mathcal{L}(X^{n_Z},X^{n_U})}\\ &\leqslant (C_3+C_5)\eta = C_4\eta\end{aligned}$$

with $C_4 = C_3 + C_5$. Consequently, the expression (19) of $q_{N,M}(\Lambda)z$ is obtained by posing $v^\ell = -i\,\xi'_\ell q_N(\xi_\ell)(\xi_\ell - \Lambda)^{-1}z$. □

Remark 6.3
The implementation of the Cauchy integral formula requires that the function is holomorphic inside the contour. In the case of an unknown function such as the function q, it is generally difficult to determine its domain of holomorphy, so it is easier to use a rational approximation q_N whose poles are under control.

Proof of Theorem 4.1
The derivation of the expression $A^* = J^{-1}\Phi_V a^*(\Lambda)\Phi_{V'}^{-1}J^{-1}$ is straightforward provided that $\langle u,v\rangle_{V',V} = (Ju,v)_V = (u,J^{-1}v)_{V'}$. Because $J = \Phi_V\Phi_{V'}^{-1}$, this expression is simplified as $A^* = \Phi_{V'}a^*\Phi_V^{-1}$. Then, (24) is equivalent to

$$\begin{aligned}&\big[\phi_{V'}(\Lambda)a^*(\Lambda)P\phi_{V'}^{-1}(\Lambda)\phi_V(\Lambda)+\phi_V(\Lambda)Pa(\Lambda)\\ &-\phi_V(\Lambda)Pb(\Lambda)b^*(\Lambda)P\phi_{V'}^{-1}(\Lambda)\phi_V(\Lambda)\\ &+\phi_Z c^*(\Lambda)c(\Lambda)\phi_Z^{-1}\phi_V\big]x = 0.\end{aligned}$$

Finally, the complete proof follows the same steps as in Theorem 3.3. □

The proof of Theorem 4.2 is similar to the one of Theorem 3.4.

7. APPLICATIONS AND NUMERICAL RESULTS

We present four applications to illustrate different aspects of the theory. In Examples 1, 3, and 4, the input operator B is bounded when in Example 2, it is not. Then, we consider cases where the operators B and C are functions of Λ (Examples 1, 2, and 4) and a case where it is not (Example 3). Most examples are devoted to internal control, nevertheless through the example of Section 7.4, it is shown how to tackle a boundary control problem. In almost all cases, efficient algorithms are described. The presentation of Examples 1, 3, and 4 follows the same plan with three subsections. The first one includes the state equation, the functional to be minimized, and some semi-decentralized controls resulting of our approach. Their derivation is detailed in the second subsection. As for the third, it discusses numerical results.

The functional analysis is carried out in Sobolev spaces defined for any integer $k \in \mathbb{N}^*$ and any domain $\Omega \subset \mathbb{R}^d$ by

$$\begin{aligned}H^k(\Omega) &= \{v \in L^2(\Omega) \quad | \quad \nabla^j v \in L^2(\Omega)^{d^j} \quad \text{for all} \quad 1 \leqslant j \leqslant k\}\\ \text{and } H_0^k(\Omega) &= \{v \in H^k(\Omega) \quad | \quad \nabla^j v = 0 \text{ on } \partial\Omega \quad \text{for all} \quad 0 \leqslant j \leqslant k-1\}.\end{aligned}$$

The boundary $\partial\Omega$ of Ω is always assumed to be sufficiently regular to avoid any singularity and thus to simplify the choice of the isomorphisms Φ. Its outward unit normal is denoted by ν. For $N \in \mathbb{N}$, $\mathbb{P}_N$ represents the set of N^{th}-order polynomials.

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7.1. Example 1: The heat equation with a bounded control operator

In this example, observation and control operators are bounded.

7.1.1. The state equation and a choice of semi-decentralized controllers. Consider a system modeled by the heat equation posed in a domain $\Omega \subset \mathbb{R}^d$, with homogeneous Dirichlet boundary conditions. The control is distributed over the whole domain, so the state $z := w$ is solution to the boundary value problem,

$$\begin{cases} \partial_t w(t,x) = \Delta w(t,x) + \beta u(t,x) & \text{in } \mathbb{R}^{+*} \times \Omega, \\ w(t,x) = 0 & \text{on } \mathbb{R}^{+*} \times \partial\Omega, \\ w(0,x) = w_0 & \text{in } \Omega, \end{cases} \tag{26}$$

and the functional $\mathcal{J}(w_0;u) = \int_0^{+\infty} \|w\|^2_{L^2(\Omega)} + \|\gamma u\|^2_{L^2(\Omega)}\, dt$ is to be minimized. Here, β and γ are two nonnegative continuous functions in $\bar{\Omega}$. We apply the theory with the self-adjoint operator $\Lambda = (-\Delta)^{-1}$, defined as the inverse of the Laplace operator $-\Delta : H^1_0(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$.

▷ *Linear approximation*: The approximation (9) with a first-degree polynomial yields

$$\mathbf{u}_1 = -\frac{\beta}{\sqrt{\gamma}}(d_0 + d_1\Lambda)w,$$

so in the special case $\gamma = \beta = 1$, $\mathbf{u}_1$ is the solution to the boundary value problem

$$-\Delta \mathbf{u}_1 = d_0 \Delta w - d_1 w \text{ in } \Omega, \quad \text{with } \mathbf{u}_1 = w = 0 \text{ on } \partial\Omega.$$

In the one-dimensional case $\Omega =]0,\pi[$, we apply Algorithm 1 described hereafter to find $d_0 = 2.23 \times 10^{-2}$ and $d_1 = 0.407$. Such $\mathbf{u}_1$ constitutes a semi-decentralized control before spatial discretization. The Laplace operator, that is, the second-order derivative, may be approximated by a three-point centered finite difference scheme, with solution $(\mathbf{u}_{1,j})_{j=0,\dots,\mathcal{N}}$ that approximates the solutions $\mathbf{u}_1(x_j)$ at the $(\mathcal{N}+1)$ nodes of a subdivision $(x_j = jh)_{j=0,\dots,\mathcal{N}}$ with $h = \frac{\pi}{\mathcal{N}}$,

$$-(\mathbf{u}_{1,j-1} - 2\mathbf{u}_{1,j} + \mathbf{u}_{1,j+1}) = d_0\left(w_{j-1} - 2w_j + w_{j+1}\right) - d_1 h^2 w_j, \quad j = 1,\dots,\mathcal{N}-1,$$

completed by the boundary conditions $\mathbf{u}_{1,0} = \mathbf{u}_{1,\mathcal{N}} = 0$. Here, $w_j = w(x_j)$ for $j = 0,\dots,\mathcal{N}$ that satisfy $w_0 = w_{\mathcal{N}} = 0$. After elimination of $\mathbf{u}_{1,0}$ and $\mathbf{u}_{1,\mathcal{N}}$, the scheme can be written in matrix form,

$$[-\Delta_h]\mathbf{u}_{1,h} = -d_0[-\Delta_h]w_h - d_1 w_h, \tag{27}$$

where $[-\Delta_h] = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$, $\mathbf{u}_{1,h} = \begin{bmatrix} \mathbf{u}_{1,1} \\ \vdots \\ \mathbf{u}_{1,\mathcal{N}-1} \end{bmatrix}$ and $w_h = \begin{bmatrix} w_1 \\ \vdots \\ w_{\mathcal{N}-1} \end{bmatrix}$. This is the fully discretized problem of the semi-decentralized control approximated by a linear polynomial.

