# Chimera in Space-Time Representation of Nonlinear Delay Dynamics

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# Outline

1. Intro: Delayed feedback loop and space-time plot

2. Selecting and designing the right DDE

3. Outro: chimera, the right suspect?

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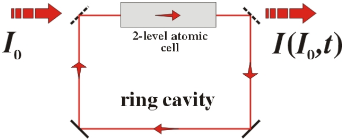
# DDE: Physics & modeling

## The delayed feedback loop dynamical concept

Closed loop oscillator architecture:

- All-optical Ikeda ring cavity



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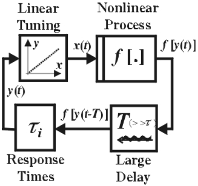

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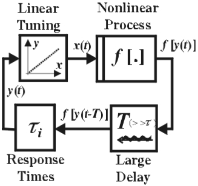


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$$\tau \frac{\mathrm{d}x}{\mathrm{d}t}(t) = -x(t) + F_{\mathrm{NL}}[x(t-\tau_D)]$$

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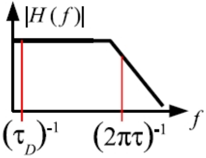


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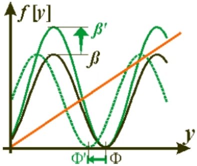


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- Nonlinear delayed driving force

$$z(t) = F_{\mathrm{NL}}[x(t-\tau_D)] = \beta \cos^2[x(t-\tau_D) + \Phi]$$

# Re-scaling and spatio-temporal viewpoint

**Normalization wrt Delay $\tau_D$:** $s = t/\tau_D$, and $\varepsilon = \tau/\tau_D$

$$\varepsilon\,\dot{x}(s) = -x(s) + f_{\mathrm{NL}}[x(s-1)], \quad \text{where} \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}s}.$$

Large delay case: $\varepsilon \ll 1$, potentially high dimensional attractor

$\infty-$dimensional phase space, initial condition: $x(s), s \in [-1, 0]$

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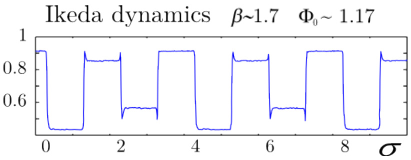
## Space-time representation

- Virtual space variable $\sigma$,

  $\sigma \in [0; 1+\gamma]$ with $\gamma = O(\varepsilon)$.



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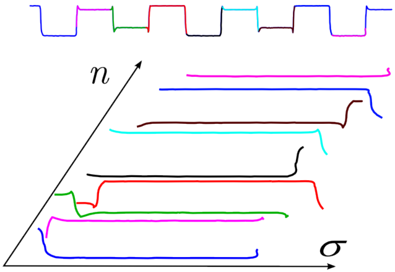


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- Discrete time $n$

$$\begin{aligned} n \quad &\rightarrow \quad (n+1) \\ s = n(1+\gamma)+\sigma \quad &\rightarrow \quad s = (n+1)(1+\gamma)+\sigma \end{aligned}$$

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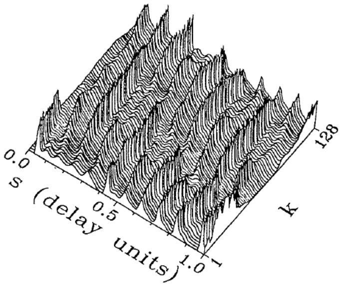


F.T. Arecchi, *et al.* Phys. Rev. A, 1992

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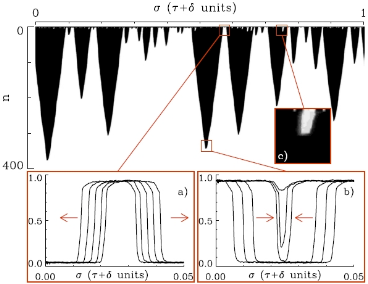


G. Giacomelli, *et al.* EPL, 2012

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## $i$ DDE: the NL delay damped oscillator viewpoint

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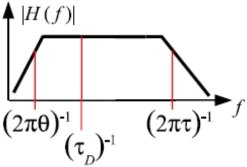


V.S. Udaltsov, *et al.* IEEE Trans. Circ. & Syst., 2002; Y.C. Kouomou *et al.*, Phys. Rev. Lett. 2005; M. Peil *et al.*, Phys. Rev. E, 2009; T.E. Murphy *et al.*, Phil. Trans. R. Soc. A, 2010; K. Callan *et al.* Phys. Rev. Lett., 2010; L. Larger & J.M. Dudley, Nature, News & Views, 2010; L. Weicker *et al.* Phys. Rev. E, 2012.

