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HOMOGENIZATION OF ELECTRICAL NETWORKS INCLUDING VOLTAGE-TO-VOLTAGE AMPLIFIERS

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We derive the homogenized model of periodic electrical networks which includes resistive devices, voltage-to-voltage amplifiers, sources of tension and sources of current. On the one hand, in considering the homogenized problem, general conditions are stated insuring the existence and uniqueness of the solution. They are formulated in function of the network topology. On the other hand, the two-scale transformation introduced by Arbogast, Douglas and Hornung is adapted to the context of electrical networks. New two-scale convergence results, inspired by the principle of Allaire's two-scale convergence, are shown in this context. In particular, the two-scale convergence for the tangential derivative on a network is established. Following these results, two models of homogenized networks are derived. The first one belongs to a general framework whereas the second one does not.

1. Introduction

This paper was written in view of the applications of the modelling of Smart Materials Systems. Let us recall that Smart Materials Systems are mechanical structures including actuators, sensors and an electronic system. We focus our attention on the case where they are many transducers and electronical devices and where they are distributed in the structure. These kinds of systems are useful in acoustics and fluid mechanics, because the sound and the perturbation in a fluid are distributed phenomena. Therefore the control needs to be distributed.

On the one hand, we have already derived models for elastic plates and shells including a great number of periodically distributed piezoelectric transducers and distributed electronics in specific configurations; see Canon and Lenczner[10,11] and Senouci-Bereksi and Lenczner.[23]

On the other hand, a general model for periodically distributed electronic network including resistors, current sources and voltage sources was announced in Ref. 14. It was based on a variational formulation of the electronical equations and on a new concept of two-scale convergence. This concept is inspired by a combination of ideas by Allaire[4] and Arbogast, Douglas and Hornung.[5] In this paper we state a general model of periodically distributed electrical network including resistors, voltage sources, current sources and voltage-to-voltage amplifiers.

This paper is divided into three parts: in the first part, the variational formulation of the electrical network equation is stated. The sufficient conditions for the existence and uniqueness of its solution are stated. In the second part, the statement of the definition of the two-scale convergence is adapted for electrical networks. The two-scale limit of the tangential derivative along a one-dimensional network is derived.

Finally, in the last part, the two-scale model for electrical networks is derived.

Part 1. We will start with the classical equations of electrical network. They are stated for example in Vlach and Singhal.[24] Then, the variational formulation equivalent to these equations is stated. The variational formulation has the form:

$$a(u,v) + b_1(v,p) = \langle f,p \rangle \,,$$

$$b_2(u,q) = \langle g,q \rangle \,,$$

where b_1 and b_2 are different. The bilinear forms $a(.,.)$, $b_1(.,.)$ and $b_2(.,.)$ are built with some partial differential operators. The use of such variational formulation for this problem seems to be new. Necessary and sufficient conditions for the existence and uniqueness of the solution for such a problem have been derived in Ref. 8. Sufficient conditions for the existence and uniqueness of the solution are given. There are graph interpretations of the conditions stated in Ref. 8. There are mainly related to the location of the various devices in the network: voltage or current sources, resistors, amplifier inputs and outputs, and earth. They use some very simple graph theory principles. The statement of the existence and uniqueness, and the equivalence between the variational formulation and the classical formulation, are stated in Theorem 1.

This approach in the electrical circuit analysis seems to be new. The results of existence and uniqueness are normally based on graph and algebra theories, see for example Recksi.[19]

In our opinion, there are two points of interest in our work. First, it gives us the possibility of having a global analysis for mechanical and electrical systems. Second, it provides the estimated solutions, necessary for the application of usual asymptotic methods. In particular, these estimates are required for the derivation of homogenized model.

The methodology which is developed here may be extended to electrical networks which includes other devices such as current to current amplifiers, voltage to current amplifiers, current to voltage amplifiers, diodes, operational amplifiers, negative resistors, capacitors and inductors. It also allows us to consider the coupling between electrical and mechanical systems in a unified framework based on graph theory as well as functional analysis.

Part 2. The two-scale transformation and the two-scale convergence that we use in this paper, were already introduced in Ref. 5. We reuse them in the one-dimensional manifold context. We establish, in two different cases, the two-scale limits of the tangential derivative of a field defined on the periodic network. These two cases refer to two kinds of field estimates which occur in our electrical network analysis. The results of two-scale convergence are stated in Theorem 2.

The results stated in Theorem 2 and their proofs are new. In particular, they were not proved in Ref. 4 or 5.

The two-scale convergence defined in Ref. 4 does not apply to networks. In Ref. 5, only the definitions of a two-scale transformation and a two-scale convergence were stated. The two-scale convergence for the gradient were not at all established. We also introduce original proofs for the two-scale convergence of the tangential derivative. These kinds of proofs may be applied to functions defined on general periodic $(n-p)$-dimensional manifolds immerted in R^n.

Let us recall the definition introduced in Ref. 4, that being the two-scale convergence of functions defined in a domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ and relative to a cell $Y =]-1/2, +1/2[^n$. A sequence $(u^\varepsilon)_{\varepsilon>0}$ in $L^2(\Omega)$, is said to be *two-scale convergent* towards a limit $u(\mathbf{z}, \mathbf{y}) \in L^2(\Omega \times Y)$ if for any function $\varphi(\mathbf{z}, \mathbf{y}) \in \mathcal{D}(\Omega; \mathcal{C}^\infty_\sharp(Y))$ we have

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(\mathbf{x}) \varphi\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) d\mathbf{x} = \int_{\Omega \times Y} u(\mathbf{z}, \mathbf{y}) \varphi(\mathbf{z}, \mathbf{y})\, d\mathbf{y}\, d\mathbf{z}\,.$$

Let us remark that this is a weak convergence. In addition, it requires the function u^ε to be defined on the whole domain Ω. Thus, this is not applicable for general electrical networks.

The principle of the two-scale convergence introduced in Ref. 5 is based on a variable change transforming Ω to $\Omega \times Y$. According to our method, the two-scale convergence is an ordinary one, concerning functions defined on $\Omega \times Y$ instead of Ω. This point of view presents two main advantages.

First of all, this two-scale convergence concept is more general in the sense that it is not restricted to L^p weak convergence. It can be easily extended to any kind of convergence concerning functions.

The second advantage being when we need the convergence of functions defined on a periodic manifold. This method may be easily adapted. In this case, Y is replaced by the reference cell which is a manifold. This method does not require any extension of the solution.

Let us also mention that in Ref. 17, the extension of the two-scale convergence[4] to a periodic $(n-1)$-dimensional manifold was carried out. It was based on an extension of the solution.

Part 3. The homogenization of the electrical network equations is based on the results stated in Theorems 1 and 2. Our goal is not to provide a general approach of electrical network homogenization. Many different models may be derived depending on the behavior of different coefficients with respect to the length ε of the period. We make some assumptions about solution estimates. This choice is led by its interest in applications and by its relative simplicity. In particular, we assume that the amplifier's coefficients are of zero order with respect to ε. The general model related to this framework is stated in Theorem 3. Finally, a particular example with coefficients at the order ε^{-1} is treated in Theorem 4.

Let us note that a homogenized model of two-dimensional electrical networks made of resistors have already been derived in Ref. 25. The method developed by Vogelius was based on an extension of the solution to an open set, which includes the electrical network. The proofs were based on some finite element techniques. The technical difference between our approach and that of Ref. 25 is that, no extension of the solution is required, and the proofs are valid for a network imbedded in an n-dimensional Euclidean space where $n \geq 1$. In addition, voltage sources, current sources and voltage-to-voltage amplifiers are taken into account in our approach. This was not the case in Ref. 25.

The two-scale convergence described in this paper may be applicable for the homogenization of trusses equations. Different approaches have already been proposed for the modelling of periodical trusses or nets, see Abrate,[1–3] Renton,[20,21] Cioranescu and Saint Jean Paulin,[12,13] Caillerie and Moreau,[9] Bakhalov and Panasenko,[6] Panasenko[18] and Maz'ya and Slutsky.[15] The approach of Refs. 12 and 13 is based on an asymptotic analysis where both the beam thickness and the truss period lengths vanish. D. Caillerie and Al. introduced the discrete homogenization method for the same problem. In this approach, the unknown are displacement of vertices and tensions of the edges. The model derivation is based on an asymptotic expansion of the solution.

The paper is divided into eight sections. In Sec. 2, we will consider an electrical network including resistors, tension sources, current sources and voltage-to-voltage amplifiers. We will provide a set of conditions on the network topology under which the problem is well-posed. In Sec. 3, two-scale convergence results concerning functions defined on electrical networks will be explained. In Sec. 4, a general framework for the homogenization of electric network based on the results of Secs. 2 and 3 will be detailed. Then, a particular example of electric network not belonging to the general framework will be described, and its homogenized model stated. In Secs. 5–8 the proof of Theorems 1–4 will be explained.

2. Variational Formulation of Electrical Networks

In this section, we state the general variational formulation which is satisfied by the electrical potential in the electrical network. The network includes resistors, current sources, voltage sources and voltage-to-voltage amplifiers. The conditions posed on the network for the existence and the uniqueness of the solution are stated. They are based on the conditions stated in Ref. 8 and are interpreted in terms of the conditions posed on the electrical network. The case of purely resistive networks was already explained in Ref. 14.

2.1. *Notations*

We use the definitions and the properties relative to electrical networks presented in Ref. 24, see Fig. 1. An electrical network is composed of vertices (or nodes) and edges (or branches). Vertices are linked by edges. The set of edges is denoted by Θ. Mathematically, Θ is a network in $\mathbb{R}^n$ where $n \in \mathbb{N}^*$. We denote by σ_0 the subset of vertices linked to the earth (i.e. where the electrical potential is equal to zero). The network Θ is divided into five disjoint parts: Θ_0, Θ_1, Θ_2, Θ_3 and Θ_4. They are occupied respectively by the voltage sources, the current sources, the resistors, the input and the output of the amplifiers. The edges included in these sets are denoted respectively by e_0^l, e_1^l, e_2^l, e_3^l and e_4^l. Here, l is an index varying from one to the number of edges belonging to the respective sets.

The network Θ is assumed to be parametrized. This parametrization defines a positive sense for each edge. s_e^+ and s_e^- represent the vertices belonging to an edge $e \subset \Theta$ such that $s_e^+ \to s_e^-$ in the positive sense. The set of edges arriving at a positive (respectively negative) sense at a vertex s is denoted by Θ_s^+ (respectively Θ_s^-). The length of an edge e is denoted by $|e|$. The function L is distributed on Θ. It is constant on each edge, and $L(\mathbf{x}) = |e|$ for all $\mathbf{x} \in e$. The tangent vector to Θ at point $\mathbf{x}$ is denoted by $\tau(\mathbf{x})$.

2.2. *Statement of equations*

In this section, the equations of electrical networks in their classical form are recalled. We also introduce the necessary notations in order to write their variational formulation.

Let us define the sets $\mathbb{P}^0(\Theta)$ or $(\mathbb{P}^0(\Theta_k))_{k=0,\dots,4}$ (respectively $\mathbb{P}^1(\Theta)$) of functions constant on each edge $e \subset \Theta$ or $(e \subset \Theta_k)_{k=0,\dots,4}$ (respectively affine on each edge $e \subset \Theta$ and continuous on Θ). The current i and the voltage u are some distributed fields belonging to $\mathbb{P}^0(\Theta)$. The electrical potential is also a distributed field, it belongs to $\mathbb{P}^1(\Theta)$. The tangential derivative of a function ψ defined on Θ is denoted by $\nabla_\tau \psi$.

An example of the network described below is represented in Fig. 1.

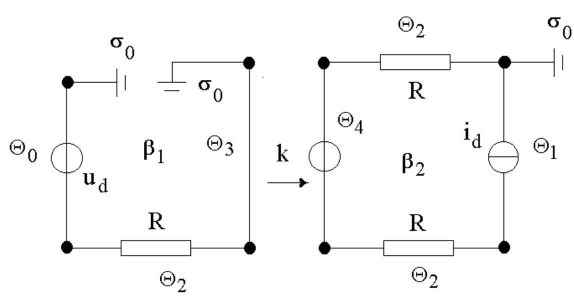

Fig. 1. An example of electrical network.

The voltage Kirchhoff law is stated on each edge $e \subset \Theta$ as follows, $u_{|e} = \varphi(s_e^+) - \varphi(s_e^-)$, or equivalently

$$-L\nabla_\tau \varphi = u \text{ on } \Theta\,. \tag{1}$$

The current Kirchhoff law is stated for each vertex s as $\sum_{e\subset\Theta_s^+} i_{|e} - \sum_{e\subset\Theta_s^-} i_{|e} = 0$. It can be equivalently written under a weak formulation:

$$\int_\Theta L\, i(\mathbf{x})\nabla_\tau \psi(\mathbf{x})\, dl(\mathbf{x}) = 0 \quad \text{for all } \psi \in \mathbb{P}^1(\Theta) \text{ such that } \psi = 0 \text{ on } \sigma_0\,. \tag{2}$$

The values of voltage, current and electrical potential are imposed respectively on Θ_0, Θ_1 and σ_0 to be equal to the voltage source $u_d \in \mathbb{P}^0(\Theta_0)$, the current source $i_d \in \mathbb{P}^0(\Theta_1)$ and 0 on σ_0:

$$u = u_d \text{ on } \Theta_0, \quad i = i_d \text{ on } \Theta_1 \quad \text{and} \quad \varphi = 0 \text{ on } \sigma_0\,.$$

Let us remark that the sign of u_d and of i_d on an edge e depends on the orientation of e.

An impedance $1/g \in \mathbb{P}^0(\Theta_2)$ is associated to Θ_2, which means that u and i are linked by the constitutive linear equation on Θ_2:

$$i = gu \text{ on } \Theta_2\,. \tag{3}$$

We assume that $g \geq g_{\min} > 0$.

We can recall that a voltage-to-voltage amplifier is a device which imposes two equations between currents and voltages of two edges. The set Θ_3 and Θ_4 are respectively the sets of amplifier's inputs and outputs. Each input edge $e_3^l \in \Theta_3$ is associated to a unique output edge $e_4^l \in \Theta_4$ where l varies from one to the number of amplifiers used.

The constitutive relations of the voltage-to-voltage amplifier are for each l:

$$u_{|e_4^l} - k_l u_{|e_3^l} = 0 \quad \text{and} \quad i_{|e_3^l} = 0\,, \tag{4}$$

where $k_l \in \mathbb{R}^*$ is the amplification coefficient. The edges e_3^l and e_4^l are respectively called the input and the output of the amplifier. Since Eq. (4) applies for each amplifier, we consider that $k \in \mathbb{P}^0(\Theta_3)$ and we write the amplifier constitutive equations as follows:

$$u_{|\Theta_4} - k_l u_{|\Theta_3} = 0 \quad \text{and} \quad i_{|\Theta_3} = 0\,. \tag{5}$$

2.3. *The variational formulation*

In this section, the variational formulation equivalent to the above equations, is introduced. Some sufficient conditions for the existence and uniqueness of the solution of the equations are also formulated. Finally, the existence and uniqueness theorem associated with the above problem is stated. This theorem is proved in Sec. 5.

The conditions stated in this section for the existence and uniqueness of the solution are graph theory interpretations of the conditions stated in Ref. 8. The conditions stated in Ref. 8 are mainly four inf-sup conditions and the continuity on the right-hand side of the variational formulation. Graph theory interpretation, means to interpret in terms of the location of the various devices such as resistors, amplifier inputs and outputs, current and voltage source and earth.

The result stated in this section is a basis for the derivation of the two-scale model stated in Secs. 4 and 5.