▷ *Approximation through the Cauchy formula combined with a polynomial approximation*: To build the approximated optimal control,

$$\mathbf{u}_{N,M,h} = -\frac{1}{2\pi}\sum_{\ell=1}^{M} \omega_\ell v^\ell_{1,h}, \tag{28}$$

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the approximation $v_{i,h}^{\ell}$ of $\left[\, v_i^{\ell}(x_1) \;\ldots\; v_i^{\ell}(x_{\mathcal{N}-1}) \,\right]^T$ is computed by solving the system (20), which we rewrite in the matrix form,

$$\begin{bmatrix} \xi_1 - [-\Delta_h]^{-1} & -\xi_2 \\ \xi_2 & \xi_1 - [-\Delta_h]^{-1} \end{bmatrix} \begin{bmatrix} v_{1,h}^{\ell} \\ v_{2,h}^{\ell} \end{bmatrix} = \begin{bmatrix} \Re e(-i\xi' p_N(\xi)) w_h \\ \Im m(-i\xi' p_N(\xi)) w_h \end{bmatrix} \tag{29}$$

for each quadrature point $\xi := \xi_{1,\ell} + i\xi_{2,\ell}$, where $p_N(\lambda)$ is a polynomial approximation of $p(\lambda)$.

7.1.2. Construction of the semi-decentralized controllers. We detail the derivation of the polynomial approximation $p_N(\Lambda)w$ of Pw required both for the linear approximation and in (29). We set $U = Z = L^2(\Omega)$; thus, $A = \Delta$ is an isomorphism from its domain $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ into Z; see [20]. Furthermore, $Y = U = Z$ and $B = C = S = I$. We set $X = Z$, $\Lambda = [-\Delta]^{-1}$, which is compact, so it has a bounded positive spectrum with an accumulation point at zero (but $0 \notin \sigma(\Lambda)$); see [21]. Thus, we can choose $\Phi_Z = \Phi_Y = \Phi_U = I$. Moreover, when $\beta = \gamma = 1$, the coefficients $a(\lambda) = -\frac{1}{\lambda}$, $b = c = s = 1$ are continuous on $I_\sigma = (0, \sigma_{\max}]$, and the ARE reads

$$p^2(\lambda) + \frac{2}{\lambda} p(\lambda) - 1 = 0. \tag{30}$$

Its exact nonnegative solution, established only to the calculations of errors, is

$$p(\lambda) = \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda}. \tag{31}$$

We observe that $p(\lambda)$ is sufficiently regular to be accurately approximated by polynomials in I_σ. The ARE (30) is equivalent to the weak formulation

$$\int_{I_\sigma} (\lambda p^2 + 2p - \lambda)\eta(\lambda)\, d\lambda = 0 \text{ for all } \eta \in \mathcal{C}^0(I_\sigma) \tag{32}$$

to which we apply the spectral method with Legendre polynomials (see [22] for instance) to find the equation satisfied by the polynomial approximation p_N. The computation of the integral is carried out exactly by using the Legendre–Gauss–Lobatto quadrature formula analyzed in [23–25]. The resolution of the nonlinear problem is achieved by the iterative semi-implicit scheme described in the following text, where ε is the stop criteria.

Algorithm 1 Semi-implicit scheme applied to (32)

1: p_N^0 given
2: $(m+1)^{\text{th}}$ step : knowing $p_N^m \in \mathbb{P}_N$, find $p_N^{m+1} \in \mathbb{P}_N$ such that

$$\int_{I_\sigma} p_N^{m+1}(\lambda) \left(\lambda p_N^m(\lambda) + 2\right) \eta(\lambda) d\lambda = \int_{I_\sigma} \lambda \eta(\lambda) d\lambda, \quad \forall \eta \in \mathbb{P}_N.$$

3: If $\| p_N^{m+1} - p_N^m \|_{L^2(I_\sigma)} \leqslant \varepsilon$, then terminate the algorithm else return to step 2.

7.1.3. Numerical results. We analyze separately the three sources of discretization error: the error of approximation of p by a polynomial p_N, the error in the quadrature of the Cauchy formula, and the spatial discretization error. We also discuss the convergence of Algorithm 1.

▷ *Polynomial approximation*: The difference between successive iterations $\| p_N^{m+1} - p_N^m \|_{L^2(I_\sigma)}$ of Algorithm 1 decays exponentially. For $N = 10$ and for the initial solution $p_N^0 = 0$, the exponential decay rate is equal to -1.80. Let us denote by p_N the polynomial obtained after convergence of p_N^m by Algorithm 1; the convergence error $\| p_N^m - p_N \|_{L^2(I_\sigma)}$ is also exponentially

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decaying with an exponential decay rate of -1.82. In addition, as it is usual for spectral methods, the relative error

$$e = \frac{\|p - p_N\|_{L^2(I_\sigma)}}{\|p\|_{L^2(I_\sigma)}}$$

of the polynomial approximation decreases exponentially with N. Here, the exponential decay rate is -1.61.

▷ *Approximation through the Cauchy formula combined with a polynomial approximation*: Because of the absence of poles in p_N, the choice of the contour of the Cauchy formula is free of constraints as long as it surrounds I_σ. We have chosen a circle parameterized by $\xi(\theta) = Re^{i\theta}$, with $\theta \in [0, 2\pi]$. Then, we have set the polynomial degree sufficiently large so that the error e can be neglected. The numerical integrations have been performed with a standard trapezoidal quadrature rule. Figure 1 represents the relative error

$$E = \frac{\|p - p_{N,M}\|_{L^2(I_\sigma)}}{\|p\|_{L^2(I_\sigma)}}$$

between p and $p_{N,M}$ for various values of the radius R. It converges exponentially with respect to M toward e, and the exponential decay rate is a decreasing function of R.

▷ *Spatial discretization*: Computations have been carried out for $\mathbf{u}_{N,h}$ defined in (27) with $N = 1$ and for $\mathbf{u}_{N,M,h}$ defined in (28) with $R = 5$, $N = 10$, and $M = 11$ so that e is in the range of 10^{-9} and is negligible compared to E. The approximation (28) is obtained from the formula (22) by substituting Φ_U and Φ_Z by the identity operator and by using the centered finite difference scheme of the second-order derivative, that is, by replacing Λ by its discretization Λ_h. The spatial discretizations are compared to the expression of the approximations $\mathbf{u}_N(t,x)$ and $\mathbf{u}_{N,M}(t,x)$ that we calculate thanks to the modal decomposition of the operator ∂^2_{xx} with homogeneous Dirichlet boundary conditions. It comes

$$\mathbf{u}_N(t,x) = -\sum_{i\in\mathbb{N}^*} w_i e^{-(\lambda_i^{-1} + p_N(\lambda_i))t}\, p_N(\lambda_i)\, \phi_i(x),$$

where λ_i, ϕ_i, and w_i represent respectively the i^{th} eigenvalue, the i^{th} eigenvector, and the i^{th} modal coefficient of the initial condition. The same expression holds for $\mathbf{u}_{N,M}(t,x)$ after replacement of p_N by $p_{N,M}$. Then, the errors,

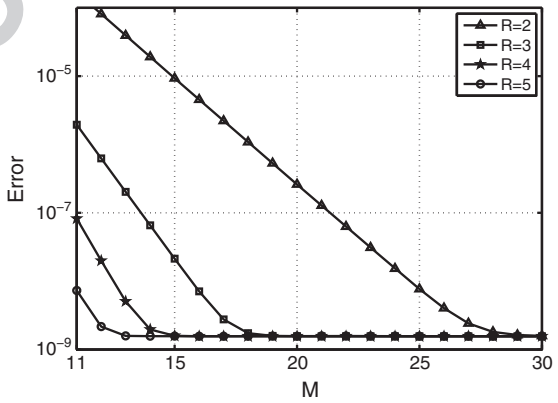

Figure 1. Error E in logarithmic scale as a function of M for different values of R and for $N = 10$.