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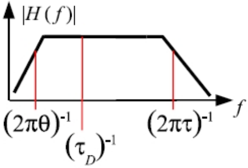


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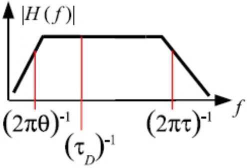


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- $z(t) = \beta\, F_{\text{NL}}[x, t, t-\tau_D]$ NL delayed self-feedback driving force

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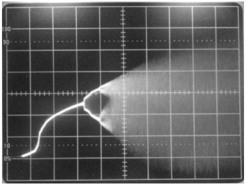
# Positive NL delayed feedback

## Standard Ikeda bifurcation: negative slope

- Period doubling route to chaos



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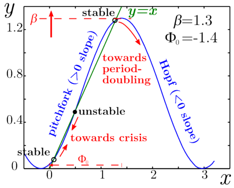

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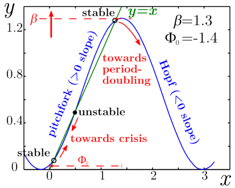

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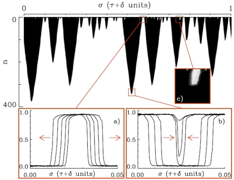


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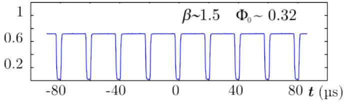


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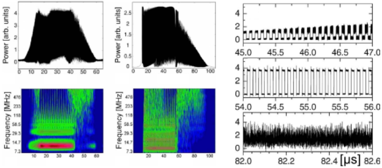


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- Multiple exotic solutions found with $i$ DDE and under positive slope operation (chaotic breathers, excitability, slow limit cycle,. . . )

$$\varepsilon \dot{x} = -\delta\, y - x + f_{\mathrm{NL}}[x(s-1)],$$
$$\dot{y} = x$$

with $\delta = \tau_D/\theta$ (Ikeda or Mackey-Glass: $\delta \equiv 0$ since $\theta \to \infty$).

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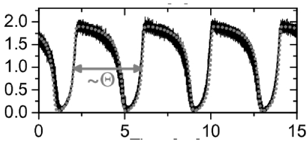


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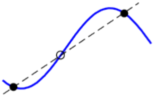
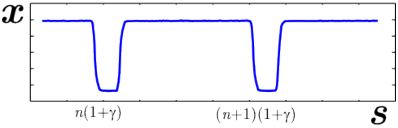
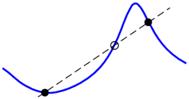
# Asymmetric non-linearity

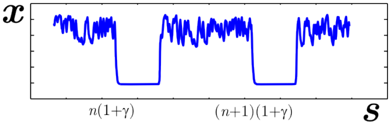
### Symmetric $f_{\text{NL}}[x]$: Similar $\sigma-$"clusters" for $x < 0$ and $x > 0$






### Asymmetric $f_{\text{NL}}[x]$: Distinct $\sigma-$clusters for $x < 0$ and $x > 0$

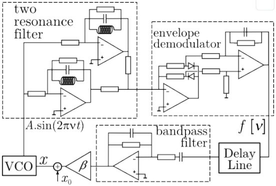




# Design of an experiment

## FM bandpass chaos

- Nonlinear delay oscillator, initially designed for chaotic secure FM radio communication.
- Experiment also used for demonstration of sub-critical Hopf bifurcation in nonlinear DDE.

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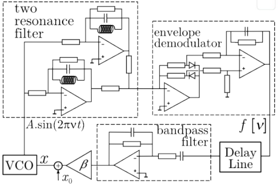
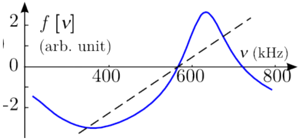



## $f_{\mathrm{NL}}[x]$: Amplitude filtering profile in the VCO tuning range

Two RLC resonant filters (tuned central frequencies and relative weights)

$$f[\nu] = \beta \left| \frac{g_{12}\, i\frac{\nu}{\nu_1}}{1-\left(\frac{\nu}{\nu_1}\right)^2 + \frac{i}{Q_1}\frac{\nu}{\nu_1}} + \frac{i\frac{\nu}{\nu_2}}{1-\left(\frac{\nu}{\nu_2}\right)^2 + \frac{i}{Q_2}\frac{\nu}{\nu_2}} \right|$$

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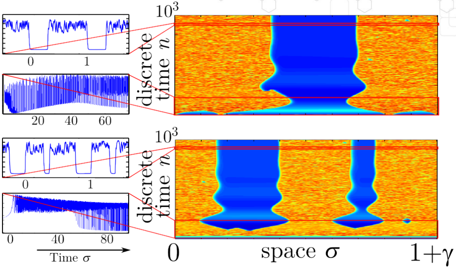
# Virtual Chimera in $(\sigma, n)-$space