For $u_d \in \mathbb{P}^0(\Theta_0)$, let us define the admissible functions set for the variational problem:

$$\Psi_{ad}(u_d) = \{(\psi, j) \in \mathbb{P}^1(\Theta) \times \mathbb{P}^0(\Theta_4), \psi = 0 \text{ on } \sigma_0 \text{ and } -|e|\nabla_\tau \psi = u_d \text{ on } \Theta_0\},$$

and the following variational formulation. Consider $(\varphi, i) \in \Psi_{ad}(u_d)$ solution of:

$$\int_{\Theta_2} Lg\nabla_\tau\varphi\nabla_\tau\psi\, dl(\mathbf{x}) + \int_{\Theta_4} i\nabla_\tau\psi\, dl(\mathbf{x}) = -\int_{\Theta_1} i_d\nabla_\tau\psi\, dl(\mathbf{x})$$

$$\int_{\Theta_3} Lk\nabla_\tau\varphi j\, dl(\mathbf{x}) - \int_{\Theta_4} L\nabla_\tau\varphi j\, dl(\mathbf{x}) = 0$$

$$\text{for all } (\psi, j) \in \Psi_{ad}(0)\,. \tag{6}$$

Let us remark that $j \in \mathbb{P}^0(\Theta_4)$ is used on Θ_3. We adopt the rule that j takes the same value on the input e_3^l and on the output e_4^l of an amplifier.

This variational formulation has the form: $(\varphi, i) \in \Psi_{ad}(u_d)$

$$a(\varphi, \psi) + b_1(i, \psi) = l(\psi)\,,$$

$$b_2(j, \varphi) = 0\,,$$

for all $(\psi, j) \in \Psi_{ad}(0)$. Here $b_1(.,.)$ and $b_2(.,.)$ are different.

Definition. (i) A path is a sequence of edges where the end of an edge is connected to the beginning of the following one.

(ii) A circuit is a path where the beginning of the first edge is connected to the end of the last one. For this definition, all vertices belonging to σ_0 (the earth) are considered as one. The circuits are denoted by the letter β.

In order to check the conditions in Ref. 8, we will introduce the following linear system. For $v \in \mathbb{P}^0(\Theta_3)$ such that

$$\int_{\beta\cap\Theta_3} v\, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta \text{ of } \Theta_0 \cup \Theta_3 \cup \Theta_4\,, \tag{7}$$

we need to construct a solution $u \in \mathbb{P}^0(\Theta - \Theta_1)$, relative to v, of the linear system:

$$\begin{aligned} &u_{|\Xi} = 0\,,\\ &u_{|e_4^l} - ku_{|e_3^l} = v_{|e_3^l} \text{ for every } l\,,\\ &\int_\beta u\, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta \text{ of } \Theta - \Theta_1\,, \end{aligned} \tag{8}$$

where Ξ is a subset of $\Theta - \Theta_1$.

Consider the class of subsets $X \subset \Theta - \Theta_1$ such that equations in (8) are independent when $\Xi = (\Theta - \Theta_1) - X$.

Definition. We say that X is minimal for the independency of equations in (8) if for any $X^* \subset X$ (with $X^* \neq X$), equations in (8) are not independent when $\Xi = (\Theta - \Theta_1) - X^*$.

Remarks. (i) For a given set of Eqs. (8), the minimal set X is not unique.

(ii) Every minimal set have the same cardinal (see Recksi[19]).

(iii) There exist algorithms for building up such minimal set X. See Recksi.[19]

Assumptions.

(H1) There exist $\overline{\Theta}_2 \subset \Theta_2$ and $\widetilde{\Theta}_2 = \Theta_2 - \overline{\Theta}_2$ such that the following two conditions are fulfilled:

(i) There exists a minimal set $X = \widetilde{\Theta}_2 \cup \Theta_3 \cup \Theta_4$ for the independency of equations in (8) such that $\Xi = (\Theta - \Theta_1) - X = \Theta_0 \cup \overline{\Theta}_2$.

(ii) For every $v \in \mathbb{P}^0(\Theta_3)$ verifying the compatibility condition (7), the linear system (8) has at most one solution $u \in \mathbb{P}^0(\Theta - \Theta_1)$.

Remark. It will be proved later that (H1)(i) is equivalent to the existence of the solution of (8). Therefore, (i) and (ii) imply that (8) has one and only one solution. That is, the system (8) has as many equations as the unknowns.

Let us consider such a minimal set X. For $e \in X$, $X^* = (X - e)$ is not a minimal set, i.e. Eqs. (8) are not independent when $\Xi = (\Theta - \Theta_1) - X^*$. After deleting some equations in (8) (except the equation $u_{|e} = 0$), the remaining equations can be independent.

Definition. (i) One says that a subset E of dependent equations of (8) with $\Xi = (\Theta - \Theta_1) - X^*$, is minimal with respect to e, when, after deleting any equation, the remaining equations are independent and when the number of equations in E is equal to the number of edges involved in E plus one.

(ii) The set of edges involved in a minimal set of dependent equations is called the minimal set of edges linked with e and is denoted by $Z(e)$.

Remarks. (i) In the above definition (i), the subset necessary contains the equation $u_{|e} = 0$, otherwise equations would be independent.

(ii) The definition of minimal subset of dependent equations leads to the existence of solution of system E. When the number of equations in E is equal to the number of edges involved in E plus one, the solution is unique.

(iii) The definition of $Z(e)$ implies that $u_{|e}$ is a unique linear combination of $(u_{|e'})_{e' \in Z(e) - \{e\}}$. Therefore, $|u|_e \leq C|u|_{Z(e)-\{e\}}$.

(H2) Let us consider $\alpha_0 \in \mathbb{R}$. One can choose a function $\alpha \in \mathbb{P}^0(\Theta)$, constant on each circuit β, such that for each $e \in \widetilde{\Theta}_2$, there exists a minimal set $Z(e)$ of edges, linked with e, such that $\alpha_{|Z(e)\cap\Theta_3} = \alpha_0$ and $\alpha_{|Z(e)\cap\widetilde{\Theta}_2} = 1$.

There is an example of the partition of $\Theta_2 = \widetilde{\Theta}_2 \cup \overline{\Theta}_2$ in Fig. 2. Relative to this example, $Z(e_3) = \{e_0, e_4, e_3, e_2^3\}$, $Z(e_4) = \{e_0, e_4, e_3, e_2^3\}$ and $Z(\widetilde{e}_2) = \{e_0, e_4, e_3, e_2^3, e_2^1, e_2^2\}$.

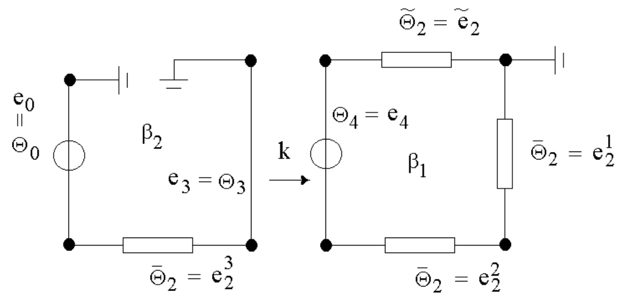

Fig. 2. The partition $\Theta_2 = \tilde{\Theta}_2 \cup \overline{\Theta}_2$.

The aim of the third assumption is to interpret the following condition: there exists a positive constant C such that for any $(\psi, 0) \in \Psi_{ad}(0)$ satisfying $B_1(\psi) = 0$ we have $|\nabla_\tau \psi|_{\Theta_3} \leq C|\nabla_\tau \psi|_{\overline{\Theta}_2}$,

(H3) Every edge $e \in \Theta_3$ belongs to a circuit $\beta \subset \{e\} \cup \Theta_0 \cup \overline{\Theta}_2 \cup \Theta_4$.

The following assumption (H4) means that there exists a positive constant C such that for every $(\psi, 0) \in \Psi_{ad}(0)$ we have $|\nabla_\tau \psi|_{\Theta_1} \leq C|\nabla_\tau \psi|_{\Theta - \Theta_1}$. It implies the continuity of the linear form $l(\psi) = \int_{\Theta_1} i_d \nabla_\tau \psi \, dl(\mathbf{x})$ with respect to the semi-norm $|\nabla_\tau \psi|_{\Theta - \Theta_1}$.

(H4) Every edge $e \in \Theta_1$ belongs to a circuit $\beta \subset \{e\} \cup (\Theta - \Theta_1)$.

The assumption (H5) means that there exists a positive constant C such that for every $(\psi, 0) \in \Psi_{ad}(0)$ we have $|\psi|_{\Theta - \Theta_1} \leq C|\nabla_\tau \psi|_\Theta$. It leads to a kind of Poincaré inequality. Combined with the assumption (H4), it insures that the semi-norm $|\nabla_\tau \psi|_\Theta$ is a norm on $\Psi_{ad}(0)$.

(H5) In each connected component of $\Theta - \Theta_1$, there is a vertex belonging to σ_0.

The assumption (H6) is a compatibility condition between the various voltage sources (the amplifier's outputs are generally called active voltage source).

(H6) There is no circuit solely made up of edges belonging to $\Theta_4 \cup \Theta_0$.

The assumption (H7) is equivalent to the following assertion. For every $j \in \mathbb{P}^0(\Theta_4)$ there exists a function $(\psi, 0) \in \Psi_{ad}(0)$ such that $\nabla_\tau \psi = j$ on Θ_4.

Consider the circuits β included in Θ satisfying $\beta \cap \Theta_4 \neq \emptyset$. There exists a subset $\Theta^* \subset \Theta$ of edges such that the network $\Theta - \Theta^*$ does not contain such a circuit β. The set Θ^* is said to be minimal if for any $\Theta^{*1} \subset \Theta^*$ $(\Theta^{*1} \neq \Theta^*)$, $\Theta - \Theta^{*1}$ contains at least one circuit β satisfying $\beta \cap \Theta_4 \neq \emptyset$ (see Recski[19]).

(H7) There exists such a minimal set Θ^* verifying $\Theta^* \cap (\Theta_0 \cup \Theta_4) = \emptyset$.

Now we are ready to state the theorem of existence and uniqueness.

Theorem 1. *If the assumptions* (H1–H7) *are fulfilled, then the variational formulation* (6) *has a unique solution.*

3. Two-Scale Convergence on One-Dimensional Periodic Manifold

In the previous section, we have derived the variational formulation for an electrical circuit. In view of the modelling of composite structures which includes periodically distributed electrical circuits, we will assume that the length of the period is small. The homogenization process consists of passing to the limit in the equations when this length vanishes. The set of equations derived from this asymptotic method is called the homogenized problem.

In this section, we describe a mathematical tool: the two-scale convergence based on the two-scale transformation introduced in Ref. 5. This tool is well-suited for the derivation of the homogenized model for electrical circuits.

3.1. *Definition of two-scale convergence*

Now Θ^ε is indexed by ε because it is a periodic network. Its period length in each direction is assumed to be equal to ε. It is assumed that $\varepsilon \in \mathbb{N}^{-1} = \{1/N, N \in \mathbb{N}^*$ such that $N > 2\}$, and that $\Theta^\varepsilon \subset \overline{\Omega} = [0,1]^n$ (see Fig. 3). For $N = 1/\varepsilon$, the square Ω and the circuit Θ^ε are divided into N^n cells indexed by $\mathbf{i} \in I^\varepsilon = \{\mathbf{i} = (i_1, \ldots, i_n) \in \{0, \ldots, N-1\}^n\}$ denoted by $Y_\mathbf{i}^\varepsilon$ and $T_\mathbf{i}^\varepsilon$. The center of $Y_\mathbf{i}^\varepsilon$ is denoted by $\mathbf{x}_\mathbf{i}^\varepsilon$. A translation and an expansion by $1/\varepsilon$ of $Y_\mathbf{i}^\varepsilon$ and $T_\mathbf{i}^\varepsilon$ give $Y =]-1/2, 1/2[^n$ and $T \subset \overline{Y}$ (see Fig. 3 for an example). Remark that every $\mathbf{x} \in T_\mathbf{i}^\varepsilon$ may be expressed as: $\mathbf{x} = \mathbf{x}_\mathbf{i}^\varepsilon + \varepsilon\mathbf{y}$ where $\mathbf{y} \in T$.

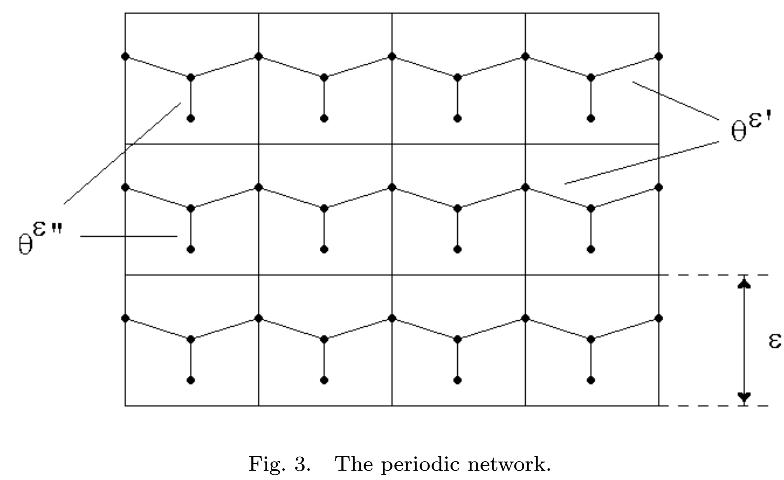

Fig. 3. The periodic network.

The Lebesgue measures on $\Omega \times T$ and on Θ^ε are denoted by $dl(\mathbf{y})\,d\mathbf{z}$ and $dl(\mathbf{x})$.

Definition. (Arbogast, Douglas and Hornung) For a function $v \in L^1(\Theta^\varepsilon)$, the two-scale transformation $\hat{v}^\varepsilon$ of v is defined on $\Omega \times T$ by $\hat{v}^\varepsilon(\mathbf{z}, \mathbf{y}) = v(\mathbf{x})$ where $\mathbf{x}, \mathbf{y}$ and

$\mathbf{z}$ are linked by the following relation: for any $\mathbf{i} \in I^\varepsilon$ and any $\mathbf{y} \in T$, $\mathbf{x} = \mathbf{x}_\mathbf{i}^\varepsilon + \varepsilon\mathbf{y}$ where $\mathbf{z}$ is any point belonging to $Y_\mathbf{i}^\varepsilon$.

Remark. The map which transforms Θ^ε into $\Omega \times T$ defined by $\mathbf{x} \mapsto (\mathbf{z}, \mathbf{y})$ is called the two-scale transformation.

The basic property of the two-scale transformation is:

Proposition 3.1. *If $v \in L^1(\Theta^\varepsilon)$, then $\hat{v}^\varepsilon \in L^1(\Omega \times T)$ and $\varepsilon^{1-n}\|\hat{v}^\varepsilon\|_{L^1(\Omega\times T)} = \|v^\varepsilon\|_{L^1(\Theta^\varepsilon)}$.*

For $p \in]0, \infty]$, let us introduce the definition of the two-scale convergence in L^p.

Corollary 3.2. *For every $p \in]0, \infty]$, if $v^\varepsilon \in L^p(\Theta^\varepsilon)$, then $\hat{v}^\varepsilon \in L^p(\Omega \times T)$ and $\|v^\varepsilon\|^p_{L^p(\Theta^\varepsilon)} = \varepsilon^{1-n}\|\hat{v}^\varepsilon\|^p_{L^p(\Omega\times T)}$.*

Definition. If $(v^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ is a sequence of functions defined on Θ^ε such that its two-scale transformation $(\hat{v}^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ converges for the $L^p(\Omega \times T)$ topology, when ε vanishes, towards some function $v \in L^p(\Omega \times T)$, then the sequence $(v^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ is said to be two-scale L^p convergent towards v. This convergence is strong if $\hat{v}^\varepsilon$ converges strongly and weak if $\hat{v}^\varepsilon$ converges weakly.