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$$\frac{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,i} - \mathbf{u}_N(x_i)|^2\right)^{\frac{1}{2}} dt}{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,i}|^2\right)^{\frac{1}{2}} dt} \quad \text{and} \quad \frac{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,M,i} - \mathbf{u}_{N,M}(x_i)|^2\right)^{\frac{1}{2}} dt}{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,M}(x_i)|^2\right)^{\frac{1}{2}} dt},$$

are known to be theoretically quadratic with respect to h the spatial discretization step, which is confirmed by our experiments.

7.2. Example 2: Heat equation with unbounded control operator

In this example, the control operator is internal and unbounded, and the observation operator is internal and bounded. We apply the theory of Section 4 without going into much detail as for other examples. We only describe the state equation and the functional analysis framework.

7.2.1. The state equation. We keep the heat equation as the state equation with the same control space $U \subset L^2(\Omega)$ and the same functional $\mathcal{J}$, but the control operator is replaced by an unbounded one defined in the distribution sense by $\langle Bu, v\rangle = -\beta \int_\Omega u\beta_1.\nabla v dx$, where β_1 is a vector of $\mathbb{R}^d$.

7.2.2. The functional framework. First, we pose $V = H_0^1(\Omega)$, so $A = \Delta$ is an isomorphism from V into V' from which we define $J = (-A)^{-1}$. It allows to give a precise definition of B: for all $v \in V$, $\langle Bu, v\rangle_{V',V} = -\int_\Omega u\beta_1.\nabla v\, dx$ for $u \in U$. Let us compute B^* defined by $(Bu, v)_{V'} = (u, B^*v)_{L^2(\Omega)}$ for $u \in U$ and $v \in V'$. Because $(Bu, v)_{V'} = \langle Bu, Jv\rangle_{V',V} = -(u, \beta_1.\nabla Jv)_{L^2(\Omega)}$, then $B^*v = -\beta_1\nabla Jv$. We introduce the kernel of B, $K_B = \{u \in L^2(\Omega) \,|\, u$ constant in the direction $\beta_1\}$, $U = L^2(\Omega)/K_B$ and the kernel of B^*, $K_{B^*} = \{v \in V' \mid Jv$ constant in the direction $\beta_1\}$. Because $Jv = 0$ on the boundary $\partial\Omega$, then $K_{B^*} = \{0\}$. Then by using classical arguments, for example, [26], one deduces that B is an isomorphism from U into V'. We pose also $Y = Z = L^2(\Omega)$, $S = C = I \in \mathcal{L}(Z, Y)$. Now, we introduce $X = V'$ and $\Lambda = J$, which is a nonnegative operator. The fact that Λ is self-adjoint, that is, that $(\Lambda v, v')_{V'} = (v, \Lambda v')_{V'}$, comes from the equality $\langle \Lambda v, Jv'\rangle_{V',V} = \langle Jv, \Lambda v'\rangle_{V,V'}$. To complete the construction, we pose $\Phi_U = (\beta_1.\nabla)^{-1}$, $\Phi_V = \Lambda$, $\Phi_{V'} = I$, $\Phi_Y = \Phi_Z = \Lambda^{1/2}$, which is an isomorphism from V' into $L^2(\Omega)$, $a(\Lambda) = I$, $b(\Lambda) = I$, and $c(\Lambda) = I$. Finally, we proceed as in the first example for the computation of $p_{N,M}$.

7.3. Example 3: Beam or plate model

Here, we deal with a second-order problem in time with distributed internal bounded observation and control.

7.3.1. The state equation and a choice of semi-decentralized controllers. The model under consideration is a fourth-order equation posed in a domain $\Omega \subset \mathbb{R}^d$, which may correspond to a Euler–Bernoulli clamped beam equation when $d = 1$ or to a Love–Kirchhoff clamped plate equation when $d = 2$. The control is still distributed over the whole domain, and the state is $z := [w \quad \partial_t w]^T$, where w is solution to the boundary value problem

$$\partial_{tt}^2 w = -\Delta^2 w + \beta u \text{ in } \mathbb{R}^{+*} \times \Omega, \tag{33}$$

$$w = \nabla w.\nu = 0 \text{ on } \mathbb{R}^{+*} \times \partial\Omega, \tag{34}$$

$$w = w_0 \text{ and } \partial_t w = w_1 \text{ in } \Omega \text{ at } t = 0, \tag{35}$$

for a given function β and given initial conditions w_0 and w_1 all defined in Ω. Choosing the cost functional $\mathcal{J}(w_0, w_1; u) = \int_0^{+\infty} \|\Delta w\|^2_{L^2(\Omega)} + \|\gamma u\|^2_{L^2(\Omega)}\, dt$, we pose $\Lambda = \left(\Delta^2\right)^{-1}$, defined as

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the inverse of the biharmonic operator $\Delta^2 : H_0^2(\Omega) \cap H^4(\Omega) \to L^2(\Omega)$. The method can handle the general case; however, in the special case $\gamma = \beta = 1$, we show in the following section that the optimal control $\mathbf{u}$ may be approached by

$$\mathbf{u}_{N,M,h} = -\frac{1}{2\pi} \sum_{\ell=1}^{M} \omega_\ell \left(\mathbf{v}_{1,h}^\ell + \overline{\mathbf{v}}_{1,h}^\ell \right), \tag{36}$$

where $\left(\mathbf{v}_{1,h}^\ell, \overline{\mathbf{v}}_{1,h}^\ell \right)$ are solution to

$$\begin{aligned} \begin{bmatrix} \xi_1 - \Lambda_h & -\xi_2 \\ \xi_2 & \xi_1 - \Lambda_h \end{bmatrix} \begin{bmatrix} \mathbf{v}_{1,h}^\ell \\ \mathbf{v}_{2,h}^\ell \end{bmatrix} &= \begin{bmatrix} \Re e\left(-i\xi' k_{1,N}(\xi)\right) w_h \\ \Im m\left(-i\xi' k_{1,N}(\xi)\right) w_h \end{bmatrix}, \\ \begin{bmatrix} \xi_1 - \Lambda_h & -\xi_2 \\ \xi_2 & \xi_1 - \Lambda_h \end{bmatrix} \begin{bmatrix} \overline{\mathbf{v}}_{1,h}^\ell \\ \overline{\mathbf{v}}_{2,h}^\ell \end{bmatrix} &= \begin{bmatrix} \Re e\left(-i\xi' k_{2,N}(\xi)\right) \partial_t w_h \\ \Im m(-i\xi' k_{2,N}(\xi)) \partial_t w_h \end{bmatrix}, \end{aligned} \tag{37}$$

for each quadrature point $\xi_\ell := \xi_{1,\ell} + i\xi_{2,\ell}$, and

$$\Lambda_h^{-1} = \frac{1}{h^4} \begin{bmatrix} 2h^3 & -\frac{1}{2}h^3 & & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & \ddots & 6 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & 6 & \ddots & 1 & \\ & & & 1 & -4 & 6 & -4 & \\ & & & & & -\frac{1}{2}h^3 & 2h^3 & \end{bmatrix}, \tag{38}$$

the vectors $\mathbf{v}_{i,h}^T$, $\overline{\mathbf{v}}_{i,h}^T$, w_h^T, $\partial_t w_h^T$ being the approximations of $[\mathbf{v}_i(x_1) \quad \dots \quad \mathbf{v}_i(x_{\mathcal{N}-1})]$, $[\overline{\mathbf{v}}_i(x_1) \quad \dots \quad \overline{\mathbf{v}}_i(x_{\mathcal{N}-1})]$, $[w(x_1) \quad \dots \quad w(x_{\mathcal{N}-1})]$, $[\partial_t w(x_1) \quad \dots \quad \partial_t w(x_{\mathcal{N}-1})]$ and $k_{i,N}$ being defined in the following section.

