Master1 PICS ET P2N Université de Franche-Comté

Quantum optics

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Introduction

Light is composed of photons: $\leftrightarrow$ wave-corpuscle duality

You have heard about it: Einstein proposed to explain the photoelectric effect by introducing quanta of light, photons of energy hν.

The simplest form of duality: probability of (detecting a photon) $\propto$ beam intensity.

More precisely, for a monochromatic light of frequency ν:

$$P_{\text{det}} = \frac{\eta}{h\nu}\int_0^{t_1} dt \int_S I(t)ds \qquad (1)$$

η : quantum efficiency, the probability of generating a photoelectron, knowing that a photon is impinging on the sensor. S: surface of the detector, powered on at t=0 and off at $t=t_1$. Unities: I in W/m^2 and hν in Joules

Points not covered in this course: the precise mechanism of the interaction of matter and radiation. We will limit ourselves to remind that the energy of the photon must be sufficient to excite an atom and make an electron "jump" in the conduction band. For example, for a silicon CCD camera, η passes from 95% for a wavelength of 0.8 μm to 5% around 1μm and much less beyond (the transition towards zero quantum efficiency is not sudden at non-zero temperatures, because thermal agitation can help an electron to cross the gap).

Covered points:

- to a constant intensity, corresponds a constant probability of photon arrival $\rightarrow$ to the absence of classical fluctuations corresponds a random arrival time of photons, therefore quantum fluctuations $\rightarrow$ definition of coherent states by photon statistics.

- Is it possible to suppress these quantum fluctuations? The answer can be yes if we take into account the conditional probabilities: probability of (detecting a photon | another photon has been detected). | means "knowing or "given"". These conditional probabilities are not described by Eq. (1) and allow us to hope for light that fluctuates less than constant intensity, and therefore specifically quantum, where the detection of one photon leads to a lower probability of detecting another; we will speak of squeezed light. In these states, the classical electric field does not provide all the statistical information on photons.

- Do photons exist outside detection (quantified nature of the interaction of matter and radiation)? We will see that these conditional probabilities can occur before detection, particularly at the level of a beamsplitter.

Examples :
- interference with one photon
- twin photons (if one is detected, the other is always present) and entanglement: the two photons behave in the same way, even at a distance
- the beam splitter: classical behavior and behavior in the presence of twin photons.
- teleportation of a quantum state
- quantum cryptography.

Knowledge assumed to have been acquired:

-the harmonic oscillator, creation and annihilation operators
- the Heisenberg's point of view (operators propagate and are applied to the input wave functions).

Table of Contents

This handout covers only the first four chapters. Reference 4 is a good introduction to quantum cryptography. For teleportation, please refer to BR p. 387-397

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Chapter 1: the beam splitter: semi-intuitive approach

I) Classical approach of coherent states

I.1) Introduction and classical definition

coherent state $\triangleq$ monomode plane wave: A(t)= Cste

In practice, the best approximation is a very monochromatic laser beam with a Gaussian transverse distribution. The theoretical frequency width of a laser can be as small as 1 Hz/10, rather 1kHz in practice.

I.2) Statistics followed by the photons

P(1 photon (detected) from t to t+dt) $\propto |A|^2 dt$

$|A|^2 dt \ll 1$ and independent events $\Rightarrow$ P(2 photons detected from t to t+dt)≈ 0

This is the definition of a Poisson law. Let $\langle N \rangle$ be the mean number of detected photons during Δt (non infinitesimal): $\langle N \rangle = E(N) = \sum_{N=0}^{\infty} N\, P(N)$

$$P(N\ sur\ \Delta t) = \frac{e^{-\langle N \rangle} \langle N \rangle^N}{N!} \tag{2}$$

Demonstration: see chapter 3.

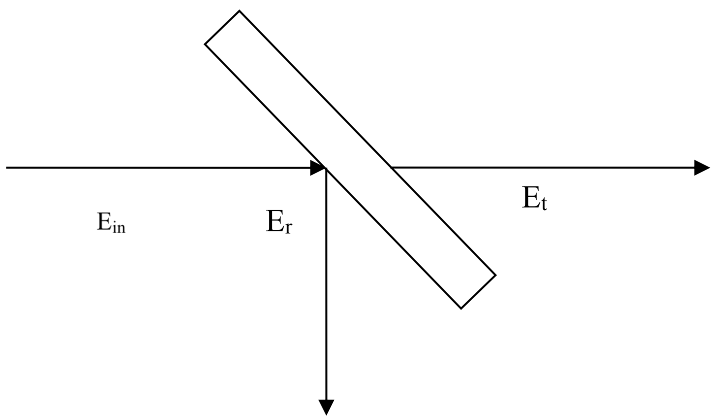
Variance: $\langle \Delta N^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$. If dt is sufficiently small, N is 0 or 1 and hence equal to N^2. Moreover $(\langle N^2 \rangle \cong \langle N \rangle) \gg \langle N \rangle^2$. Hence $\langle \Delta N^2 \rangle \cong \langle N \rangle$. This property, demonstrated on the infinitesimal dt, remains valid over any interval Δt: the events are independent, so the variances are added, as are the means.