3.2. *Two-scale convergence of a derivative*

In this section, we give the expression of the limit of the tangential derivative of a function defined on Θ^ε. It is useful because it allows one to pass to the limit in the variational formulation of the electrical network.

In each point $\mathbf{x} \in \Theta^\varepsilon$ or $\mathbf{y} \in T$, the tangential derivatives of a function ψ on Θ^ε and T are denoted by the same notation $\nabla_\tau\psi$.

Let us define some functional spaces:

$$H^1(\Theta^\varepsilon) = \{\psi \in L^2(\Theta^\varepsilon), \nabla_\tau\psi \in L^2(\Theta^\varepsilon)\}\,,$$

$$L^2(\Omega; H^1_\sharp(T)) = \{\psi \in L^2(\Omega \times T), \nabla_\tau\psi \in L^2(\Omega \times T), \psi \text{ is } Y\text{-periodic}\}\,.$$

The subset $T' \subset T$ is composed of all paths t' going through Y from one side to the opposite one and being periodic. The complementary set of T' in T is denoted by $T'' = T - T'$. The subsets $\Theta^{\varepsilon\prime}$, $\Theta^{\varepsilon\prime\prime}$ of Θ^ε are such that $\Omega \times T'$ and $\Omega \times T''$ are the ranges of $\Theta^{\varepsilon\prime}$ and $\Theta^{\varepsilon\prime\prime}$ by the two-scale transformation. We denote by i the index of the normal direction of the faces Y where T' meets ∂Y. The ith component of the external normal $\mathbf{n}_\Omega$ to $\partial\Omega$ is denoted by $n_{\Omega i}$. The path T'' is such that its extremities do not belong to ∂Y. For an example of such paths, see Figs. 3 and 4.

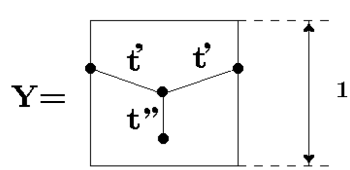

Fig. 4. The reference cell.

Let us define:

$$H^1_\tau(\Omega, T') = \{\psi \in L^2(\Omega \times T') \text{ such that on each path } t' \subset T',$$
$$\nabla_{\mathbf{z}}\psi_{|t'} \cdot \tau^0_{|t'} \in L^2(\Omega \times t') \text{ and } \psi \text{ is independent of } \mathbf{y} \text{ on } t'\}$$

where t' crosses Y from one side to the opposite side and is periodic, and $\tau^0_{|t'} = \int_{t'} \tau(\mathbf{y})\, dl(\mathbf{y})$.

Theorem 2. (i) *Consider a sequence* $(\varphi^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ *defined on* $\Theta^{\varepsilon\prime}$ *such that*

$$\varepsilon^{n-1}||\varphi^\varepsilon||^2_{H^1(\Theta^{\varepsilon\prime})} \leq C\,,$$

then there exists an extracted subsequence $(\varphi^\varepsilon)_\varepsilon$ *such that*:

$$\varphi^\varepsilon \rightharpoonup \varphi^0$$

and on each path $t' \subset T'$ *which crosses* Y *and is periodic*

$$\nabla^\varepsilon_\tau \varphi_{|t'} \rightharpoonup \nabla_{\mathbf{z}}\varphi^0_{|t'} \cdot \tau^0_{|t'} + \nabla_\tau \varphi^1_{|t'}\,,$$

where the convergences are two-scale weak in $L^2(\Omega \times T')$. *Here* $\varphi^0 \in H^1_\tau(\Omega, T')$ *and* $\varphi^1 \in L^2(\Omega; H^1_\sharp(t'))$.
(ii) *Consider a function* $\varphi_d \in L^2(\partial\Omega)$. *Moreover if*

$$\varphi^\varepsilon = \varphi_d \text{ on } \partial\Theta^{\varepsilon\prime} \cap \partial\Omega\,,$$

then

$$\varphi^0(\mathbf{z}, \mathbf{y}) n_{\Omega i} = \varphi_d(\mathbf{z}) n_{\Omega i} \text{ for } (\mathbf{z}, \mathbf{y}) \in \partial\Omega \times T'\,.$$

(iii) *Consider a sequence* $(\eta^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ *defined on* $\Theta^{\varepsilon\prime\prime}$ *such that*

$$(\varepsilon^{n-1}||\varepsilon\nabla_\tau\eta^\varepsilon||^2_{L^2(\Theta^{\varepsilon\prime\prime})} + \varepsilon^{n-1}||\eta^\varepsilon||^2_{L^2(\Theta^{\varepsilon\prime\prime})})_{\varepsilon\in\mathbb{N}^{-1}} \leq C\,.$$

There exists an extracted subsequence $(\eta^\varepsilon)_\varepsilon$ *such that*

$$(\eta^\varepsilon)_\varepsilon \rightharpoonup \eta^0 \text{ and } (\varepsilon\nabla_\tau\eta^\varepsilon)_\varepsilon \rightharpoonup \nabla_\tau\eta^0\,,$$

where the convergences are two-scale weak in $\Omega \times T''$. *Here* $\eta^0 \in L^2(\Omega; H^1(T''))$.
(iv) *Moreover, if*

$$T' \cap T'' \neq \emptyset, \text{ and } \varphi^\varepsilon = \eta^\varepsilon \text{ on } \Theta^{\varepsilon\prime} \cap \Theta^{\varepsilon\prime\prime}\,,$$

then

$$\varphi^0 = \eta^0 \text{ on } \Omega \times (T' \cap T'')\,.$$

4. Homogenization of Electrical Network Equations

We now consider that the electrical network is periodic and that its period is small. Different classes of assumptions leading to different classes of models may be discussed. It is out of our scope to derive all the possible models. We consider first a class of assumptions formulated in a general framework. We derive the general homogenized model related to this general class. Secondly, we consider a particular case which does not belong to the preceding general class, and we derive its homogenized model. Both models are based on results stated in previous sections.

4.1. *A general model*

In this section, we will consider some electrical circuits fulfilling the assumptions required for the existence and uniqueness of the solution stated in Sec. 2. We will formulate additional assumptions in order to insure that the solution is bounded in the sense of Theorem 2. Thus, using Theorem 2, we will pass to the limit in the variational formulation and will derive the homogenized model.

Let us assume that the network Θ and the subnetworks $(\Theta_k)_{k=0,\ldots,4}$ are εY-periodic. They are denoted by Θ^ε and $(\Theta_k^\varepsilon)_{k=0,\ldots,4}$. All the notations stated in Sec. 2 are now attached with an index ε. We also use the notations of Sec. 3 relative to two-scale convergence. Voltage sources, current sources, resistors, amplifier's inputs and outputs in T are denoted by $(T_k)_{k=0,\ldots,4}$.

The set σ_0^ε of nodes linked to the earth is also assumed to be periodic. In addition, there may exist a set γ_0^ε of nodes located on $\Theta^\varepsilon \cap \partial\Omega$ where the electrical potential is also equal to zero. The two-scale transformation of σ_0^ε is denoted by $\Omega \times S_0$. The two-scale transformation of γ_0^ε is defined on each face $(\Gamma^{k+} \cup \Gamma^{k-})_{k=1,\ldots,n}$ of the boundary $\partial\Omega$ and is denoted by $(\Gamma^{k+} \times S_0^{k+} \cup \Gamma^{k-} \times S_0^{k-})_{k=1,\ldots,n}$. Here Γ^{k+} (respectively Γ^{k-}) are the faces belonging to $\partial\Omega$ which are normal to the kth vector of the basis, and such that their external normals are oriented in positive (respectively negative) direction.

The assumptions for the existence, uniqueness and convergence of the solution are based on the same notations as those introduced in Sec. 2.3.

Definition. The definition of the circuit which is used for paths belonging to T coincide with the definition given in Sec. 2 taking into account the following exception. Two vertices located periodically on the boundary ∂Y are considered as one. Such circuits are denoted by $\beta_\sharp$.

First, let us introduce some restrictions on the configuration of the periodic network.

In this paper, a path which goes through a cell, is assumed to go from one side to the other. Other situations are out of the scope of this paper. Any edge belongs to ∂Y.

The set Θ^ε (respectively $(\Theta_k^\varepsilon)_{k=0,\ldots,4}$) is divided into two sets $\Theta^{\varepsilon\prime}$ and $\Theta^{\varepsilon\prime\prime}$ (respectively $(\Theta_k^{\varepsilon\prime})_{k=0,\ldots,4}$ and $(\Theta_k^{\varepsilon\prime\prime})_{k=0,\ldots,4}$). These two kinds of subsets are formed of paths t' and t'' defined in Sec. 3.

Remark. Other two-scale models (simpler ones) may be derived without such assumption made on the partition of Θ^ε in $\Theta^{\varepsilon\prime}$ and $\Theta^{\varepsilon\prime\prime}$. However, the above choice is motivated by the applications that we have in mind.

The following estimates have to be verified case by case. For their verification, we use the method described in Sec. 2 based on graph theory. We consider a partition of $\Theta_2^\varepsilon = \overline{\Theta}_2^\varepsilon \cup \widetilde{\Theta}_2^\varepsilon$ based on the assumptions (H1–H3). For $v^\varepsilon \in \mathbb{P}^0(\Theta_3^\varepsilon)$ and $f^\varepsilon \in \mathbb{P}^0(\overline{\Theta}_2^\varepsilon)$ such that

$$\int_{\beta^\varepsilon \cap \Theta_3^\varepsilon} v^\varepsilon \, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta^\varepsilon \text{ of } \Theta_0^\varepsilon \cup \Theta_3^\varepsilon \cup \Theta_4^\varepsilon$$

and

$$\int_{\beta^\varepsilon \cap \overline{\Theta}_2^\varepsilon} f^\varepsilon \, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta^\varepsilon \text{ of } \Theta_0^\varepsilon \cup \overline{\Theta}_2^\varepsilon ,$$

we need to construct a solution $w^\varepsilon \in \mathbb{P}^0(\Theta^\varepsilon - \Theta_1^\varepsilon)$, relative to v^ε and f^ε, of the linear system:

$$\begin{gathered} L^\varepsilon \nabla_\tau w^\varepsilon_{|\overline{\Theta}_2^\varepsilon} = f^\varepsilon_{|\overline{\Theta}_2^\varepsilon} , \\ L^\varepsilon \nabla_\tau w^\varepsilon_{|\Theta_0^\varepsilon} = 0 , \\ L^\varepsilon \nabla_\tau w^\varepsilon_{|e_4^\varepsilon l} - k L^\varepsilon \nabla_\tau w^\varepsilon_{|e_3^\varepsilon l} = v^\varepsilon_{|e_3^\varepsilon l} \text{ for every } l , \\ \int_{\beta^\varepsilon} L^\varepsilon \nabla_\tau w^\varepsilon \, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta^\varepsilon \text{ of } \Theta^\varepsilon - \Theta_1^\varepsilon . \end{gathered} \tag{9}$$

(H1bis) Consider w^ε the solution of (9). There exists a positive constant C such that

$$|w^\varepsilon|^2_{\Theta_2^{\varepsilon\prime} \cup \Theta_3^{\varepsilon\prime} \cup \Theta_4^{\varepsilon\prime}} + |\varepsilon w^\varepsilon|^2_{\Theta_2^{\varepsilon\prime\prime} \cup \Theta_3^{\varepsilon\prime\prime} \cup \Theta_4^{\varepsilon\prime\prime}} \leq C\Big(|f^\varepsilon|^2_{\overline{\Theta}_2^{\varepsilon\prime}} + |\varepsilon f^\varepsilon|^2_{\overline{\Theta}_2^{\varepsilon\prime\prime}} + |v^\varepsilon|^2_{\Theta_3^{\varepsilon\prime}} + |\varepsilon v^\varepsilon|^2_{\Theta_3^{\varepsilon\prime\prime}} \Big) .$$

(H3bis) There exists a positive constant C such that for every $(\psi, 0) \in \Psi^\varepsilon_{ad}(0)$ we have

$$|\nabla_\tau \psi|^2_{\Theta_3^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \psi|^2_{\Theta_3^{\varepsilon\prime\prime}} \leq C\Big(|\nabla_\tau \psi|^2_{\Theta_2^{\varepsilon\prime} \cup \Theta_4^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \psi|^2_{\Theta_2^{\varepsilon\prime\prime} \cup \Theta_4^{\varepsilon\prime\prime}} \Big) .$$

(H4bis) There exists a positive constant C such that for every $(\psi, 0) \in \Psi^\varepsilon_{ad}(0)$ we have

$$|\nabla_\tau \psi|^2_{\Theta_1^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \psi|^2_{\Theta_1^{\varepsilon\prime\prime}} \leq C\Big(|\nabla_\tau \psi|^2_{\Theta^{\varepsilon\prime} - \Theta_1^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \psi|^2_{\Theta^{\varepsilon\prime\prime} - \Theta_1^{\varepsilon\prime\prime}} \Big) .$$

(H5bis) There exists a positive constant C such that for every $(\psi, 0) \in \Psi^\varepsilon_{ad}(0)$ we have

$$|\psi|^2_{\Theta^{\varepsilon\prime} - \Theta_1^{\varepsilon\prime}} \leq C\Big(|\nabla_\tau \psi|^2_{\Theta^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \psi|^2_{\Theta^{\varepsilon\prime\prime}} \Big) .$$

(H6bis) There is no circuit $\beta_\sharp$ included in $T_4 \cup T_0$.

The following assumption is related to the two-scale convergence of the data.

(H8) The data k^ε, g^ε, i_d^ε and u_d^ε satisfy the following estimates and two-scale convergence.

For current sources,

$$\begin{gathered} \varepsilon^{n-1} |\varepsilon^{-1} i_d^\varepsilon|^2_{\Theta_1^{\varepsilon\prime}} + \varepsilon^{n-1} |\varepsilon^{-2} i_d^\varepsilon|^2_{\Theta_1^{\varepsilon\prime\prime}} + \varepsilon^{n-1} |u_d^\varepsilon|^2_{\Theta_0^{\varepsilon\prime}} + \varepsilon^{n-1} |\varepsilon u_d^\varepsilon|^2_{\Theta_0^{\varepsilon\prime\prime}} \leq C , \\ \varepsilon^{-1} \widehat{i_d^\varepsilon} \to i_d \text{ in } L^2(\Omega \times T_1'), \quad \varepsilon^{-2} \widehat{i_d^\varepsilon} \to i_d \text{ in } L^2(\Omega \times T_1'') \text{ weak} . \end{gathered}$$

For voltage sources,

$$\widehat{u}_d^\varepsilon \to u_d \text{ in } L^2(\Omega \times T_0'), \quad \varepsilon \widehat{u}_d^\varepsilon \to u_d \text{ in } L^2(\Omega \times T_0'') .$$

For immitances,

$$g^\varepsilon = \varepsilon^2 g^{\varepsilon\prime\prime} \text{ on } \Theta_2^{\varepsilon\prime\prime}, \widehat{g}^\varepsilon \to g \text{ in } L^\infty(\Omega \times T_2'') \text{ weak}*\,,$$

$$\widehat{g}^\varepsilon \to g \text{ in } L^\infty(\Omega \times T_2') \text{ weak} * \,.$$

For the amplifier coefficients:

$$\widehat{k}^\varepsilon \to k \text{ in } L^\infty(\Omega \times T_3) \text{ weak} * \,.$$

And for the length of the edges:

$$\varepsilon^{-1}\widehat{L}^\varepsilon \to L \text{ in } L^\infty(\Omega \times T) \text{ weak} * \,.$$

For $u_d \in L^2(\Omega; \mathbb{P}^0(T_0))$, let us introduce the admissible functions set $\Psi_{ad\sharp}(u_d)$ of functions (ψ^0, ψ^1, j) verifying:

$$\psi^0 \in L^2(\Omega; \mathbb{P}^1(T'')) \cap H^1_\tau(\Omega, T'), \psi^1 \in L^2(\Omega; \mathbb{P}^1_\sharp(T'))/\mathbb{R}$$

and

$$j \in L^2(\Omega; \mathbb{P}^0(T_4))\,.$$

In addition, (ψ^0, ψ^1) satisfy the voltage imposed conditions,

$$LD(\psi^0, \psi^1) = u_d \text{ on } \Omega \times T_0\,,$$

and the earth condition,

$$\psi^0 = 0 \text{ on } \{(\Omega \times S_0) \cup (\cup_{k=1}^n \Gamma^{k+} \times S_0^{k+} \cup \Gamma^{k-} \times S_0^{k-})\}\,.$$

Here the notation

$$D(\psi^0, \psi^1) = \nabla_\tau \psi^0 \text{ on } \Omega \times T'' \text{ and } = \nabla_{\mathbf{z}}\psi^0(\mathbf{z}, \mathbf{y}) \cdot \tau^0 + \nabla_\tau \psi^1(\mathbf{z}, \mathbf{y}) \text{ on } \Omega \times T'$$

is used.