7.3.2. Construction and study of the semi-decentralized controllers. First, the plate equation must be formulated under the form of a first-order system. We set $z^T = \begin{bmatrix} w & \partial_t w \end{bmatrix}$, so we find that $A = \begin{bmatrix} 0 & I \\ -\Delta^2 & 0 \end{bmatrix}$, the operators $B^T = \begin{bmatrix} 0 & I \end{bmatrix}$, $C = \begin{bmatrix} \Delta & 0 \end{bmatrix}$, $S = I$ and the functional spaces $U = L^2(\Omega)$, $Y \subset L^2(\Omega)$. The usual state space is $Z = H_0^2(\Omega) \times L^2(\Omega)$; thus, B and C are bounded. We pose $X = L^2(\Omega)$, $\Lambda = (\Delta^2)^{-1}$ an isomorphism from X into $H^4(\Omega) \cap H_0^2(\Omega)$, $\Phi_Z = \begin{bmatrix} \Lambda^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}$, $\Phi_U = I$ and $\Phi_Y = \Delta\Lambda^{1/2}$, so $Y = \Delta\Lambda^{\frac{1}{2}} L^2(\Omega) = \Delta H_0^2(\Omega)$ and $a(\lambda) = \begin{bmatrix} 0 & \lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{bmatrix}$, $b^T(\lambda) = [0 \quad 1]$, $c(\lambda) = [1 \quad 0]$ and $s(\lambda) = 1$.

Remark 7.1

1. We indicate how isomorphisms Φ_Y and Φ_Z have been chosen. The choice of Φ_Z directly comes from the expression of the inner product $(z, z')_Z = \left(\Phi_Z^{-1} z, \Phi_Z^{-1} z'\right)_{X^2}$ and from $\left(z_1, z_1'\right)_{H_0^2(\Omega)} = \left((\Delta^2)^{\frac{1}{2}} z_1, (\Delta^2)^{\frac{1}{2}} z_1'\right)_{L^2(\Omega)}$. For Φ_Y, we start from $C = \Phi_Y c(\Lambda) \Phi_Z^{-1}$ and from the relation $(y, y')_Y = \left(\Phi_Y^{-1} y, \Phi_Y^{-1} y'\right)_X$, which imply that $\Delta = \Phi_Y c_1 \Lambda^{-\frac{1}{2}}$. The expression of Φ_Y follows.
2. The isomorphisms Φ_Z and Φ_U are some matrices of functions of Λ, and so is Q. Thus, the approximation is directly developed on $Q = k(\Lambda)$.

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The controller Q is a 1×2 matrix of operators $k = [k_1 \quad k_2]$, with $k_1 = p_{21}\Lambda^{-\frac{1}{2}}$ and $k_2 = p_{22}$. So $(k_i)_{i=1,2}$ is solution to the system

$$\lambda k_1^2 + 2k_1 - 1 = 0 \quad \text{and} \quad 2k_1 - k_2^2 = 0. \tag{39}$$

As in the first example, a nonnegative exact solution

$$k_1(\lambda) = \frac{-1 + \sqrt{1+\lambda}}{\lambda} \quad \text{and} \quad k_2(\lambda) = \sqrt{2\frac{-1 + \sqrt{1+\lambda}}{\lambda}}$$

can be exhibited, so it is used to discuss numerical validation. Again, the functions $k_i(\lambda)$ are sufficiently regular to be accurately approximated by polynomials, in which the computation is carried out using the spectral method with Legendre polynomials and the Legendre–Gauss–Lobatto quadrature formulae. The weak formulation equivalent to (39) is

$$\int_{I_\sigma} \left(\lambda k_1^2 + 2k_1 - 1\right) \eta_1(\lambda) d\lambda = 0 \quad \text{and} \quad \int_{I_\sigma} \left(2k_1 - k_2^2\right) \eta_2(\lambda) d\lambda = 0 \tag{40}$$

for all $\eta_1, \eta_2 \in \mathcal{C}^0(I_\sigma)$, it is solved by the semi-implicit Algorithm 2.

Algorithm 2 Semi-implicit scheme for (40)

1: k^0_{1,N_1}, k^0_{2,N_2} are given.

2: $(m+1)^{\text{th}}$ step: Knowing $\left(k^m_{1,N_1}, k^m_{2,N_2}\right) \in \mathbb{P}_{N_1} \times \mathbb{P}_{N_2}$, find $\left(k^{m+1}_{1,N_1}, k^{m+1}_{2,N_2}\right) \in \mathbb{P}_{N_1} \times \mathbb{P}_{N_2}$ such that $\forall (\eta_1, \eta_2) \in \mathbb{P}_{N_1} \times \mathbb{P}_{N_2}$,

$$\int_{I_\sigma} k^{m+1}_{1,N_1}(\lambda) \left(\lambda k^m_{1,N_1}(\lambda) + 2\right) \text{"}_1(\lambda) d\lambda = \int_{I_\sigma} \eta_1(\lambda)\, d\lambda,$$

$$\int_{I_\sigma} k^{m+1}_{2,N_2}(\lambda) \left(k^m_{2,N_2}(\lambda) + 1\right) \eta_2(\lambda)\, d\lambda = \int_{I_\sigma} \left(2k^{m+1}_{1,N_1}(\lambda) + k^m_{2,N_2}(\lambda)\right) \eta_2(\lambda) d\lambda.$$

3: If $\|k^{m+1}_{i,N_i} - k^m_{i,N_i}\|_{L^2(I_\sigma)} \leq \varepsilon_i$, then terminate the algorithm else return to step 2.

7.3.3. Numerical results. The simulations are conducted for a Euler–Bernoulli beam model with length $L = 4.73$ m so that all eigenvalues of Λ are included in $I_\sigma = (0, 1)$.

▷ *Polynomial approximation*: Numerical tests show an exponential convergence of Algorithm 2. For $N_1 = N_2 = 10$ and for null initial conditions, the exponential decay rate is about -1.80, and this of the differences of successive iterates $\left(\left\|k^{m+1}_{i,N_i} - k^m_{i,N_i}\right\|_{L^2(I_\sigma)}\right)_{i=1,2}$ is about -1.83. The two polynomial approximation errors

$$e_i = \frac{\|k_{i,N_i} - k_i\|_{L^2(I_\sigma)}}{\|k_i\|_{L^2(I_\sigma)}}$$

are in the order of 10^{-10} and 10^{-11}.