Coherent state $\Rightarrow$ Poisson statistics of photons $\Rightarrow$ Variance=mean.

Of course, variance=mean is only possible for a number of discrete events and makes no sense for a continuous quantity.

II Beam splitter and photons

II.1) One incident coherent beam



Transmission coefficient: $T=|t|^2=0.5$
Reflection coefficient: $R=|r|^2=0.5$

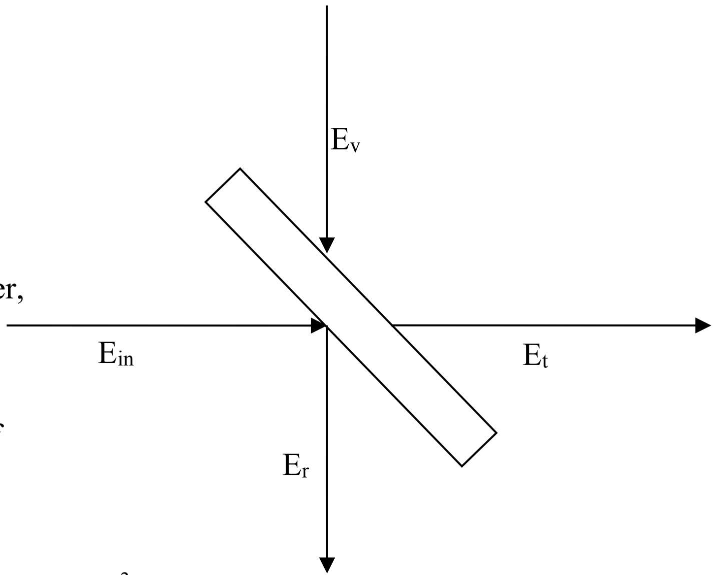
I_{in}= Cste(t) $\Rightarrow I_r = I_t = 0.5\ I_{in}$= Cste(t)

Constant output intensities $\Rightarrow$ Poisson photon statistics

For each incident photon, the beamsplitter flips a coin to decide if this photon is transmitted or reflected. This random behaviour of the beamsplitter leads to a division by $\sqrt{2}$ of the signal-to-noise ratio $\frac{<N>}{\sqrt{<\Delta N^2>}}$.

II.2) Two incident coherent beams

E_v and E_{in} have a phase relationship with each other (for example because issued from the same laser) and will interfere constructively or destructively on the outputs according to their phases. However, energy conservation requires that a bright fringe on t is always associated with to a dark fringe on r, which imposes a phase relationship between the coefficients of transmission and reflection in amplitude t_{in}, t_v, r_{in}, r_v., i.e.:



$$\begin{aligned}|E_{in}|^2+|E_v|^2 &= |E_t|^2+|E_r|^2 = |t_{in}E_{in}+r_vE_v|^2+|t_vE_v+r_{in}E_{in}|^2\\ &= T|E_{in}|^2+R|E_v|^2+\left[t_{in}r_v^*E_{in}E_v^*+c.c.\right]\\ &\quad +T|E_v|^2+R|E_{in}|^2+\left[r_{in}t_v^*E_{in}E_v^*+c.c.\right]\end{aligned} \tag{3}$$

...which requires that the sum of the terms in square brackets must be zero, i.e.:

$$t_{in}r_v^*+r_{in}t_v^*=0 \Rightarrow \varphi_{tin}-\varphi_{rv}=\pi+\varphi_{rin}-\varphi_{tv} \tag{4}$$

where φ stands for the phase values. A solution is: $t_{in}=t_v=r_v=\sqrt{2}/2$,
$r_{in}=-\sqrt{2}/2$,
giving, at the outputs:

$$\begin{aligned}|E_t|^2 &= T|E_{in}+E_v|^2\\ |E_r|^2 &= R|E_{in}-E_v|^2\end{aligned} \tag{5}$$

E_{in} and E_v in phase, constructive interferences on T, destructives on R
E_{in} and E_v with a π phase shift, constructive interferences on R, destructives on T

II.2) Two entangled photons at the input (MW p.646, BR p.105)

The incident state is a pair of entangled photons, with the wavefunction:

$\psi = ||1_{in}\ ,\ 1_v\rangle$, meaning that a photon is present in v **if and only if** a photon is present in in.