Now we are ready for the statement of the main result. This is the formulation of the general homogenized model related to the periodic electrical network. Let us define the three bilinear forms,

$$a^0((\varphi^0, \varphi^1), (\psi^0, \psi^1)) = \int_{\Omega \times T_2} gLD(\varphi^0, \varphi^1) D(\psi^0, \psi^1)\, dl(\mathbf{y})\, d\mathbf{z}\,,$$

$$b_1^0(i, (\psi^0, \psi^1)) = \int_{\Omega \times T_4} iD(\psi^0, \psi^1)\, dl(\mathbf{y})\, d\mathbf{z},$$

$$b_2^0(j, (\varphi^0, \varphi^1)) = \int_{\Omega \times T_3} LkD(\varphi^0, \varphi^1) j\, dl(\mathbf{y})\, d\mathbf{z} - \int_{\Omega \times T_4} LD(\varphi^0, \varphi^1) j\, dl(\mathbf{y})\, d\mathbf{z}\,,$$

and the linear form,

$$l^0((\psi^0, \psi^1)) = -\int_{\Omega \times T_1} i_d D(\psi^0, \psi^1)\, dl(\mathbf{y})\, d\mathbf{z}\,.$$

Consider $(\varphi^\varepsilon, i^\varepsilon)_{\varepsilon \in \mathbb{N}^{-1}}$ solution of (6) where Θ and Θ_k are replaced by Θ^ε and Θ_k^ε.

Theorem 3. *Under the assumptions* (H1–H8) *and* (H2bis–H6bis), *the following L^2 two-scale convergences hold*

$$\varphi^\varepsilon \rightharpoonup \varphi^0,\ \nabla_\tau^\varepsilon \varphi^\varepsilon \rightharpoonup \nabla_{\mathbf{z}}\varphi^0 + \nabla_\tau \varphi^1 \quad \textit{and} \quad \varepsilon^{-1} i^\varepsilon \rightharpoonup i \text{ on } \Omega \times T' .$$

and

$$\varepsilon \nabla_\tau^\varepsilon \varphi^\varepsilon \rightharpoonup \nabla_\tau \varphi^0 \quad \textit{and} \quad \varepsilon^{-2} i^\varepsilon \rightharpoonup i \textit{ on } \Omega \times T'' ,$$

where $(\varphi^0, \varphi^1, i) \in \Psi_{ad\sharp}(u_d)$ *is the unique solution of*

$$\begin{aligned} a^0((\varphi^0,\varphi^1),(\psi^0,\psi^1)) + b_1^0(i,(\psi^0,\psi^1)) &= l^0((\psi^0,\psi^1)) , \\ b_2^0(j,(\varphi^0,\varphi^1)) &= 0 , \end{aligned} \tag{10}$$

for all $(\psi^0, \psi^1, j) \in \Psi_{ad\sharp}(0)$.

4.2. *A particular model of homogenized circuit*

In this section we exhibit a particular example which does not belong to the general framework that we have considered in the previous section. Here, the coefficients of the amplifiers are not bounded.

Consider the periodic network in Fig. 5. This network is two-dimensional, i.e. $n = 2$, but is periodic only in the direction z_1. In order to apply our theory to this case, we consider an ε-periodic repetition of this network in the second direction z_2. This leads to a two-dimensional model, which will be independent of z_2.

Let us assume that, for each $e \subset T$, $|e| = 1$. Here $T' = \{e_2^1, e_3^1, e_3^2\}$ and $T'' = \{e_1, e_2^2, e_4^1, e_4^2\}$. Consider the sequence of solutions $(\varphi^\varepsilon, i^\varepsilon)_{\varepsilon \in \mathbb{N}^{-1}}$ of (6) related to this electrical network.

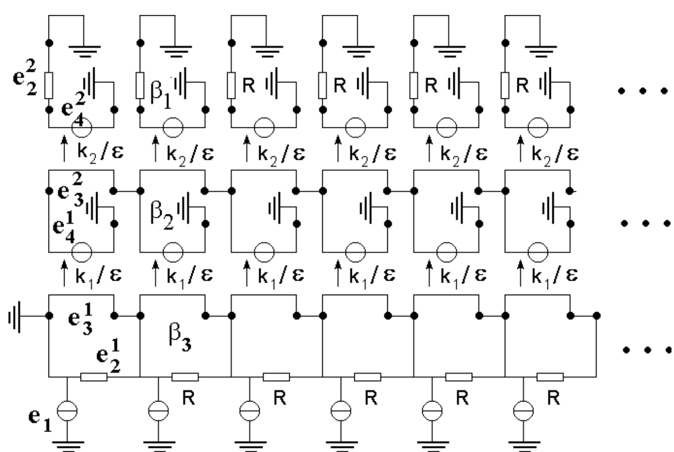

Fig. 5. An example of periodic network.

Theorem 4. *The conclusions stated in Theorem* 3 *are still true, and the transfer function between* $\Omega \times e_2^1$ *and* $\Omega \times e_2^2$ *is*:

$$\nabla_\tau \varphi^0_{|\Omega \times e_2^2} = k_1 k_2 \partial^2_{z_1 z_1} \varphi^0_{|\Omega \times e_2^1} .$$

5. Proof of Theorem 1

The proof is divided into two steps. First, we prove that the variational formulation admits a unique solution if and only if the assumptions (H1–H7) are satisfied. Then, we prove the equivalence between the variational formulation and Eqs. (1)–(4).

5.1. *Step 1*

Since (H6) is satisfied, there does not exist any circuit included in Θ_0. Therefore, for all $u_d \in \mathbb{P}^0(\Theta_0)$, $\Psi_{ad}(u_d) \neq \emptyset$. Consider $\widetilde{\varphi} \in \Psi_{ad}(u_d)$ and the problem verified by $\overline{\varphi} = \varphi - \widetilde{\varphi} \in \Psi_{ad}(0)$:

$$a(\overline{\varphi}, \psi) + b_1(i, \psi) = l(\psi) - a(\widetilde{\varphi}, \psi) ,$$

$$b_2(j, \overline{\varphi}) = -b_2(j, \widetilde{\varphi})$$

$$\text{for every } (\psi, j) \in \Psi_{ad}(0) .$$

The problems of existence and uniqueness of φ or $\overline{\varphi}$ are equivalent. Thus, in the following, we consider only the case where $u_d = 0$.

The variational formulation admits a unique solution if and only if the following properties are satisfied there:
(i) Consider the norm $||\psi||^2 = \int_\Theta |\nabla_\tau \psi|^2 + |\psi|^2 \, dl(\mathbf{x})$ on the $(\psi, .) \in \Psi_{ad}(0)$. For all $k \in \mathbb{P}^0(\Theta_3)$, there exists a positive constant α such that $\forall \, (\psi, 0) \in \Psi_{ad}(0)$ verifying $\nabla_\tau \psi = 0$ on Θ_4, there exists $(\varphi, 0) \in \Psi_{ad}(0)$ different from zero such that for every couple $(e_3^l, e_4^l) \in \Theta_3 \times \Theta_4$, φ satisfies $L \nabla_\tau \varphi_{|e_4^l} = k L \nabla_\tau \varphi_{|e_3^l}$ and

$$\int_{\Theta_2} g L \nabla_\tau \varphi \nabla_\tau \psi \, dl(\mathbf{x}) \geq \alpha ||\varphi|| \cdot ||\psi|| . \tag{11}$$

(ii) For all $k \in \mathbb{P}^0(\Theta_3)$, there exists a positive constant β such that $\forall \, (\varphi, 0) \in \Psi_{ad}(0)$ verifying $L \nabla_\tau \varphi_{|e_4^l} = k L \nabla_\tau \varphi_{|e_3^l}$ on each couple $(e_3^l, e_4^l) \in \Theta_3 \times \Theta_4$, there exists $(\psi, 0) \in \Psi_{ad}(0)$ different from zero verifying $\nabla_\tau \psi = 0$ on Θ_4 and

$$\int_{\Theta_2} g L \nabla_\tau \varphi \nabla_\tau \psi \, dl(\mathbf{x}) \geq \beta ||\varphi|| \cdot ||\psi|| . \tag{12}$$

(iii) There exists a strictly positive constant γ_1 such that for every $j \in \mathbb{P}^0(\Theta_4)$, there exists $(\varphi, 0) \in \Psi_{ad}(0)$ such that

$$\int_{\Theta_4} j L \nabla_\tau \varphi \, dl(\mathbf{x}) \geq \gamma_1 ||\varphi|| \cdot |j|_{L^2(\Theta_4)} . \tag{13}$$

(iv) There exists a strictly positive constant γ_2 such that for every $j \in \mathbb{P}^0(\Theta_4)$, there exists $(\varphi, 0) \in \Psi_{ad}(0)$ such that

$$\int_{\Theta_4} jL\nabla_\tau\varphi\, dl(\mathbf{x}) - \int_{\Theta_3} kjL\nabla_\tau\varphi\, dl(\mathbf{x}) \geq \gamma_2||\varphi|| \cdot |j|_{L^2(\Theta_4)}\,, \tag{14}$$

where the values of j on each e_3^l and e_4^l are the same.
(v) The linear form $l(\psi)$ is continuous.

The point (v) is a straightforward consequence of the assumption (H4).

Lemma 5.1. *Let us assume that* (H1) *is fulfilled, and let us consider a node* $e \in \widetilde{\Theta}_2 \cup \Theta_3 \cup \Theta_4$.
(i) *For every* $(\varphi, 0) \in \Psi_{ad}(0)$ *such that* $(L\nabla_\tau\varphi)_{|\Theta_4} - (kL\nabla_\tau\varphi)_{|\Theta_3} = 0$ *we have*:

$$|L\nabla_\tau\varphi|_e \leq \sum_{e' \in \overline{\Theta}_2} C_0(e, e')|L\nabla_\tau\varphi|_{e'} \quad \textit{thus } |L\nabla_\tau\varphi|^2_{\widetilde{\Theta}_2 \cup \Theta_3 \cup \Theta_4} \leq C_1|L\nabla_\tau\varphi|^2_{\overline{\Theta}_2}\,, \tag{15}$$

where $C_0(e, e')$ *and* C_1 *are some positive constants.*
(ii) *Let us assume, in addition, that* (H2) *is fulfilled. For each* $e \in \widetilde{\Theta}_2$ *there exists some constants* $C_2(e, e')$ *related to* $e' \in \Theta_3 \cap Z(e)$ *such that*:

$$|u|_e \leq \sum_{e' \in \Theta_3 \cap Z(e)} C_1(e, e')|v|_{e'} \quad \textit{and} \quad |u|_{\widetilde{\Theta}_2} \leq C|v|_{\Theta_3^1}\,, \tag{16}$$

where Θ_3^1 *is the subset of* Θ_3 *when* $\alpha = \alpha_0$.

Proof. Let us denote by $f_{|\overline{\Theta}_2}$ the derivative $L\nabla_\tau\varphi_{|\overline{\Theta}_2}$. Since $(\varphi, 0) \in \Psi_{ad}(0)$, φ is solution of

$$L\nabla_\tau\varphi_{|\overline{\Theta}_2} = f_{|\overline{\Theta}_2}\,,$$

$$L\nabla_\tau\varphi_{|\Theta_0} = 0\,,$$

$$L\nabla_\tau\varphi_{|e_4^l} - kL\nabla_\tau\varphi_{|e_3^l} = 0 \text{ for every } l\,,$$

$$\int_\beta L\nabla_\tau\varphi\, dl(\mathbf{x}) = 0 \text{ for each circuit } \beta \text{ of } \Theta - \Theta_1\,.$$

Using the assumption (H1)(i), one knows that this system has a unique solution $\nabla_\tau\varphi_{|\Theta-(\Theta_1\cup\widetilde{\Theta}_2)}$ which is continuous with respect to $f_{|\overline{\Theta}_2}$.

For $e \in \widetilde{\Theta}_2 \cup \Theta_3 \cup \Theta_4$, the assumption (H1)(i) means that $L\nabla_\tau\varphi_{|e}$ is a linear combination of $\nabla_\tau\varphi_{|\Theta-(\Theta_1\cup\widetilde{\Theta}_2)}$. It implies in turn that $L\nabla_\tau\varphi_{|e}$ is also continuous with respect to $f_{|\overline{\Theta}_2}$. This proves (i).

Let us prove (ii). From the definition of $Z(e)$, $u_{|e}$ is a unique linear combination of $(u_{|e'})_{e' \in Z(e)-e}$. Here continuity with respect to $f_{|\overline{\Theta}_2}$ is replaced by continuity with respect to $v_{|\Theta_3\cap Z(e)}$. ☐

Lemma 5.2. (i) *Let us assume that* (H3) *is fulfilled. For every* $(\psi, 0) \in \Psi_{ad}(0)$ *satisfying* $\nabla_\tau \psi_{|\Theta_4} = 0$, *there exists a constant* C *such that*

$$|\nabla_\tau \psi|^2_{\Theta_3} \leq C|\nabla_\tau \psi|^2_{\overline{\Theta}_2} \quad \text{and} \quad |\nabla_\tau \psi|^2_{\Theta_3^1} \leq C|\nabla_\tau \psi|^2_{\overline{\Theta}_2^1} . \tag{17}$$

(ii) *Let us assume that* (H4) *is fulfilled. There exists a constant* C *such that for every* $(\psi, 0) \in \Psi_{ad}(0)$:

$$|\nabla_\tau \psi|^2_{\Theta_1} \leq C|\nabla_\tau \psi|^2_{\Theta - \Theta_1} .$$

Proof. Let us prove (i). The first estimate is a straightforward consequence of (H3). Let us prove the second estimate of (i). From (H3), for any $e \in \Theta_3^1$ there exists a circuit $\beta \subset \{e\} \cup \Theta_0 \cup \overline{\Theta}_2^1 \cup \Theta_4$. The inequality follows easily. The proof of (ii) is straightforward. □

Lemma 5.3. *If the assumption* (H5) *is satisfied, then the semi-norm* $|\nabla_\tau \psi|^2_{\Theta - \Theta_1}$ *is a norm on* $\Psi_{ad}(0)$.

Proof. The proof is straightforward. □

Lemma 5.4. *If the assumptions* (H1–H5) *are satisfied then the properties* (11) *and* (12) *are satisfied.*

Proof. Let us prove (11). Let us consider the function α defined in (H2).