▷ *Approximation through the Cauchy formula combined with a polynomial approximation*: The numerical integrations, carried out as in the first example, yield relative errors

$$E_i = \frac{\|k_{i,N_i,M} - k_i\|_{L^2(I_\sigma)}}{\|k_i\|_{L^2(I_\sigma)}}$$

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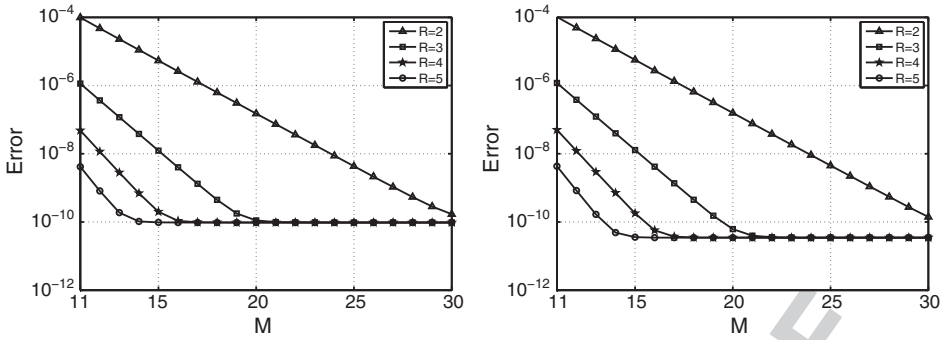

Figure 2. Relative errors (a) E_1 and (b) E_2 in logarithmic scale with respect to M_i for different values of R and for $N_1 = N_2 = 10$.

parameterized by the number M of integration nodes. They decrease exponentially with respect to M as shown in Figure 2, where both M are varying from 11 to 30. Several values of the radius R have been tested showing that the convergence rate is increasing with R.

▷ *Spatial discretization*: Taking the same notation as in Example 1, the finite difference discretization of the one-dimensional fourth-order boundary value problem

$$\Delta^2 \mathbf{v} = f \text{ in } \Omega, \quad \mathbf{v} = \nabla \mathbf{v}.\nu = 0 \text{ on } \partial\Omega, \tag{41}$$

is

$$\frac{1}{h^4}\left(\mathbf{v}_{i-2} - 4\mathbf{v}_{i-1} + 6\mathbf{v}_i - 4\mathbf{v}_{i+1} + \mathbf{v}_{i+2}\right) = f(x_i), \quad \text{for } i = 2, \ldots, \mathcal{N}-2$$

for the equation in Ω, and $\mathbf{v}_0 = 0, \mathbf{v}_{\mathcal{N}} = 0$ for the boundary conditions on $\mathbf{v}$. This scheme is consistent at the order 2. To not deteriorate the error, we use a second-order scheme for the boundary conditions on ∂_x. From Taylor's theorem, $\mathbf{v}(x_1) = \mathbf{v}(0) + h\partial_x \mathbf{v}(0) + \frac{h^2}{2}\partial^2_{xx}\mathbf{v}(0) + \mathcal{O}(h^3)$ and $\mathbf{v}(x_2) = \mathbf{v}(0) + 2h\partial_x \mathbf{v}(0) + 2h^2\partial^2_{xx}\mathbf{v}(0) + \mathcal{O}(h^3)$. By eliminating the term in $\partial^2_{xx}\mathbf{v}(0)$, it comes $\partial_x \mathbf{v}(0) = \frac{-3\mathbf{v}(0)+4\mathbf{v}(x_1)-\mathbf{v}(x_2)}{2h} + \mathcal{O}\left(h^2\right)$. The same is carried out for $\partial_x \mathbf{v}(L)$; we find $\partial_x \mathbf{v}(L) = \frac{-3\mathbf{v}(x_{\mathcal{N}})+4\mathbf{v}(x_{\mathcal{N}-1})-\mathbf{v}(x_{\mathcal{N}-2})}{2h} + \mathcal{O}(h^2)$. In total, the discretization of the problem (41) after elimination of $\mathbf{v}_0$ and $\mathbf{v}_{\mathcal{N}}$ is written in matrix form $[\Delta_h^2]\mathbf{v}_h = f_h$, where $\mathbf{v}_h^T = [\,\mathbf{v}_1 \ \ldots \ \mathbf{v}_{\mathcal{N}-1}\,]$, $f_h^T = [\,f(x_1) \ \ldots \ f(x_{\mathcal{N}-1})\,]$ and $\left[\Delta_h^2\right]$ is the matrix in (38). The full optimal control approximation (36) is obtained by using the formulae (20) and (21) and the formula of $\mathbf{u}_{N,M,h}$ in (23) with $\Lambda_h = \left[\Delta_h^2\right]^{-1}$. To validate this full strategy, we have carried a computation with $R = 5$, $M = 11$, 10^2 points in the mesh of Ω and for the time $t \in (0, T)$ with $T = 15s$.

The spatial discretization is compared to the expression of the approximation $\mathbf{u}_{N,M}(t, x)$ that we calculate thanks to the modal decomposition of the operator ∂^4_{xxxx} with homogeneous Dirichlet boundary conditions. Its expression is too big to be presented; it has been detailed in [4]. Denoting by $\mathbf{u}_{N,M,i} = \left(\mathbf{u}_{N,M,h}\right)_i$ the discrete values of the control, the spatial discretization relative error

$$\frac{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,M,i}(t) - \mathbf{u}_{N,M}(x_i, t)|^2\right)^{\frac{1}{2}} dt}{\int_0^T \left(\sum_{i=0}^{\mathcal{N}} |\mathbf{u}_{N,M}(x_i, t)|^2\right)^{\frac{1}{2}} dt}$$

between $u_{N,M}$ and $u_{N,M,h}$ is equal to 1.10×10^{-4}.

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7.4. Example 4: Two-dimensional heat equation with a boundary control

This example deals with a special case of boundary control.

7.4.1. The state equation and a choice of semi-decentralized controller. Let Ω be the rectangle $(0,1)\times(0,\pi)\subset\mathbb{R}^2$ and $\Gamma_0=\{(0,y):0<y<\pi\}$ a part of its boundary. Let us consider the heat equation with a control $v(t,y)$ applied to the boundary Γ_0,

$$\begin{cases} \partial_t w(t,x,y)-\partial^2_{xx}w(t,x,y)-\partial^2_{yy}w(t,x,y) = 0 & \text{in}\mathbb{R}^{+*}\times\Omega,\\ w(t,0,y) = v(t,y) & \text{on } \mathbb{R}^{+*}\times\Gamma_0,\\ w(t,x,y) = 0 & \text{on } \mathbb{R}^{+*}\times\partial\Omega\backslash\Gamma_0,\\ w(0,x,y) = w_0(x,y) & \text{in } \Omega. \end{cases}$$

Because our method is not directly applicable, we reduce the problem to an internal control problem. We introduce $\overline{w}(t,x,y) = w(t,x,y)-(1-x)v(t,y)$ solution to the heat equation with homogeneous boundary conditions,

$$\begin{cases} \partial_t \overline{w}(t,x,y) = \partial^2_{xx}\overline{w}(t,x,y)+\partial^2_{yy}\overline{w}(t,x,y)-(1-x)u(t,y) & \text{in } \mathbb{R}^{+*}\times\Omega,\\ \overline{w}(t,x,y) = 0 & \text{on } \mathbb{R}^{+*}\times\partial\Omega,\\ \overline{w}(0,x,y) = \overline{w}_0(x,y) = w_0(x,y)-(1-x)w_0(0,y) & \text{in } \Omega, \end{cases}$$

with $u(t,y)=\partial_t v(t,y)-\partial^2_{yy}v(t,y)$ that allows for easy computation of v once u is known. For simplicity, we define the cost function and the control space with u instead of v. So, we chose the control space $U\subset L^2(\Gamma_0)$ and the cost functional

$$\mathcal{J}(\overline{w}_0;u)=\int_0^{+\infty}\|\overline{w}(t,x,y)\|^2_{L^2(\Omega)}+\|u(t,y)\|^2_{L^2(\Gamma_0)}\,dt. \tag{42}$$