## Numerics:

- $\beta = 0.6$, $\nu_0 = 1$, $\varepsilon = 5.10^{-3}$, $\delta = 1.6 \times 10^{-2}$ ($m = 56$)
- Initial conditions: small amplitude white noise (repeated several times with different noise realizations)
- Calculated durations: Thousands of $n$

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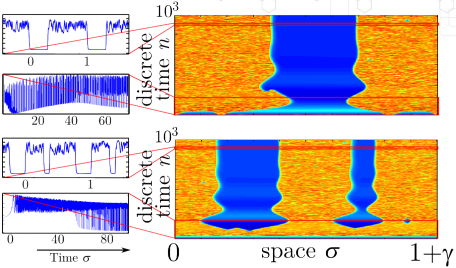
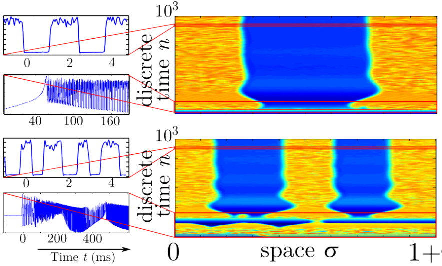



## Experiment

- Very close amplitude and time parmeters, $\tau_D = 2.54$ms, $\theta = 0.16$s, $\tau = 12.7\mu$s
- Initial condition forced by background noise
- Recording of up to $16 \times 10^6$ points, allowing for a few thousands

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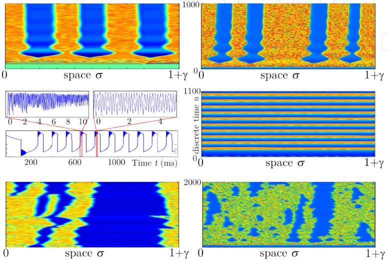
# Is chimera the right suspect?

## High dimension is here

- As in PDE, or network of long range coupled oscillators;
- Strong dynamical diversity is observed, with multiple headed “virtual chimera”;
- Bistability with coexisting solutions (e.g. chaotic breathers)

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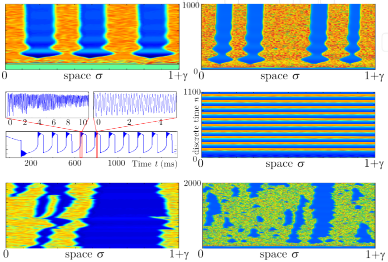


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- Bistability with coexisting solutions (e.g. chaotic breathers)

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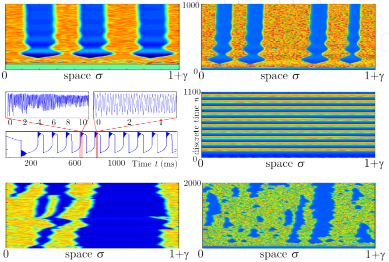
- Limited feedback gain (around $\beta = 0.6$), quiescent steady state, or period-1 cycles for lower $\beta$, and turbulent or even chaotic global state for larger $\beta$
- Chimera also loses stability when $\delta$ is decreased

L. Larger, B. Penkovsky, Y. Maistrenko, Submitted, 2013

# Is chimera the right suspect?

## High dimension is here

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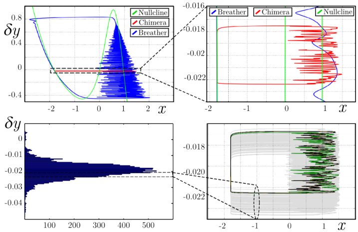



## Wandering feature

- Clusters frontiers are wandering
- Nullcline plots also reveals it

L. Larger, B. Penkovsky, Y. Maistrenko, Submitted, 2013

# Modeling interpretation

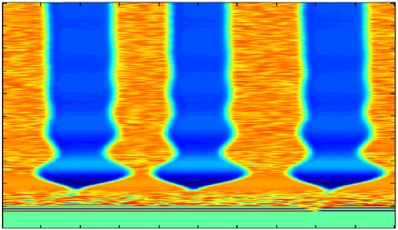
## Analogy with spatio-temporal dynamics showing chimera

Instead of the delay differential writing of the DDE, re-write globally over the virtual space $\sigma$ (convolution product with impulse response $h(t) = \mathsf{FT}^{-1}[H(f)]$: linear filter)

$$x_n(s) = \int_0^s h(\xi) f_{\mathsf{NL}}[x_{n-1}(s-\xi)]\,\mathsf{d}\xi$$

- Discrete time dynamics: $x_{n-1} \to x_n$
- Distant coupling coefficient between positions $s$ and $s-\xi$: $h(\xi)$
- Extension of this non-local coupling: width of the impulse response $h(t)$

# Three headed virtual chimera...



# Three headed virtual chimera...

***Kerberos***

# Three headed virtual chimera...

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Conseil régional de Franche-Comté

SEVENTH FRAMEWORK PROGRAMME

Thank you for attention!