Let us pose $v = k\alpha L \nabla_\tau \psi$ on Θ_3. Consider the solution $u \in \mathbb{P}^0(\Theta - \Theta_1)$ of (8) and the unique $(\varphi, 0) \in \Psi_{ad}(0)$ such that

$$L\nabla_\tau \varphi = u + \alpha L \nabla_\tau \psi \text{ on } \Theta - \Theta_1 .$$

The uniqueness of φ results in (H5). The existence of φ will be a consequence of $\nabla_\tau \varphi_{|\Theta_0} = 0$ and $\int_\beta L\nabla_\tau \varphi \, dl(\mathbf{x}) = 0$. The equality $\nabla_\tau \varphi_{|\Theta_0} = 0$ is immediate. In another way, since α is constant on each circuit β:

$$\int_\beta L\nabla_\tau \varphi \, dl(\mathbf{x}) = \int_\beta u \, dl(\mathbf{x}) + \alpha \int_\beta L \nabla_\tau \psi \, dl(\mathbf{x}) = 0 .$$

Let us verify that $(L\nabla_\tau \varphi)_{|\Theta_4} - (kL\nabla_\tau \varphi)_{|\Theta_3} = 0$.

$$\begin{aligned}(L\nabla_\tau \varphi)_{|\Theta_4} - (kL\nabla_\tau \varphi)_{|\Theta_3} &= u_{|\Theta_4} - ku_{|\Theta_3} - (kL\alpha\nabla_\tau \psi)_{|\Theta_3} \\ &= (k\alpha L \nabla_\tau \psi)_{|\Theta_3} - (k\alpha L\nabla_\tau \psi)_{|\Theta_3} = 0 .\end{aligned}$$

Now, let us derive the inequality (11). Let us denote by $\overline{\Theta}_2^1$ and Θ_3^1 (respectively $\overline{\Theta}_2^2$ and Θ_3^2) the subsets of $\overline{\Theta}_2$ and Θ_3 where $\alpha = \alpha_0$ (respectively where $\alpha = 1$).

$$\begin{aligned}\int_{\Theta_2} gL\nabla_\tau \psi \nabla_\tau \varphi \, dl(\mathbf{x}) &= \int_{\Theta_2} g\alpha L |\nabla_\tau \psi|^2 \, dl(\mathbf{x}) + \int_{\widetilde{\Theta}_2} g \nabla_\tau \psi u \, dl(\mathbf{x}) \\ &\geq \frac{1}{2} |(gL)^{1/2} \nabla_\tau \psi|^2_{\widetilde{\Theta}_2} + \alpha_0 |(gL)^{1/2} \nabla_\tau \psi|^2_{\overline{\Theta}_2^1} \\ &\quad + |(gL)^{1/2} \nabla_\tau \psi|^2_{\overline{\Theta}_2^2} - \frac{1}{2} |(L^{-1}g)^{1/2} u|^2_{\widetilde{\Theta}_2} .\end{aligned}$$

From Lemmas 5.1(ii) and 5.2(i) there exist some constants C_2 and C_3 such that:

$$|(L^{-1}g)^{1/2}u|^2_{\tilde{\Theta}_2} \leq C_2\alpha_0^2|L\nabla_\tau\psi|^2_{\Theta_3^1} \leq C_3\alpha_0^2|(gL)^{1/2}\nabla_\tau\psi|^2_{\overline{\Theta}_2^1}$$

thus

$$\int_{\Theta_2} gL\nabla_\tau\psi\nabla_\tau\varphi\, dl(\mathbf{x}) \geq \frac{1}{2}|(gL)^{1/2}\nabla_\tau\psi|^2_{\tilde{\Theta}_2}$$
$$+ (\alpha_0 - C_3\alpha_0^2)|(gL)^{1/2}\nabla_\tau\psi|^2_{\overline{\Theta}_2^1} + |(gL)^{1/2}\nabla_\tau\psi|^2_{\overline{\Theta}_2^2}\,.$$

Let us pose $\alpha_0 < 1/C_3$, thus there exists a constant C_4 such that:

$$\int_{\Theta_2} gL\nabla_\tau\psi\nabla_\tau\varphi\, dl(\mathbf{x}) \geq C_4|L\nabla_\tau\psi|^2_{\Theta_2}\,.$$
$$\geq C_4|L\nabla_\tau\psi|_{\Theta_2}|L\nabla_\tau\psi|_{\overline{\Theta}_2} \geq C_5|L\nabla_\tau\psi|_{\Theta_2}|L\nabla_\tau\varphi|_{\overline{\Theta}_2}\,.$$

Applying Lemmas 5.1(i) and 5.2(i), there exists a constant C_6 such that:

$$\int_{\Theta_2} gL\nabla_\tau\psi\nabla_\tau\varphi\, dl(\mathbf{x}) \geq C_6|L\nabla_\tau\psi|_\Theta \cdot |L\nabla_\tau\varphi|_\Theta\,.$$

In conclusion,

$$\int_{\Theta_2} gL\nabla_\tau\psi\nabla_\tau\varphi\, dl(\mathbf{x}) \geq C||\psi|| \cdot ||\varphi||\,.$$

This is (11).

For the proof of (12), we pose

$$L\nabla_\tau\psi = \alpha^{-1}(L\nabla_\tau\varphi - u) \text{ on } \Theta\,.$$

The end of the derivation of (12) is the same derivation of (11). This ends the proof of Lemma 5.5. □

Let us prove (13). For every $i \in \mathbb{P}^0(\Theta_4)$ there exists φ such that $(\varphi, i) \in \Psi_{ad}(0)$ and $L\nabla_\tau\varphi = i$ on Θ_4, if and only if, for every circuit $\beta \subset \Theta_4 \cup \Theta_0$, i satisfy the compatibility condition $\int_{\beta\cap\Theta_4} i\ dl(\mathbf{x}) = 0$. Since there exists no circuit in $\Theta_4 \cup \Theta_0$, this compatibility condition never occurs. Using the assumption (H7) we can pose $\nabla_\tau\varphi = 0$ on $(\Theta_1 \cup \Theta_2 \cup \Theta_3) - \Theta^*$. The value of $\nabla_\tau\varphi$ on Θ^* is determined by the circuit relations $\int_\beta L\nabla_\tau\varphi\ dl(\mathbf{x}) = 0$ for each β such that $\beta \cap \Theta_4 \neq \emptyset$. Thus, (13) results from the inequality $|\nabla_\tau\varphi|_\Theta \leq |\nabla_\tau\varphi|_{\Theta_4}$.

Finally, let us prove (14). Using the assumption (H1), we pose $v_{|\Theta_3} = j$ and $|e|\nabla_\tau\varphi = u$ on Θ, this implies that $|\nabla_\tau\varphi|_\Theta \leq C|j|_{\Theta_3}$, which leads to (14).

5.2. *Step 2*

Equivalence between the variational formulation and Eqs. (1)–(4).

Consider (2). Since $i = g|e|\nabla_\tau \varphi$ on Θ_2, $i = 0$ on Θ_3 and $i = i_d$ on Θ_1, we have

$$\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi\, dl(\mathbf{x}) + \int_{\Theta_0} i\nabla_\tau\psi\, dl(\mathbf{x}) + \int_{\Theta_4} i\nabla_\tau\psi\, dl(\mathbf{x}) = -\int_{\Theta_1} i_d\nabla_\tau\psi\, dl(\mathbf{x})$$

$$\text{for all } \psi \in \mathbb{P}^1(\Theta) \text{ such that } \psi = 0 \text{ on } \sigma_0\,.$$

The variational formulation of (4_2) and of the condition $u = u_d$ on Θ_0 are:

$$\int_{\Theta_3} Lk\nabla_\tau\varphi j\, dl(\mathbf{x}) + \int_{\Theta_4} L\nabla_\tau\varphi j\, dl(\mathbf{x}) = 0 \text{ for every } j \in \mathbb{P}^0(\Theta_3)$$

$$\text{and } \int_{\Theta_0} L\nabla_\tau\varphi j_0\, dl(\mathbf{x}) = \int_{\Theta_0} u_d j_0\, dl(\mathbf{x}) \text{ for every } j_0 \in \mathbb{P}^0(\Theta_0)\,,$$

where j takes the same value on each e_3^l and e_4^l belonging to the same amplifier. Here i on Θ_0 plays the role of a Lagrange multiplier. Equivalently, $(\varphi, i) \in \Psi_{ad}(u_d)$ is the unique solution of:

$$\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi\, dl(\mathbf{x}) + \int_{\Theta_4} i\nabla_\tau\psi\, dl(\mathbf{x}) = -\int_{\Theta_1} i_d\nabla_\tau\psi\, dl(\mathbf{x})$$

$$\int_{\Theta_3} Lk\nabla_\tau\varphi j\, dl(\mathbf{x}) + \int_{\Theta_4} L\nabla_\tau\varphi j\, dl(\mathbf{x}) = 0 \quad \text{for every } (\psi, j) \in \Psi_{ad}(0)\,.$$

The proof of the converse is straightforward. □

6. Proof of Theorem 2

First, let us prove property 1 and give some of its consequences.

6.1. *Proof of Proposition* 3.1

If $v \in L^1(\Theta^\varepsilon)$,

$$\|v\|_{L^1(\Theta^\varepsilon)} = \int_{\Theta^\varepsilon} |v(\mathbf{x})|\, dl(\mathbf{x}) = \sum_{\mathbf{i}\in I^\varepsilon} \int_{T_\mathbf{i}^\varepsilon} |v(\mathbf{x})|\, dl(\mathbf{x})\,,$$

since $\hat{v}^\varepsilon$ is independent of $\mathbf{z}$ in each set $T_\mathbf{i}^\varepsilon$, then

$$\begin{aligned}\sum_{\mathbf{i}\in I^\varepsilon} \varepsilon \int_T |\hat{v}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon, \mathbf{y})|\, dl(\mathbf{y}) &= \varepsilon^{1-n} \sum_{\mathbf{i}\in I^\varepsilon} \int_{Y_\mathbf{i}^\varepsilon} d\mathbf{z} \int_T |\hat{v}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon, \mathbf{y})|\, dl(\mathbf{y}) \\ &= \varepsilon^{1-n} \int_\Omega \int_T |\hat{v}^\varepsilon(\mathbf{z}, \mathbf{y})|\, dl(\mathbf{y})\, d\mathbf{z}\,.\end{aligned}$$

This proves property 3.1. □

Let us consider $(\varepsilon^{(n-1)/2} v^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ a bounded sequence of $L^2(\Theta^\varepsilon)$. Using Corollary 3.2 and the two-scale convergence definition, one can extract a subsequence $(v^\varepsilon)_\varepsilon$ of $(v^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ which two-scale converges in L^2 weakly towards some $v \in L^2(\Omega \times T)$. The mean value $v^{0\varepsilon}(\mathbf{z}) = \int_T \widehat{v}^\varepsilon(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})$ is also bounded in $L^2(\Omega)$, then one can extract another subsequence still denoted by $(v^\varepsilon)_\varepsilon$ which converges in $L^2(\Omega)$ weakly towards some $v^0 \in L^2(\Omega)$.

Lemma 6.1. *For every* $\mathbf{z} \in \Omega$,

$$v^0(\mathbf{z}) = \int_T v(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})\,.$$

Proof. Using the two-scale convergence of $(v^\varepsilon)_\varepsilon$, for every $w \in L^2(\Omega)$,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_T \widehat{v}^\varepsilon(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y}) w(\mathbf{z}) d\mathbf{z} = \int_\Omega \int_T v(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y}) w(\mathbf{z})\, d\mathbf{z}\,.$$

Using the weak convergence of $(v^{0\varepsilon})_\varepsilon$,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_T \widehat{v}^\varepsilon(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y}) w(\mathbf{z})\, d\mathbf{z} = \lim_{\varepsilon\to 0} \int_\Omega v^{0\varepsilon}(\mathbf{z}) w(\mathbf{z})\, d\mathbf{z} = \int_\Omega v^0(\mathbf{z}) w(\mathbf{z})\, d\mathbf{z}\,.$$

Thus, $v^0(\mathbf{z}) = \int_T v(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})$. □

6.2. *Proof of Theorem* 2($\mathbf{i}_1$)

Let us prove part (i_1) of Theorem 2. Corollary 3.2 applied to $\Theta^{\varepsilon\prime}$ implies that there exists an extracted subsequence $(\varphi^\varepsilon)_\varepsilon$ of $(\varphi^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ which two-scale converges in L^2 weakly towards a $\varphi^0(\mathbf{z}, \mathbf{y})$. In addition,

$$\varepsilon^{(n-1)/2} \|\nabla_T^\varepsilon \varphi^\varepsilon\|_{L^2(\Theta^{\varepsilon\prime})} = \|\varepsilon^{-1} \nabla_\tau \widehat{\varphi}^\varepsilon\|_{L^2(\Omega\times T')} \leq C\,,$$

thus, $(\nabla_\tau \widehat{\varphi}^\varepsilon)_\varepsilon$ strongly converges towards 0 in $L^2(\Omega \times T')$. Thus $\nabla_\tau \varphi^0 = 0$ in $\Omega \times T'$. This means that φ^0 is independent of $\mathbf{y}$. The fact that $\nabla_\mathbf{z} \varphi^0(\mathbf{z}, \mathbf{y}) \tau^0 \in L^2(\Omega \times T')$ will be proved later.

6.3. *Proof of Theorem* 2($\mathbf{i}_2$)

Let us establish the two-scale limit of $f^\varepsilon = \varepsilon^{(n-1)/2} \nabla_T^\varepsilon \varphi^\varepsilon$. The extremities of T' are denoted by $\mathbf{s}^-$ and $\mathbf{s}^+$. They are located periodically on ∂Y. We start from the equality for every $\psi \in L^2(\Omega; H^1(T'))$:

$$\begin{aligned}
\int_{\Omega\times T'} \widehat{f^\varepsilon}(\mathbf{z}, \mathbf{y}) \psi(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})\, d\mathbf{z} &= \int_{\Omega\times T'} \frac{1}{\varepsilon} \nabla_\tau \widehat{\varphi}^\varepsilon(\mathbf{z}, \mathbf{y}) \psi(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})\, d\mathbf{z} \\
&= -\int_{\Omega\times T'} \frac{1}{\varepsilon} \widehat{\varphi}^\varepsilon(\mathbf{z}, \mathbf{y}) \nabla_\tau \psi(\mathbf{z}, \mathbf{y})\, dl(\mathbf{y})\, d\mathbf{z} \\
&\quad + \int_\Omega \frac{1}{\varepsilon} [\widehat{\varphi}^\varepsilon(\mathbf{z}, \mathbf{y}) \psi(\mathbf{z}, \mathbf{y})]_{\mathbf{s}^-}^{\mathbf{s}^+}\, d\mathbf{z}\,. \qquad (18)
\end{aligned}$$

Let us consider the last term. For the sake of simplicity, we assume that the extremities $\mathbf{s}^+$ and $\mathbf{s}^-$ are located on the faces having their normal in the direction of the first vector $\mathbf{e}_1$ of the Euclidean basis. Let us consider $I^{*\varepsilon} = \{1, \ldots, N-2\} \times \{0, \ldots, N-1\}^{n-1} \subset I^\varepsilon$. The set $C_\sharp^\infty(T')$ is constituted of functions belonging to $C^\infty(T')$ which are Y-periodic. The set of indices $I^\varepsilon - I^{*\varepsilon}$ related to cells located on the boundary is partitioned into $I^{\varepsilon+} = \{N-1\}\times\{0,\ldots,N-1\}^{n-1}$ and $I^{\varepsilon-} = \{0\}\times\{0,\ldots,N-1\}^{n-1}$. In the following, τ_1 and $n_{\Omega 1}$ denote the first components of τ and of the external normal $\mathbf{n}_\Omega$ to Ω. In addition, $\Gamma^+ = \{\mathbf{z}\in\partial\Omega, z_1 = 0\}$ and $\Gamma^- = \{\mathbf{z}\in\partial\Omega, z_1 = 1\}$.