Then, the approximation of the control is carried out by using J terms in a modal decomposition of ∂^2_{xx}. Without entering into much details, which are given in Section 7.4.2, the state vector is comprised with J components $\overline{w}_j(t,y)=\sqrt{2}\int_0^1\overline{w}(t,x,y)\sin(j\pi x)dx$, and the associated control is therefore $\mathbf{u}=-k(\Lambda)z$, where k is a J-row vector of functions and Λ is the isomorphism $\left(-\partial^2_{yy}\right)^{-1}: L^2(\Gamma_0)\to H^2(\Gamma_0)\cap H^1_0(\Gamma_0)$. A semi-decentralized control is built from a rational approximation k_{j,N_j} of each component k_j and from a quadrature rule in the Cauchy formula,

$$\mathbf{u}_{N,M,h}=-\frac{1}{2\pi}\sum_{\ell=1}^{M}\omega_\ell\sum_{j=1}^{J}\mathbf{v}^{\ell,j}_{1,h}, \tag{43}$$

where each $\mathbf{v}^{\ell,j}_{1,h}$ is solution to a system such as (29) with $\overline{w}_{j,h}$ instead of w_h and k_{j,N_j} instead of p_N.

7.4.2. Construction and study of the semi-decentralized controller. We start with projecting the model on the J first components of the orthonormal basis $\psi_j(x)=\sqrt{2}\sin(j\pi x)$ in $L^2(0,1)$. Because $\int_0^1(1-x)\psi_j(x)\,dx=\frac{\sqrt{2}}{j\pi}$, the components $\overline{w}_j(t,y)$ are solution to the equations posed on Γ_0,

$$\begin{cases} \partial_t\overline{w}_j(t,y) = -j^2\pi^2\overline{w}_j(t,y)+\partial^2_{yy}\overline{w}_j(t,y)-\frac{\sqrt{2}}{j\pi}u(t,y) & \text{in } \mathbb{R}^{+*}\times\Gamma_0,\\ \overline{w}_j(t,0) = \overline{w}_j(t,\pi)=0 & \text{in } \mathbb{R}^{+*},\\ \overline{w}_j(0,y) = \overline{w}_{j,0}(y)=\int_0^1\overline{w}_0(x,y)\psi_j(x)dx & \text{in } \Gamma_0. \end{cases} \tag{44}$$

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This is the system of state equations coupled by a common internal control $u(t, y)$. The cost functional (42) is reduced to

$$\mathcal{J}(\overline{w}_0; u) \simeq \mathcal{J}(\overline{w}_{.,0}; u) = \int_0^{+\infty} \sum_{j=1}^{J} ||\overline{w}_j(t, y)||^2_{L^2(\Gamma_0)} + ||u(t, y)||^2_{L^2(\Gamma_0)}\, dt.$$

Then, the state variable is $z^T = [\,\overline{w}_1 \ \ldots \ \overline{w}_J\,]$, $A = -\mathrm{diag}[(j^2\pi^2 + \Lambda^{-1})_{j=1\ldots J}]$, $B^T = \frac{\sqrt{2}}{\pi}\left[\,\frac{I}{1} \ \ldots \ \frac{I}{J}\,\right]$, and C is the identity operator. The control and the observation spaces are $U = L^2(\Gamma_0)$ and $Y = \left(L^2(\Gamma_0)\right)^J$. In addition, we pose $X = L^2(\Gamma_0)$ and the state space $Z = (L^2(\Gamma_0))^J$; thus, B and C are bounded. Thus, $\Phi_Z = \Phi_Y = I_{J\times J}$, $\Phi_U = I$, and $a(\lambda) = -\mathrm{diag}\left[\left(j^2\pi^2 + \frac{1}{\lambda}\right)_{j=1\ldots J}\right]$, $b^T(\lambda) = \frac{\sqrt{2}}{\pi}\left[\,\frac{1}{1} \ \ldots \ \frac{1}{J}\,\right]$ and $c(.)$ is the identity operator on $\mathbb{R}^J$. Because Φ_Z and Φ_U are the identity operators, the approximation is developed on $Q = k(\Lambda)$ with $k(\Lambda) = q(\Lambda)$, and the exact optimal control is $\mathbf{u} = -k(\Lambda)z$.

To build a rational interpolation $k_N(\lambda)$ of the form (15), the interval $I_\sigma = (0, 1]$ is meshed with $L+1$ distinct nodes $\lambda_0, \ldots, \lambda_L$, and each $p(\lambda_n)$ solutions to the ARE is accurately computed with a standard solver. The exact expression of $k(\lambda_n) = b^T p(\lambda_n)$ follows, and the coefficients of the rational approximation are solution to the $L+1$ equations $k_{j,N_j}(\lambda_n) = k_j(\lambda_n)$, that is, to

$$\sum_{m=0}^{N_j^N} d_m \lambda_n^m - k_j(\lambda_n) \sum_{m'=0}^{N_j^D} d'_{m'} \lambda_n^{m'} = 0 \text{ for } n = 0, \ldots, L.$$

The number L of equations is taken sufficiently large so that the system with $N^N + N^D + 2$ unknowns is over-determined and is solved in the mean square sense by using the singular value decomposition.

7.4.3. Numerical results. The simulation have been conducted with four modes, that is, for $J = 4$. The shape of the four first functions $k_j(\lambda)$ are represented in Figure 3, which shows that they exhibit a singular behavior at the origin. Thus, they cannot be accurately approximated by polynomials but may be by rational functions.

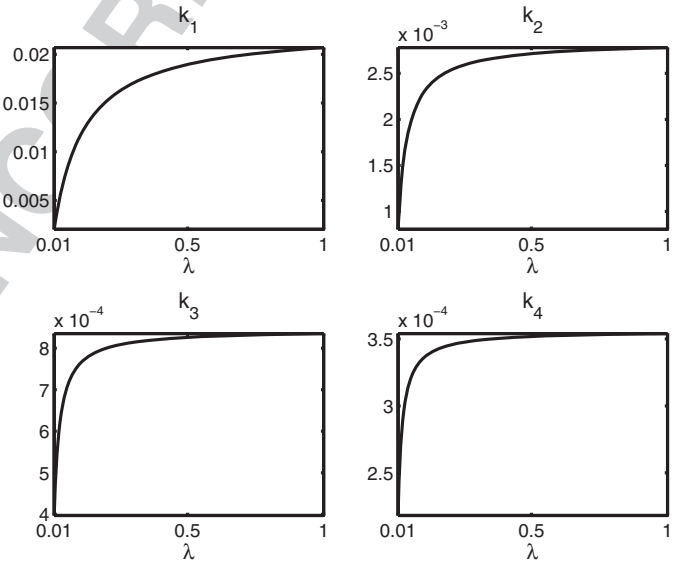

Figure 3. Shapes of the spectral functions k_1, k_2, k_3, and k_4.

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▷ *Rational approximation*: In order to obtain an accurate approximation, we choose a logarithmic distribution of 100 nodes in $(10^{-2}, 1)$, which corresponds to a truncation of high frequencies. In Table I, we report the relative errors in the discrete ℓ^2-norm on the other set $\{\lambda_n\}_{n=0...200}$ **T1**

$$e_j = \frac{\left(\sum_{n=0}^{200} \left|k_{j,N_j}(\lambda_n) - k_j(\lambda_n)\right|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=0}^{200} \left|k_j(\lambda_n)\right|^2\right)^{\frac{1}{2}}}, \text{ with } \lambda_n = 10^{-2+\frac{n}{100}} \quad \text{for } j = 1, \ldots, 4,$$

between the exact function k_j and its rational approximation k_{j,N_j} for special values of numerator's and denominator's polynomial degrees $N_j = \left(N_j^N, N_j^D\right)$

▷ *The Cauchy formula combined with rational approximations*: Then, according to Remark 6.3, numerical integrations are performed with a standard trapezoidal quadrature rule along the ellipse defined by the two radii in the real and imaginary directions $R_1 = 1.02$ and $R_2 = 0.07$. The relative errors

$$E_j = \frac{\left(\sum_{n=0}^{200} \left|k_{j,N_j,M}(\lambda_n) - k_j(\lambda_n)\right|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=0}^{200} \left|k_j(\lambda_n)\right|^2\right)^{\frac{1}{2}}} \text{ with } \lambda_n = 10^{-2+\frac{n}{100}} \quad \text{for } j = 1, \ldots, 4,$$

between the exact functions and final approximations are plotted in logarithmic scale in Figure 4 **F4** for M varying from 10 to 5×10^2. Figure 4 reports the errors E_1, E_2, E_3, and E_4 showing an exponential decay with respect to M. Note that the parameters R_1 and R_2 of the ellipse affects the rate of convergence errors, which is confirmed by our numerical calculation.

Table I. Errors of the rational approximations with numerator's and denominator's degrees $N_j = \left(N_j^N, N_j^D\right)$.

j	1	2	3	4
N_j	(19, 3)	(18, 3)	(17, 1)	(20, 2)
$e_j \times 10^{-10}$	0.003	0.013	1.003	0.182

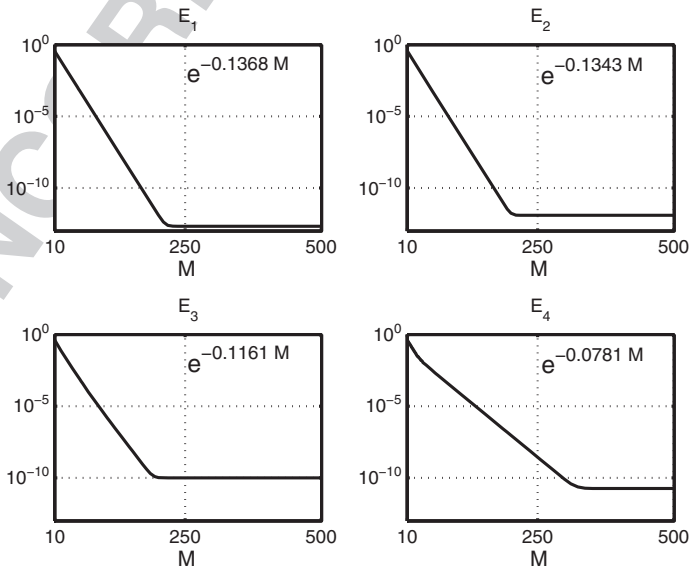

Figure 4. Errors E_1, E_2, E_3, and E_4.

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▷ *Spatial discretization*: The approximation (43) is obtained from the formula of $\mathbf{u}_{N,M,h}$ in (23) and by using the centered finite difference scheme of the second-order derivative ∂^2_{yy}. The expression of Λ_h is the same as in Example 1, and the error between $\mathbf{u}_{N,M,h}$ and $\mathbf{u}$ is quadratic in the space step h.

8. CONCLUSION

We have proposed a method to compute distributed control applied to linear distributed systems with a control operator that is bounded or not. It has been conceived for architectures of semi-decentralized processors. Its construction uses a functional calculus for matrices of functions of an operator, based on spectral theory and Cauchy formula. In the case of polynomial approximation of k, we have noticed that the numerical integration needs few integration points and that the radius of the contour affects the accuracy of the numerical integration of the Cauchy formula. If the approximation is rational, we have concluded that numerical integration requires more integration points in the ellipse in which its parameters have been chosen heuristically. We think that the performance of the method could be further improved by finding optimal contour parameters depending on the number of quadrature nodes following the ideas in J. A. C. Weideman and L. N. Trefethen [27]. Finally, the method can be extended to other frameworks for distributed control and for functional calculus.

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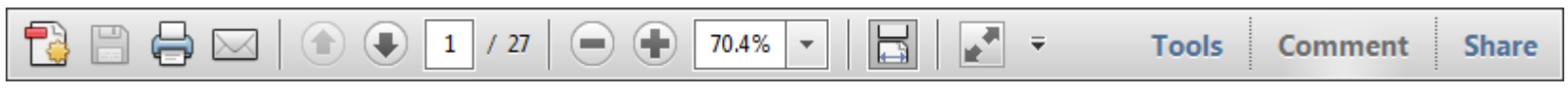
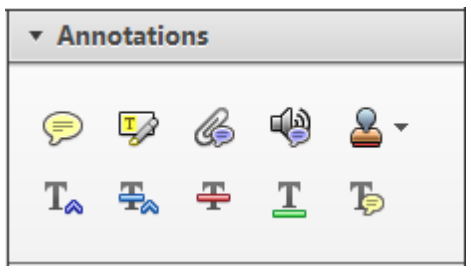
Once you have Acrobat Reader open on your computer, click on the Comment tab at the right of the toolbar:



This will open up a panel down the right side of the document. The majority of tools you will use for annotating your proof will be in the Annotations section, pictured opposite. We've picked out some of these tools below:

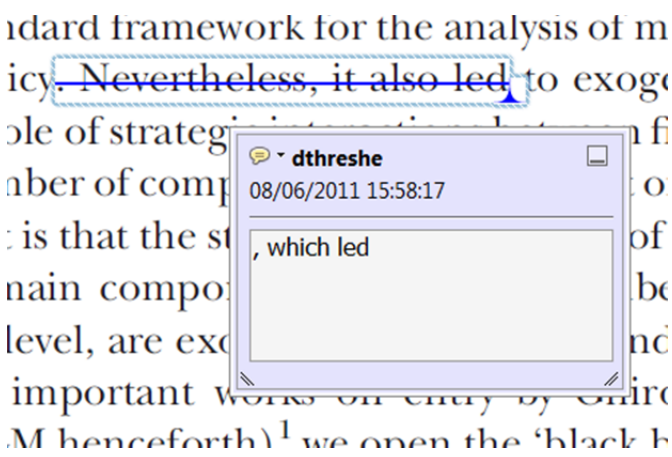


1. Replace (Ins) Tool – for replacing text.

Strikes a line through text and opens up a text box where replacement text can be entered.

How to use it

- Highlight a word or sentence.
- Click on the Replace (Ins) icon in the Annotations section.
- Type the replacement text into the blue box that appears.

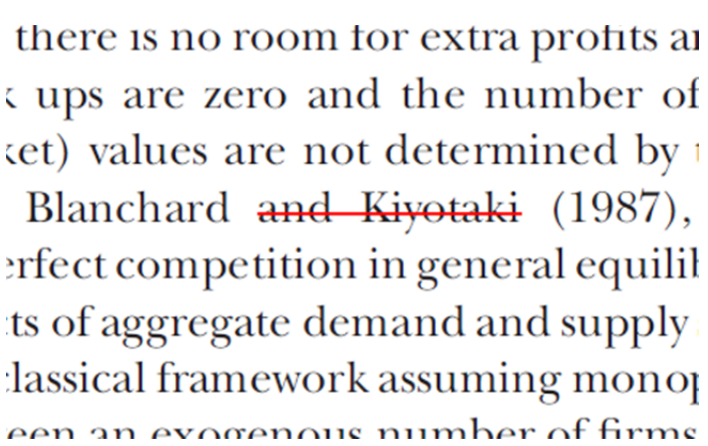


2. Strikethrough (Del) Tool – for deleting text.

Strikes a red line through text that is to be deleted.