Lemma 6.2. *If the sequence φ^ε satisfies the estimate $\varepsilon^{n-1}||\varphi^\varepsilon||_{H^1(\Theta^\varepsilon)} \le C$, then one may extract a subsequence denoted by φ^ε, such that for any $\psi \in H^1(\Omega\times T')$ verifying $\nabla_\tau\psi = 0$ on T', φ^ε satisfies*

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon}[\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s})\psi(z,\mathbf{s})]_{\mathbf{s}^-}^{\mathbf{s}^+}\,d\mathbf{z} = -\int_{\Omega\times T'}\varphi^0(\mathbf{z})\tau_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z}$$
$$+\int_{(\Gamma^+\cup\Gamma^-)\times T'}\varphi^0(\mathbf{z})\psi(\mathbf{z},\mathbf{y})\cdot\tau_1 n_{\Omega 1}\,ds(\mathbf{z})\,.$$

Proof of Lemma 6.2. First, let us prove that for every function $\psi\in H^1(\Omega;H^1_\sharp(T'))$ verifying $\nabla_\tau\psi = 0$ we have

$$\int_\Omega \frac{1}{\varepsilon}[\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})]_{\mathbf{s}^-}^{\mathbf{s}^+}\,d\mathbf{z} = -\int_{\Omega^\varepsilon\times T'}\nabla_{\boldsymbol{\tau}}(\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})\mathbf{y}\cdot\nabla_{\mathbf{z}}\psi(\mathbf{z},\mathbf{y}))\,dl(\mathbf{y})\,d\mathbf{z}$$
$$+\int_{\Gamma^+}\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\,ds(\mathbf{z}) - \int_{\Gamma^-}\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\,ds(\mathbf{z}) + O(\varepsilon)\,. \quad (19)$$

Here $\Omega^\varepsilon = \cup_{\mathbf{i}\in I^{*\varepsilon}}Y^\varepsilon_{\mathbf{i}}$. Since $\Omega = \cup_{\mathbf{i}\in I^\varepsilon}Y^\varepsilon_{\mathbf{i}}$, we have

$$\int_\Omega \frac{1}{\varepsilon}[\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})]_{\mathbf{s}^-}^{\mathbf{s}^+}\,d\mathbf{z}$$
$$= \sum_{\mathbf{i}\in I^\varepsilon}\int_{Y^\varepsilon_{\mathbf{i}}}\frac{1}{\varepsilon}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)\psi(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+) - \frac{1}{\varepsilon}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\psi(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\,d\mathbf{z}$$
$$= \sum_{\mathbf{i}\in I^\varepsilon}\varepsilon^{n-1}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+) - \varepsilon^{n-1}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\,,$$

where $\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{y}) = \frac{1}{\varepsilon^n}\int_{Y^\varepsilon_{\mathbf{i}}}\psi(\mathbf{x},\mathbf{y})\,d\mathbf{x}$ for every $(\mathbf{z},\mathbf{y})\in Y^\varepsilon_{\mathbf{i}}\times T'$. Since $\varphi^\varepsilon\in H^1(\Theta^\varepsilon)$,

$$\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-) = \widehat{\varphi}^\varepsilon\circ\mathcal{T}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-) \text{ and } \widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+) = \widehat{\varphi}^\varepsilon\circ\mathcal{T}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)\,,$$

where for $(\mathbf{z},\mathbf{y})\in\Omega\times\{\mathbf{s}^-,\mathbf{s}^+\}$, $\mathcal{T}^\varepsilon(\mathbf{z},\mathbf{y}) = (\mathbf{z}+\varepsilon\mathbf{n}_Y(\mathbf{y}),\mathbf{y}-\mathbf{n}_Y(\mathbf{y}))$. Since ψ is Y-periodic with respect to $\mathbf{y}$, we have

$$= \sum_{\mathbf{i}\in I^{*\varepsilon}}\varepsilon^n\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)\frac{\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+) - \overline{\psi}^\varepsilon\circ\mathcal{T}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)}{2\varepsilon}$$
$$- \sum_{\mathbf{i}\in I^{*\varepsilon}}\varepsilon^n\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\frac{\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-) - \overline{\psi}^\varepsilon\circ\mathcal{T}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)}{2\varepsilon}$$
$$+ \sum_{\mathbf{i}\in I^{\varepsilon+}}\varepsilon^{n-1}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^+) - \sum_{\mathbf{i}\in I^{\varepsilon-}}\varepsilon^{n-1}\widehat{\varphi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{s}^-)\,.$$

Using the periodicity condition on ψ we find that $\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^+) = \overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^-)$. Using this equality leads to

$$= \sum_{\mathbf{i}\in I^{*\varepsilon}} \varepsilon^n \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^+) \frac{\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^+) - \overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon + \varepsilon\mathbf{n}_Y(\mathbf{s}^+),\mathbf{s}^+)}{2\varepsilon}$$

$$- \sum_{\mathbf{i}\in I^{*\varepsilon}} \varepsilon^n \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-) \frac{\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-) - \overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon + \varepsilon\mathbf{n}_Y(\mathbf{s}^-),\mathbf{s}^-)}{2\varepsilon}$$

$$+ \sum_{\mathbf{i}\in I^{\varepsilon+}} \varepsilon^{n-1} \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^+) - \sum_{\mathbf{i}\in I^{\varepsilon-}} \varepsilon^{n-1} \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-)\,.$$

For $(\mathbf{z},\mathbf{y}) \in Y_\mathbf{i}^\varepsilon \times T'$, and $\mathbf{i} \in I^\varepsilon$, let us remark that

$$\frac{\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{y}) - \overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon + \varepsilon\mathbf{n}_Y(\mathbf{y}),\mathbf{y})}{2\varepsilon} = -y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y}) + O(\varepsilon)\,.$$

Thus

$$= -\int_{\Omega^\varepsilon} [\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y}) y_1 \partial_{z_1}\psi(\mathbf{z},\mathbf{y})]_{y=\mathbf{s}^-}^{y=\mathbf{s}^+}\, d\mathbf{z} + \varepsilon^{n-1} \sum_{\mathbf{i}\in I^{\varepsilon+}} \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^+)\overline{\psi}^\varepsilon(x_\mathbf{i}^\varepsilon,\mathbf{s}^+)$$

$$-\varepsilon^{n-1} \sum_{\mathbf{i}\in I^{\varepsilon-}} \widehat{\varphi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{x}_\mathbf{i}^\varepsilon,\mathbf{s}^-) + O(\varepsilon)$$

$$= -\int_{\Omega^\varepsilon} [\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y}) y_1 \partial_{z_1}\psi(\mathbf{z},\mathbf{y})]_{y=\mathbf{s}^-}^{y=\mathbf{s}^+}\, d\mathbf{z} + \int_{\Gamma^+} \widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\, ds(\mathbf{z})$$

$$-\int_{\Gamma^-} \widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\, ds(\mathbf{z}) + O(\varepsilon)$$

$$= -\int_{\Omega^\varepsilon\times T'} \nabla_\tau(\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y}) y_1 \partial_{z_1}\psi(\mathbf{z},\mathbf{y}))\, dl(\mathbf{y})\, d\mathbf{z} + \int_{\Gamma^+} \widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\, ds(\mathbf{z})$$

$$-\int_{\Gamma^-} \widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\, ds(\mathbf{z}) + O(\varepsilon)\,. \tag{20}$$

This is (19). Since $\nabla_\tau(\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})\ y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y})) = \nabla_\tau\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y}) y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y}) + \widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})$ $\nabla_\tau(y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y}))$, and since $\nabla_\tau\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})$ tends to zero in $L^2(\Omega\times T')$, one may pass to the limit in the above term. Since φ^0 and ψ are constants with respect to $\mathbf{y}\in T'$ and since $\int_{T'} \tau_1\ dl(\mathbf{y}) = 1$, where $\tau_1 = \nabla_\tau y_1$, one may pass to the limit. This leads to:

$$= -\int_{\Omega\times T'} \varphi^0(\mathbf{z})\nabla_\tau(y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y}))\, dl(\mathbf{y})\, d\mathbf{z} + \int_{(\Gamma^+\cup\Gamma^-)\times T'} \varphi^0(\mathbf{z})\psi(\mathbf{z},\mathbf{y})\cdot\tau_1 n_{\Omega 1}\, ds(\mathbf{z})\,.$$

This ends the proof of Lemma 6.2. □

Let us denote by $f(\mathbf{z},\mathbf{y})$ the two-scale limit in L^2 of f^ε. For every $\psi \in H^1(\Omega; H^1_\sharp(T'))$ such that $\nabla_\tau\psi(\mathbf{z},\mathbf{y}) = 0$ for $(\mathbf{z},\mathbf{y}) \in \Omega\times T'$, one has:

$$\int_{\Omega\times T'} f(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})\, dl(\mathbf{y})\, d\mathbf{z} = \int_{\Omega\times T'} \partial_{z_1}\varphi^0(\mathbf{z})\tau_1\psi(\mathbf{z},\mathbf{y})\, dl(\mathbf{y})\, d\mathbf{z}$$

or equivalently:

$$\int_{\Omega\times T'} f(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z} = \int_{\Omega} \nabla_{\mathbf{z}}\varphi^0(\mathbf{z})\cdot\tau^0(\mathbf{z})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z}\,,$$

where $\tau^0(\mathbf{z}) = \int_{T'} \tau(z,y)\ dl(\mathbf{y})$.

This proves that $\partial_{z_1}\varphi^0(\mathbf{z})\tau_1 \in L^2(\Omega)$ and means that there exists a function φ^1 such that

$$\begin{aligned}\int_{\Omega\times T'} f(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z} &= \int_{\Omega\times T'} \nabla_{\mathbf{z}}\varphi^0(\mathbf{z})\cdot\tau^0(\mathbf{z})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z}\\ &\quad - \int_{\Omega\times T'} \varphi^1(\mathbf{z},\mathbf{y})\nabla_\tau\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z}\end{aligned}$$

for every $\psi \in H^1(\Omega; H^1_\sharp(T'))$. Since ψ is T'-periodic, this equality is equivalent to

$$\begin{aligned}&\nabla_\tau\varphi^1(\mathbf{z},\mathbf{y}) + \nabla_{\mathbf{z}}\varphi^0(\mathbf{z})\cdot\tau^0(\mathbf{z}) = f(\mathbf{z},\mathbf{y}) \text{ in } \Omega\times T',\\ &[\varphi^1(\mathbf{z},\mathbf{y})]_{\mathbf{y}=\mathbf{s}^-}^{\mathbf{y}=\mathbf{s}^+} = 0 \text{ in } \Omega\,.\end{aligned}$$

Since $\varphi^0 \in H^1(\Omega)$, we see that $\varphi^1 \in L^2(\Omega; H^1_\sharp(T'))$. This ends the proof of Theorem 2(i). □

6.4. *Proof of Theorem* 2(ii)

Using (20), we have

$$\begin{aligned}\int_{\Omega\times T'} \widehat{f^\varepsilon}(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z} &= -\int_{\Omega^\varepsilon\times T'} \nabla_\tau(\widehat{\varphi}^\varepsilon(\mathbf{z},\mathbf{y})y_1\partial_{z_1}\psi(\mathbf{z},\mathbf{y}))\,dl(\mathbf{y})\,d\mathbf{z}\\ &\quad + \int_{\Gamma^+} \varphi_d(\mathbf{z})\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^+)\,ds(\mathbf{z})\\ &\quad - \int_{\Gamma^-} \varphi_d(\mathbf{z})\overline{\psi}^\varepsilon(\mathbf{z},\mathbf{s}^-)\,ds(\mathbf{z}) + O(\varepsilon)\,.\end{aligned}$$

Passing to the limit we obtain:

$$\begin{aligned}\int_{\Omega\times T'} f(\mathbf{z},\mathbf{y})\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z} &= \int_{\Omega\times T'} \partial_{z_1}\varphi^0(\mathbf{z})\tau_1\psi(\mathbf{z},\mathbf{y})\,dl(\mathbf{y})\,d\mathbf{z}\\ &\quad + \int_{(\Gamma^+\cup\Gamma^-)\times T'} (\varphi_d(\mathbf{z}) - \varphi^0(\mathbf{z}))\psi(\mathbf{z},\mathbf{y})\cdot\tau_1 n_{\Omega 1}\,ds(\mathbf{z})\,,\end{aligned}$$

which implies in particular that $\varphi^0(\mathbf{z}) = \varphi_d(\mathbf{z})$ on $\Gamma^+\cup\Gamma^-$. This is part (ii). □

6.5. *Proof of Theorem* 2(iii)

Corollary 3.2 implies that there exists an extracted subsequence $(\eta^\varepsilon)_\varepsilon$ of $(\eta^\varepsilon)_{\varepsilon\in\mathbb{N}^{-1}}$ which two-scale converges in L^2 weakly towards a $\eta^0(\mathbf{z},\mathbf{y})$. In addition, $\varepsilon^{(n-1)/2}\|\varepsilon\nabla_\tau\eta^\varepsilon\|_{L^2(\Theta^{\varepsilon\prime\prime})} = \|\nabla_\tau\widehat{\eta}^\varepsilon\|_{L^2(\Omega\times T'')} \le C$, thus $(\nabla_\tau\widehat{\eta}^\varepsilon)_\varepsilon$ converges weakly towards $\nabla_\tau\eta^0$ in $L^2(\Omega\times T'')$. □

6.6. *Proof of Theorem* 2(iv)

Since $\varphi^\varepsilon = \eta^\varepsilon$ on $\theta^\varepsilon \cap \theta^{\varepsilon\prime}$ thus $\widehat{\varphi}^\varepsilon = \widehat{\eta}^\varepsilon$ on $\Omega \times (T' \cap T'')$. But $\widehat{\varphi}^\varepsilon$ converges towards φ^0 in $\Omega \times T'$ and $\widehat{\eta}^\varepsilon$ converges towards η^0 in $\Omega \times T''$ thus $\varphi^0 = \eta^0$ on $\Omega \times (T' \cap T'')$. $\square$

7. Proof of Theorem 3

The proof consists of two steps. First, we derive a uniform estimate of the solution $(\varphi^\varepsilon, i^\varepsilon)$. Second, we pass to the limit in the variational formulation. The estimates are written on the set Θ^ε, which is dependent on ε. After applying the two-scale transform, they are written on $\Omega \times T$ which is independent of ε. This allows one to use the classical compactness arguments for the extraction of convergent subsequences.

7.1. *Estimate of the solution*

Let us define the norm $||\cdot||_\varepsilon^2$ on $(\varphi^\varepsilon, i^\varepsilon) \in \Psi_{ad}^\varepsilon(0)$ by

$$||(\varphi^\varepsilon, i^\varepsilon)||_\varepsilon^2 = |\nabla_\tau \varphi^\varepsilon|^2_{\Theta^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \varphi^\varepsilon|^2_{\Theta^{\varepsilon\prime\prime}} + |\varphi^\varepsilon|^2_{\Theta^\varepsilon} + |\varepsilon^{-1} i^\varepsilon|^2_{\Theta_4^{\varepsilon\prime}} + |\varepsilon^{-2} i^\varepsilon|^2_{\Theta_4^{\varepsilon\prime\prime}} \,.$$

Lemma 7.1. *If the assumptions* (H1–H7) *and* (H1bis–H6bis) *and* (H8) *are fulfilled, then* $||(\varphi^\varepsilon, i^\varepsilon)||_\varepsilon^2 \leq C$.