How to use it

- Highlight a word or sentence.
- Click on the Strikethrough (Del) icon in the Annotations section.

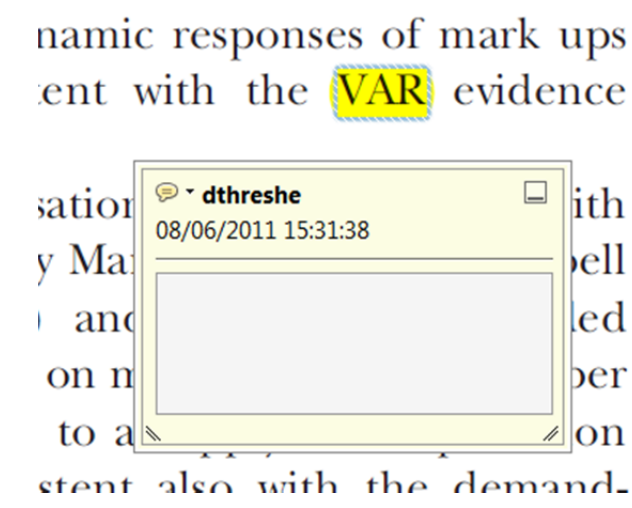


3. Add note to text Tool – for highlighting a section to be changed to bold or italic.

Highlights text in yellow and opens up a text box where comments can be entered.

How to use it

- Highlight the relevant section of text.
- Click on the Add note to text icon in the Annotations section.
- Type instruction on what should be changed regarding the text into the yellow box that appears.

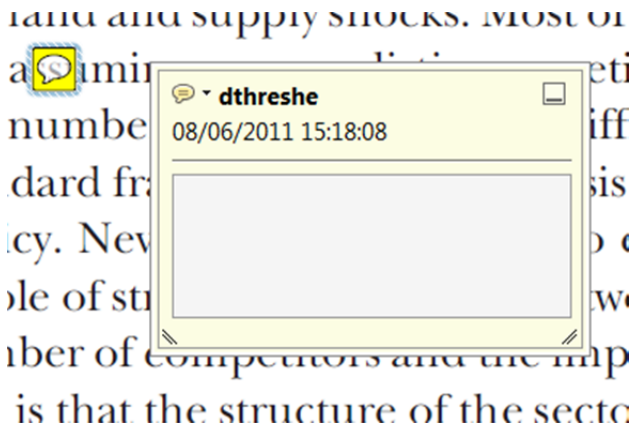


4. Add sticky note Tool – for making notes at specific points in the text.

Marks a point in the proof where a comment needs to be highlighted.

How to use it

- Click on the Add sticky note icon in the Annotations section.
- Click at the point in the proof where the comment should be inserted.
- Type the comment into the yellow box that appears.



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5. Attach File Tool – for inserting large amounts of text or replacement figures.

Inserts an icon linking to the attached file in the appropriate pace in the text.

How to use it

- Click on the Attach File icon in the Annotations section.
- Click on the proof to where you'd like the attached file to be linked.
- Select the file to be attached from your computer or network.
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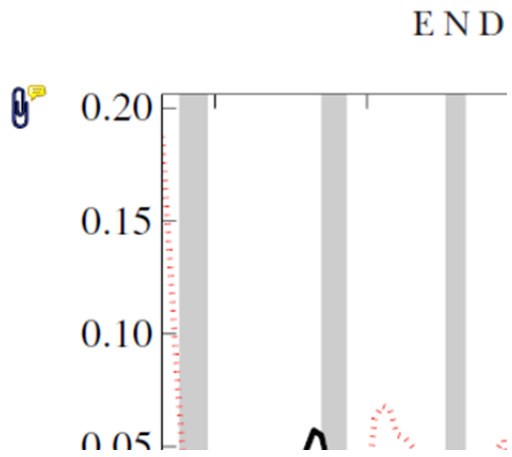

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How to use it

- Click on the Add stamp icon in the Annotations section.
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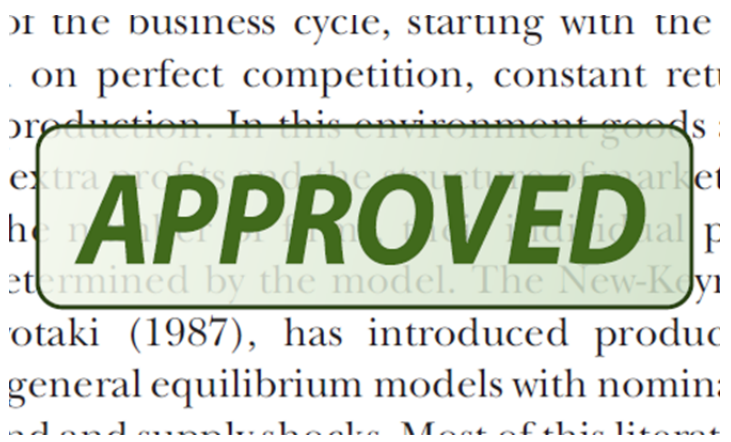
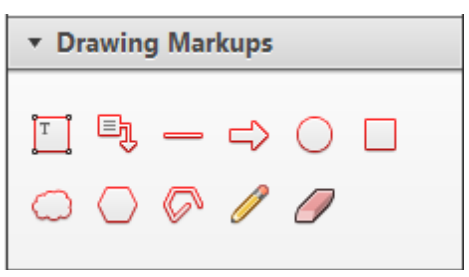


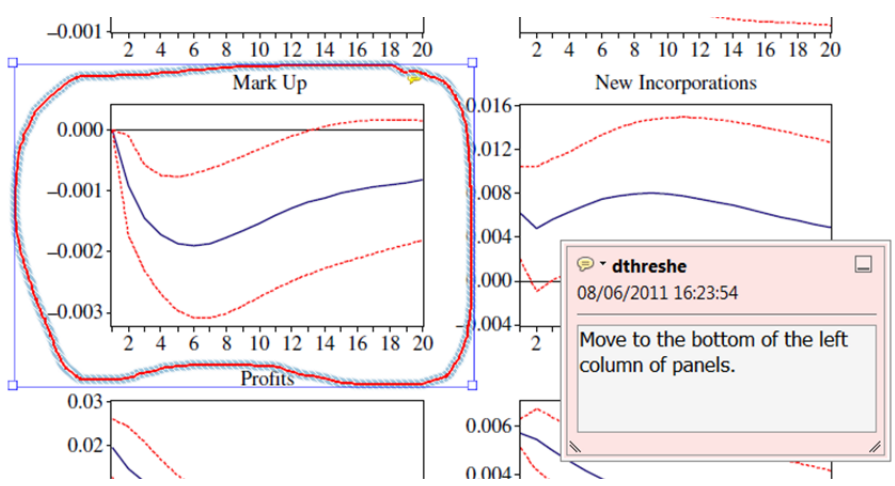


7. Drawing Markups Tools – for drawing shapes, lines and freeform annotations on proofs and commenting on these marks.

Allows shapes, lines and freeform annotations to be drawn on proofs and for comment to be made on these marks..

How to use it

- Click on one of the shapes in the Drawing Markups section.
- Click on the proof at the relevant point and draw the selected shape with the cursor.
- To add a comment to the drawn shape, move the cursor over the shape until an arrowhead appears.
- Double click on the shape and type any text in the red box that appears.



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