Proof of Lemma 7.1. The proof is based on the same arguments as that of the existence and uniqueness of the solution. We establish successively the following four estimates:

(i) There exists a positive constant δ (independent of ε) such that $\forall\, (\psi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$ verifying $\nabla_\tau \psi^\varepsilon = 0$ on Θ_4^ε, there exists $(\varphi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$ different from zero such that for every couple $(e_3^{l\varepsilon}, e_4^{l\varepsilon}) \in \Theta_3^\varepsilon \times \Theta_4^\varepsilon$, φ^ε satisfies $(L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|e_4^{l\varepsilon}} = (k^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|e_3^{l\varepsilon}}$ and

$$\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \nabla_\tau \psi^\varepsilon \, dl(\mathbf{x}) \geq \delta\varepsilon ||(\varphi^\varepsilon, 0)||_\varepsilon \cdot ||(\psi^\varepsilon, 0)||_\varepsilon \,. \tag{21}$$

(ii) There exists a positive constant β such that $\forall\, (\varphi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$, for every couple $(e_3^{l\varepsilon}, e_4^{l\varepsilon}) \in \Theta_3^\varepsilon \times \Theta_4^\varepsilon$, such that $(L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|e_4^{l\varepsilon}} = (k^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|e_3^{l\varepsilon}}$, there exists $(\psi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$ different from zero verifying $\nabla_\tau \psi^\varepsilon = 0$ on Θ_4^ε and

$$\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \nabla_\tau \psi^\varepsilon \, dl(\mathbf{x}) \geq \beta\varepsilon ||(\varphi^\varepsilon, 0)||_\varepsilon \cdot ||(\psi^\varepsilon, 0)||_\varepsilon \,. \tag{22}$$

(iii) There exists a strictly positive constant γ_1 such that for every $j^\varepsilon \in \mathbb{P}^0(\Theta_4^\varepsilon)$, there exists $(\varphi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$ such that:

$$\int_{\Theta_4^\varepsilon} j^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) \geq \gamma_1 ||(\varphi^\varepsilon, 0)||_\varepsilon \cdot ||(0, j^\varepsilon)||_\varepsilon \,. \tag{23}$$

(iv) There exists a strictly positive constant γ_2 such that for every $j^\varepsilon \in \mathbb{P}^0(\Theta_4^\varepsilon)$, there exists $(\varphi^\varepsilon, 0) \in \Psi_{ad}^\varepsilon(0)$ such that:

$$\int_{\Theta_4^\varepsilon} j^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) - \int_{\Theta_3^\varepsilon} k^\varepsilon j^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) \geq \varepsilon\gamma_2 ||(\varphi^\varepsilon, 0)||_\varepsilon \cdot ||(0, j^\varepsilon)||_\varepsilon \,, \quad (24)$$

where the values of j^ε on each $e_3^{l\varepsilon}$ and $e_4^{l\varepsilon}$ are the same.

From Ref. 8, these estimates imply the estimate of Lemma 7.1.

Let us prove the estimate (21). This proof follows the same steps as that of Lemma 5.4. We pose $v^\varepsilon = \alpha k^\varepsilon \nabla_\tau \psi^\varepsilon_{|\Theta_3^\varepsilon}$ where α is determined as in the proof of Theorem 1. Consider a solution $u^\varepsilon \in \mathbb{P}^0(\Theta^\varepsilon)$ of (8) and φ^ε such that

$$L^\varepsilon \nabla_\tau \varphi^\varepsilon = u^\varepsilon + \alpha L \nabla_\tau \psi^\varepsilon \quad \text{on } \Theta^\varepsilon \,.$$

Following the same arguments as in Sec. 5, we establish that there exists a positive constant C such that:

$$\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \psi^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) \geq C\varepsilon ||(\psi^\varepsilon, 0)||_\varepsilon \cdot ||(\varphi^\varepsilon, 0)||_\varepsilon \,.$$

This is (21).

For the proof of (22), we pose

$$|e^\varepsilon| \nabla_\tau \psi^\varepsilon = \alpha^{-1}(|e^\varepsilon| \nabla_\tau \varphi^\varepsilon - u^\varepsilon) \text{ on } \Theta^\varepsilon \,.$$

Thus, the derivation of (22) is the same as that the derivation of (21).

Let us prove (23). For $i^\varepsilon \in \mathbb{P}^0(\Theta_4^\varepsilon)$ let us pose $(\varphi^\varepsilon, i^\varepsilon) \in \Psi_{ad}^\varepsilon(0)$ such that

$$L^\varepsilon \nabla_\tau \varphi^\varepsilon = i^\varepsilon \text{ on } \Theta_4^{\varepsilon\prime} \quad \text{and} \quad L^\varepsilon \nabla_\tau \varphi^\varepsilon = \varepsilon^{-1} i^\varepsilon \text{ on } \Theta_4^{\varepsilon\prime\prime} \,.$$

Then, the derivation of (23) is similar to that of (13). □

Finally, let us prove (24). Using the assumption (H1), we pose j^ε such that $\varepsilon^{-1} j^\varepsilon = v^\varepsilon$ on $\Theta_3^{\varepsilon\prime}$, $\varepsilon^{-3} j^\varepsilon = v^\varepsilon$ on $\Theta_3^{\varepsilon\prime\prime}$, and φ^ε such that $L^\varepsilon \nabla_\tau \varphi^\varepsilon = u^\varepsilon$ on Θ^ε. The assumption (H1bis) leads to the estimate

$$|\nabla_\tau \varphi^\varepsilon|^2_{\Theta^{\varepsilon\prime}} + |\varepsilon \nabla_\tau \varphi^\varepsilon|^2_{\Theta^{\varepsilon\prime\prime}} \leq C|\varepsilon^{-1} j^\varepsilon|^2_{\Theta_3^{\varepsilon\prime}} + |\varepsilon^{-2} j^\varepsilon|^2_{\Theta_3^{\varepsilon\prime\prime}} \,.$$

Replacing $(L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|\Theta_4^\varepsilon} - (k^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon)_{|\Theta_3^\varepsilon}$ by $\varepsilon v^\varepsilon$ we find that

$$\int_{\Theta_4^\varepsilon} j^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) - \int_{\Theta_3^\varepsilon} k^\varepsilon j^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) = \varepsilon^2(|\varepsilon^{-1} j^\varepsilon|^2_{\Theta_3^{\varepsilon\prime}} + |\varepsilon^{-2} j^\varepsilon|^2_{\Theta_3^{\varepsilon\prime\prime}})$$

which leads to (24) because $j^\varepsilon_{|\Theta_3^\varepsilon} = j^\varepsilon_{|\Theta_4^\varepsilon}$. This ends the proof of Lemma 7.1. □

7.2. *Passing to the limit in the variational formulation*

Let us derive the two-scale variational formulation. Applying Theorem 2 implies that there exists a subsequence $(\varphi^\varepsilon, i^\varepsilon)_\varepsilon$ of $(\varphi^\varepsilon, i^\varepsilon)_{\varepsilon \in \mathbb{N}^{-1}}$, such that $(\varphi^\varepsilon, i^\varepsilon)_\varepsilon$ two-scale converges in L^2 weakly towards some limits φ^0 on $\Omega \times T$ and i on $\Omega \times T_4$. In other words, $(\nabla_\tau \varphi^\varepsilon)_\varepsilon$ two-scale converges in L^2 weakly towards some limits

$\nabla_{\mathbf{z}}\varphi^0 \cdot \tau + \nabla_\tau \varphi^1$ on $\Omega \times T'$ and $\nabla_\tau\varphi^0$ on $\Omega \times T''$. Here $(\varphi^0_{|\Omega\times T'}, \varphi^0_{|\Omega\times T''}, \varphi^1) \in H^1_\tau(\Omega, T') \times L^2(\Omega; \mathbb{P}^1(T'')) \times L^2(\Omega; \mathbb{P}^1_\sharp(T'))$. In particular, this means that

$$D(\varphi^0, \varphi^1) = u_d \text{ in } \Omega \times T_0 .$$

In conclusion $(\varphi^0,\, \varphi^1, i) \in \Psi_{ad\sharp}(u_d)$.

In the following two lemmas, we establish the strong convergence of some particular test functions useful to pass to the limit in the variational formulation.

Let us consider a given function ψ^0 defined on $\Omega \times T$ which has a restriction independent of $\mathbf{y}$ on every $\Omega \times T_c$, Here T_c represents any connected component T_c of T'. That is

$$\psi^0 \in \mathcal{C}^2(\Omega; \mathbb{P}^1(T)) \text{ and } \psi^0_{|\Omega\times T_c} \in \mathcal{C}^2(\Omega) .$$

Let ψ^0 be given. Let us define the test function $\psi^{0\varepsilon}$ on Θ^ε by

$$\psi^{0\varepsilon}(\mathbf{x}) = \psi^0(\mathbf{x}) \text{ for each vertex } \mathbf{x} \in \Theta^{\varepsilon\prime} ,$$

$$\psi^{0\varepsilon}(\mathbf{x}) = \psi^0(\mathbf{x}^\varepsilon_{\mathbf{i}}, \mathbf{y}), \text{ for each vertex } \mathbf{x} = \mathbf{x}^\varepsilon_{\mathbf{i}} + \varepsilon\mathbf{y} \in Y^\varepsilon_{\mathbf{i}} \cap \Theta^{\varepsilon\prime\prime} .$$

In addition, $\psi^{0\varepsilon}$ is assumed to be affine on each edge of Θ^ε.

Lemma 7.2. (i) *$\psi^{0\varepsilon}$ and $\nabla_\tau\psi^{0\varepsilon}$ two-scale converge in L^2 strongly towards some ψ^0 and $\nabla_{\mathbf{z}}\psi^0 \cdot \tau$ on $\Omega \times T'$.*
(ii) *$\psi^{0\varepsilon}$ and $\varepsilon\nabla_\tau\psi^{0\varepsilon}$ two-scale converge in L^2 strongly towards some ψ^0 and $\nabla_\tau\psi^0$ on $\Omega \times T''$.*

Proof. (i) Consider an edge $e \subset T'$ with extremities $\mathbf{s}^-$ and $\mathbf{s}^+$ and consider the subset $\theta^\varepsilon \subset \Theta^\varepsilon$ such that its two-scale transformation is equal to $\Omega \times e$. Since $\psi^{0\varepsilon}$ is affine on each edge of θ^ε, we have

$$\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{s}^-) + (\mathbf{y}b - \mathbf{s}^-)\nabla_{\mathbf{y}}\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{s}^-) \text{ for every } (\mathbf{z},\mathbf{y}) \in \Omega \times e .$$

For $\mathbf{z} \in Y^\varepsilon_{\mathbf{i}}$,

$$\nabla_{\mathbf{y}}\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{s}^-) = \varepsilon\nabla_{\mathbf{z}}\psi^0(\mathbf{x}^\varepsilon_{\mathbf{i}}) + \varepsilon O(\varepsilon) .$$

thus

$$\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \psi^0(\mathbf{x}^\varepsilon_{\mathbf{i}}) + O(\varepsilon) \quad \text{and} \quad \varepsilon^{-1}\nabla_\tau\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \tau(y) \cdot \nabla_{\mathbf{z}}\psi^0(\mathbf{x}^\varepsilon_{\mathbf{i}}) + O(\varepsilon) .$$

Passing to the limit in these expressions gives

$$\lim_{\varepsilon\to 0}\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \psi^0(\mathbf{z}) \text{ and } \lim_{\varepsilon\to 0}\varepsilon^{-1}\nabla_\tau\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \nabla_{\mathbf{z}}\psi^0(\mathbf{z}) \cdot \tau(y) .$$

Since the domain $\Omega \times T_c$ is bounded, this leads to Lemma 7.2(i).
(ii) Since $T'' \cap \partial Y = \emptyset$, $\psi^{0\varepsilon}$ is well defined on $\Theta^{\varepsilon\prime\prime}$. For $(\mathbf{z},\mathbf{y}) \in Y^\varepsilon_{\mathbf{i}} \times T''$, $\widehat{\psi}^{0\varepsilon}(\mathbf{z},\mathbf{y}) = \widehat{\psi}^{0\varepsilon}(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{y}) + O(\varepsilon)$ converges strongly towards ψ^0 and $(\widehat{\varepsilon\nabla_\tau\psi^{0\varepsilon}}) = \nabla_\tau\widehat{\psi}^{0\varepsilon}(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{y}) + O(\varepsilon)$ converges strongly towards $\nabla_\tau\psi^0$. $\square$

Let us define the test function $\psi^{1\varepsilon}$. For a given $\psi^1 \in H^1_\sharp(T')$ such that $(0,\psi^1,0) \in \Psi^0_{ad\sharp}(0)$, and a given $\rho \in \mathcal{D}(\Omega)$, let us consider the function $\psi^\varepsilon \in H^1(\Theta^{\varepsilon\prime})$ defined by its two-scale transformation $\widehat{\psi}^\varepsilon(\mathbf{z},\mathbf{y}) = \psi^1(\mathbf{y})$ for every $(\mathbf{z},\mathbf{y}) \in \Omega \times T'$. The test function $\psi^{1\varepsilon}$ associated to ψ^1 and ρ is

$$\psi^{1\varepsilon}(\mathbf{x}) = \rho(\mathbf{x})\psi^\varepsilon(\mathbf{x})\,.$$

For a given $j \in \mathcal{C}^0(\Omega;\mathbb{P}^0(T_4))$, consider the function $j^\varepsilon \in \mathbb{P}^0(\Theta^\varepsilon_4)$ defined as follows:

$$j^\varepsilon(\mathbf{x}) = j(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{y}) \text{ on } \Theta^{\varepsilon\prime}_4 \text{ and } j^\varepsilon(\mathbf{x}) = \varepsilon j(\mathbf{x}^\varepsilon_{\mathbf{i}},\mathbf{y}) \text{ on } \Theta^{\varepsilon\prime\prime}_4\,,$$

for $\mathbf{x} = \mathbf{x}^\varepsilon_{\mathbf{i}} + \varepsilon\mathbf{y} \in e^\varepsilon \subset \Theta^\varepsilon_4 \cap Y^\varepsilon_{\mathbf{i}}$.

Lemma 7.3. (i) *The sequences $\psi^{1\varepsilon}$ and $\varepsilon\nabla_\tau\psi^{1\varepsilon}$ two-scale converge in L^2 strongly towards some $\rho(\mathbf{z})\psi^1(\mathbf{z},\mathbf{y})$ and $\rho(\mathbf{z})\nabla_\tau\psi^1(\mathbf{z},\mathbf{y})$ on $\Omega\times T'$.*
(ii) *The sequence equal to j^ε on Θ'_4 and $\varepsilon^{-1}j^\varepsilon$ on Θ'_4, two-scale converges in L^2 strongly towards a j.*

Proof. (i) The convergence of $\psi^{1\varepsilon}$ is immediate. The convergence of $\varepsilon\nabla_\tau\psi^{1\varepsilon}$ results from the fact that $\nabla_\tau\rho$ two-scale converges towards $\nabla_{\mathbf{z}}\rho$. Thus $\varepsilon\nabla_\tau\rho$ two-scale converges towards 0. The proof of (ii) is evident. □

Let us pass to the limit in the variational formulation. Let us consider $(\psi^0,\psi^1,j) \in \Psi_{ad\sharp}(0)$ satisfying the continuity assumptions stated in Lemmas 7.2 and 7.3 and let us consider $(\psi^{0\varepsilon},\psi^{1\varepsilon},j^\varepsilon) \in \Psi^\varepsilon_{ad}(0)$ associated to (ψ^0,ψ^1,j), given by Lemmas 7.2 and 7.3. Thus,

$$\begin{aligned}\int_{\Theta^{\varepsilon\prime}_2} & L^\varepsilon g^\varepsilon \nabla_\tau\varphi^\varepsilon(\nabla_\tau\psi^{0\varepsilon} + \varepsilon\nabla_\tau\psi^{1\varepsilon})\,dl(\mathbf{x}) + \int_{\Theta^{\varepsilon\prime\prime}_2} L^\varepsilon g^\varepsilon\nabla_\tau\varphi^\varepsilon\nabla_\tau\psi^{0\varepsilon}\,dl(\mathbf{x})\\ &+ \int_{\Theta^{\varepsilon\prime}_4} i^\varepsilon(\nabla_\tau\psi^{0\varepsilon} + \varepsilon\nabla_\tau\psi^{1\varepsilon})\,dl(\mathbf{x}) + \int_{\Theta^{\varepsilon\prime\prime}_4} i^\varepsilon\nabla\tau\psi^{0\varepsilon}\,dl(\mathbf{x})\\ &= -\int_{\Theta^{\varepsilon\prime}_1} i^\varepsilon_d(\nabla_\tau\psi^{0\varepsilon} + \varepsilon\nabla_\tau\psi^{1\varepsilon})\,dl(\mathbf{x}) - \int_{\Theta^{\varepsilon\prime\prime}_1} i^\varepsilon_d\nabla_\tau\psi^{0\varepsilon}\,dl(\mathbf{x})\end{aligned}$$

and

$$\int_{\Theta^\varepsilon_3} L^\varepsilon k^\varepsilon\nabla_\tau\varphi^\varepsilon j^\varepsilon\,dl(\mathbf{x}) - \int_{\Theta^\varepsilon_4} L^\varepsilon\nabla_\tau\varphi^\varepsilon j^\varepsilon\,dl(\mathbf{x}) = 0\,.$$

Let us divide the first equation by ε, using the two-scale transformation of the expressions and using Lemmas 7.1, 7.2 and 7.3, one may pass to the limit when ε vanishes. The test functions for the first equation is

$$\nabla_{\mathbf{z}}\psi^0\cdot\boldsymbol{\tau} + \rho\nabla_{\boldsymbol{\tau}}\psi^1 \text{ on } \Omega\times T'\,.$$

Let us remark that it is equivalent to consider the variational formulation with the test function $\nabla_{\mathbf{z}}\psi^0\cdot\boldsymbol{\tau} + \nabla_{\boldsymbol{\tau}}\psi^1$ or with the test function $\nabla_{\mathbf{z}}\psi^0\cdot\tau^0 + \nabla_{\boldsymbol{\tau}}\psi^1$. Finally, the variational formulation is

$$\int_{\Omega\times T_2'} LgD(\varphi^0,\varphi^1)D(\psi^0,\psi^1)\,dl(\mathbf{y})\,d\mathbf{z} + \int_{\Omega\times T_4} iD(\psi^0,\psi^1)\,dl(\mathbf{y})\,d\mathbf{z}$$
$$= -\int_{\Omega\times T_1} i_d D(\psi^0,\psi^1)\,dl(\mathbf{y})\,d\mathbf{z}, \int_{\Omega\times T_3} kD(\varphi^0,\varphi^1)j\,dl(\mathbf{y})\,d\mathbf{z}$$
$$-\int_{\Omega\times T_4} D(\varphi^0,\varphi^1)j\,dl(\mathbf{y})\,d\mathbf{z} = 0\,.$$

This completes the proof of Theorem 3. □

8. Proof of Theorem 4

The proof will be complete after proving estimates (21)–(24) with a convenient norm $||\cdot||_\varepsilon$ and we will pass to the limit in the variational formulation. However, we will first check the assumptions (H1–H7) in order to show that these assumptions are also satisfied.

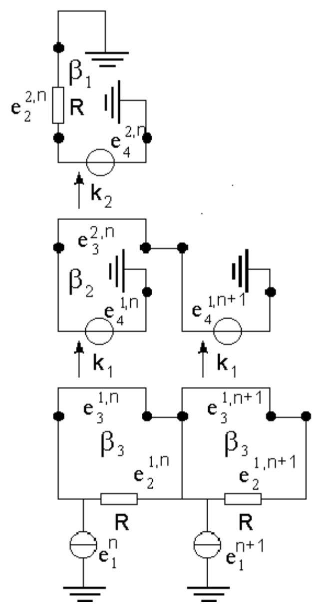

Fig. 6. The cells n and $n+1$.

Let us check the assumptions (H1–H7). We denote by $\Theta_k^{j\varepsilon} = \{e_k^{j,n}\}_{n=1,\ldots,\varepsilon^{-1}}$. Here

$$\overline{\Theta}_2^\varepsilon = \Theta_2^{1\varepsilon}, \quad \widetilde{\Theta}_2^\varepsilon = \Theta_2^{2\varepsilon}\,,$$

$$\Theta^{\varepsilon\prime} = \Theta_1^\varepsilon \cup \Theta_2^{1\varepsilon} \cup \Theta_3^{1\varepsilon} \cup \Theta_3^{2\varepsilon}$$

and

$$\Theta^{\varepsilon\prime\prime} = \Theta_4^{1\varepsilon} \cup \Theta_4^{2\varepsilon} \cup \Theta_2^{2\varepsilon} .$$

For $n \in \{1, \ldots, \varepsilon^{-1}\}$, we have

$$u^\varepsilon_{|e_2^{1,n}} = u_{|e_3^{1,n}} = 0 ,$$

thus

$$u^\varepsilon_{|e_4^{1,n}} = v^\varepsilon_{|e_3^{1,n}} ,$$

$$u^\varepsilon_{|e_3^{2,n}} = u^\varepsilon_{|e_4^{1,n+1}} - u^\varepsilon_{|e_4^{1,n}} = v^\varepsilon_{|e_3^{1,n+1}} - v^\varepsilon_{|e_3^{1,n}} ,$$

$$\varepsilon u^\varepsilon_{|e_4^{2,n}} = k_2 u^\varepsilon_{|e_3^{2,n}} + \varepsilon v^\varepsilon_{|e_3^{2,n}} = k_2\big(v^\varepsilon_{|e_3^{1,n+1}} - v^\varepsilon_{|e_3^{1,n}}\big) + \varepsilon v^\varepsilon_{|e_3^{2,n}} ,$$

and

$$u^\varepsilon_{|e_2^{2,n}} = -u^\varepsilon_{|e_4^{2,n}} .$$

Thus, the solution u^ε exists and is unique. Hence, (H1) is satisfied.

Let us prove (H2). Let us pose $\alpha^\varepsilon = \alpha_0$ on $\Theta_3^\varepsilon \cup \Theta_2^{1\varepsilon} \cup \Theta_4^{1\varepsilon}$ and $\alpha^\varepsilon = 1$ on $\Theta_2^{2\varepsilon} \cup \Theta_4^{2\varepsilon}$. Here $Z(e_2^{2,n}) = e_2^{2,n} \cup e_4^{2,n}$ thus (H2) is clearly satisfied.

The assumptions (H3–H7) are clearly satisfied.

Now, we consider the norm

$$||(\psi, j)||_\varepsilon^2 = \int_{\Theta_2^{1\varepsilon}\cup\Theta_3^{1\varepsilon}} (\nabla_\tau \psi)^2 \, dl(\mathbf{x}) + \int_{\Theta_4^{1\varepsilon}} (\varepsilon\nabla_\tau\psi)^2 dl(\mathbf{x})$$

$$+ \int_{\Theta_3^{2\varepsilon}} \left(\int_0^{x_1} \psi dx_1\right)^2 dl(\mathbf{x}) + \int_{\Theta_4^{2\varepsilon}\cup\Theta_2^{2\varepsilon}} \left(\varepsilon\int_0^{x_1} \psi dx_1\right)^2 dl(\mathbf{x}) + |\varepsilon^{-2} j|^2_{\Theta_4^\varepsilon},$$

and we will prove that

$$||(\nabla_\tau \varphi^\varepsilon, i^\varepsilon)||_\varepsilon \le C|\varepsilon^{-2} i_d^\varepsilon|^2_{\Theta_1^\varepsilon} .$$

For this purpose, we prove the estimate (21)–(24) using this norm. Let us prove (21). We choose u^ε and φ^ε as in the proof of Theorem 3:

$$L^\varepsilon \nabla_\tau \varphi^\varepsilon = u^\varepsilon + \alpha^\varepsilon L^\varepsilon \nabla_\tau \psi^\varepsilon \text{ on } \Theta^\varepsilon .$$

Since

$$\nabla_\tau \psi^\varepsilon = 0 \text{ on } \Theta_4^\varepsilon \quad \text{and} \quad u^\varepsilon = 0 \text{ on } \Theta_2^{1\varepsilon} ,$$

thus

$$\nabla_\tau \psi^\varepsilon = 0 \text{ on } \Theta_3^{2\varepsilon} \cup \Theta_2^{2\varepsilon} \quad \text{and} \quad u^\varepsilon = 0 \text{ on } \Theta_3^{1\varepsilon} .$$

Then,

$$\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \psi^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) = \int_{\Theta_2^{1\varepsilon}} \alpha^\varepsilon g^\varepsilon L^\varepsilon \nabla_\tau \psi^\varepsilon \nabla_\tau \psi^\varepsilon \, dl(\mathbf{x})$$

$$\ge |(\alpha^\varepsilon g^\varepsilon L^\varepsilon)^{1/2} \nabla_\tau \psi^\varepsilon|^2_{\Theta_2^{1\varepsilon}} \ge C\varepsilon |\nabla_\tau \psi^\varepsilon|_{\Theta^\varepsilon} \cdot |\nabla_\tau \varphi^\varepsilon|_{\Theta_2^{1\varepsilon}} .$$

For $n \in \{1, \ldots, \varepsilon^{-1} - 1\}$,

$$\begin{aligned}
\varepsilon \nabla_\tau \varphi^\varepsilon_{|e_2^{2,n}} &= -\varepsilon \nabla_\tau \varphi^\varepsilon_{|e_4^{2,n}} = -k_2 \nabla_\tau \varphi^\varepsilon_{|e_3^{2,n}} \\
&= k_2 \big(\nabla_\tau \varphi^\varepsilon_{|e_4^{1,n}} - \nabla_\tau \varphi^\varepsilon_{|e_4^{1,n+1}} \big) \\
&= \varepsilon^{-1} k_2 k_1 \big(\nabla_\tau \varphi^\varepsilon_{|e_3^{1,n}} - \nabla_\tau \varphi^\varepsilon_{|e_3^{1,n+1}} \big) \\
&= -\varepsilon^{-1} k_2 k_1 \big(\nabla_\tau \varphi^\varepsilon_{|e_2^{1,n}} - \nabla_\tau \varphi^\varepsilon_{|e_2^{1,n+1}} \big) .
\end{aligned} \tag{25}$$

Or equivalently,

$$\begin{aligned}
\nabla_\tau \varphi^\varepsilon_{|e_2^{1,n+1}} &= -\nabla_\tau \varphi^\varepsilon_{|e_3^{1n+1}} = k_1^{-1} \nabla_\tau \varphi^\varepsilon_{|e_4^{1n+1}} = k_1^{-1} \int_0^{x_1^n} \nabla_\tau \varphi^\varepsilon_{|e_3^2}\, dx_1 \\
&= (k_2 k_1)^{-1} \int_0^{x_1^n} \varepsilon \nabla_\tau \varphi^\varepsilon_{|e_4^2} dx_1 = -(k_2 k_1)^{-1} \int_0^{x_1^n} \varepsilon \nabla_\tau \varphi^\varepsilon_{|e_2^2}\, dx_1 .
\end{aligned}$$

Thus, there exists a positive constant C such that:

$$\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \psi^\varepsilon \nabla_\tau \varphi^\varepsilon \, dl(\mathbf{x}) \geq C\varepsilon ||(\nabla_\tau \psi^\varepsilon, 0)||_\varepsilon \cdot ||(\nabla_\tau \varphi^\varepsilon, 0)||_\varepsilon .$$

This is (21).

For the proof of (22) it is sufficient to pose $L^\varepsilon \nabla_\tau \psi^\varepsilon = (\alpha^\varepsilon)^{-1}(L^\varepsilon \nabla_\tau \varphi^\varepsilon - u^\varepsilon)$ on Θ^ε.

The proofs of (23)–(24) and of the uniform continuity on the right-hand side of the variational formulation are similar to their proofs of Theorem 3. This ends the proofs of (21)–(24).

The additional estimate

$$|\varphi^\varepsilon|^2_{\Theta^\varepsilon} \leq C |\varepsilon^{-2} i_d^\varepsilon|^2_{\Theta_1^\varepsilon}$$

is necessary for the application of Theorem 2. It directly results from:

$$-\varepsilon^{-1} k_2 k_1 (\nabla_\tau \varphi^\varepsilon_{|e_2^{1,n}} - \nabla_\tau \varphi^\varepsilon_{|e_2^{1,n+1}}) = g^{-1} k_2 k_1 \varepsilon^{-2} (i_d^\varepsilon)_{|e_1^n}$$

and from (25). In conclusion,

$$||(\nabla_\tau \varphi^\varepsilon, i^\varepsilon)|| \leq C |\varepsilon^{-2} i_d^\varepsilon|^2_{\Theta_1^\varepsilon} ,$$

which allows one to pass to the limit in the variational formulation, and to get the same variational formulation as that in Theorem 3.

Here, $\nabla_\tau \varphi^1 = 0$ on $\Omega \times (e_2^1 \cup e_3^1 \cup e_3^2)$ because φ^1 is Y-periodic. Thus $(\varphi^0, 0, i) \in \Psi_{ad\sharp}(0)$ is the unique solution of:

$$\int_{\Omega\times e_2^2} g\nabla_\tau\varphi^0\nabla_\tau\psi^0\,dl(\mathbf{y})\,d\mathbf{z} + \int_{\Omega\times e_2^1} g\nabla_{\mathbf{z}}\varphi^0\nabla_{\mathbf{z}}\psi^0\,dl(\mathbf{y})\,d\mathbf{z} + \int_{\Omega\times\{e_4^1\cup e_4^2\}} i\nabla_\tau\psi^0\,dl(\mathbf{y})\,d\mathbf{z}$$

$$= -\int_{\Omega\times e_1} i_d\nabla_\tau\psi^0\,dl(\mathbf{y}) \int_{\Omega\times\{e_4^1,e_4^2\}} \nabla_\tau\varphi^0 j\,dl(\mathbf{y})\,d\mathbf{z}$$

$$-\int_{\Omega\times\{e_3^1,e_3^2\}} k\nabla_{\mathbf{z}}\varphi^0 j\,dl(\mathbf{y})\,d\mathbf{z} = 0 \quad \text{for all } (\psi^0, 0, j) \in \Psi_{ad\sharp}(0)\,. \tag{26}$$

Since $(\varphi^0, 0, i) \in \Psi_{ad\sharp}(0)$,

$$\nabla_\tau\varphi^0_{|\Omega\times e_2^2} = -\nabla_\tau\varphi^0_{|\Omega\times e_4^2}\,.$$

From (26_2),

$$= -k_2\nabla_{\mathbf{z}}\varphi^0_{|\Omega\times e_3^2}\,.$$

Since $(\varphi^0, 0, i) \in \Psi_{ad\sharp}(0)$,

$$\varphi^0_{|\Omega\times e_3^2} = \nabla_\tau\varphi^0_{|\Omega\times e_4^1} = k_1\nabla_{\mathbf{z}}\varphi^0_{|\Omega\times e_3^1} = -k_1\nabla_{\mathbf{z}}\varphi^0_{|\Omega\times e_2^1}\,,$$

it follows that

$$\nabla_\tau\varphi^0_{|\Omega\times e_2^2} = k_2k_1\Delta_{\mathbf{z}}\varphi^0\,.$$

This ends the proof of Theorem 4. □

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