In other words, photons can be (are usually) produced randomly, but always in pairs in,v. Among the possible methods, some of which use radiative cascades of an excited atom. Let us take for example the Spontaneous down conversion (SPDC) in non-linear optics, where a pump photon of frequency ω_p is annihilated and gives rise to two signal and idler photons, of frequency ω_s and ω_i ,ω_s and ω_i , $\omega_s + \omega_i = \omega_p$. The conversion is random, but energy conservation requires the emission of pairs of photons:

P(1 signal photon | 1 idler photon) =1 . (6)

Of course, this state is purely quantum, since a classical field cannot describe conditional probabilities.

It can be shown (see Chapter 3) that the probability amplitude associated with an output possibility is the product of the probability amplitudes for each process. There are 4 possibilities:

- 2 reflected photons : amplitude $r_{in}\, r_v = -1/2 \rightarrow |1_t, 1_r>$
- 2 transmitted photons: amplitude $t_{in}\, t_v = 1/2 \rightarrow |1_t, 1_r>$

photon v transmitted, photon in reflected, amplitude $r_{in}\, t_v = -1/2 \rightarrow |0_t, 2_r>$
photon in transmitted, photon v reflected, amplitude $r_v\, t_{in} = 1/2 \rightarrow |2_t, 0_r>$

giving : $|\psi_{out}\rangle \propto (1/2-1/2)|1_t,1_r\rangle + 1/2\big[|2_t,0_r\rangle - |0_t,2_r\rangle\big]$ (7)

The first term being zero, we conclude that the photons both exit on the same path, with a probability 1/2 for each of the paths (N.B.: this "intuitive" calculation leads to a non-standardized wave function, contrary to Chapter 3).

In the 1987 Hong-Ou-Mandel experiment, coincidences with a temporal resolution of about a hundred nanoseconds were detected: if the two photons of a pair arrive with a slight shift, much less than this resolution, on the beamsplitter, they do not interfere and emerge randomly on one or the other of the channels, giving rise to coincidence if they leave on different channels, which occurs with every other chance. On the other hand, if the two photons arrive together, they exit on the same path, which results experimentally in a drop in of coincidences to 0 in the absence of parasitic coincidences, due either to electronics or to the arrival of two pairs in the detector's resolution window. The width of this "coincidence dip", of the order of a hundred fs, gives access to the width of the photon's wave function, proportional in SPDC to the inverse of the frequency width of the pair emission process, which is itself related to the phase matching conditions (see the non-linear optics course). In this original experiment, the arrival delay was obtained by modifying the optical paths. In more recent versions of the experiment (see for example J. Beugnon et al, Nature 4628, April 2006, p.779), the width of the wave function due to an atomic transition is about ten nanoseconds, i.e. much higher than the resolution of the detector. We can then record directly in time the interference of the wave functions of the two photons, produced here by two distinct atoms, trapped by laser and excited simultaneously.

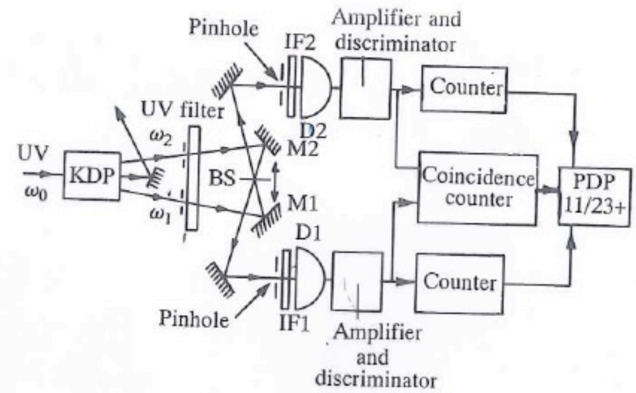


Fig. 22.5 Outline of an experiment to measure the time separation between two photons by interference at a beam splitter BS. KDP is a nonlinear crystal of potassium dihydrogen phosphate functioning as down-converter and PDP 11/23 is a computer. (Reproduced from Hong, Ou and Mandel, 1987.)

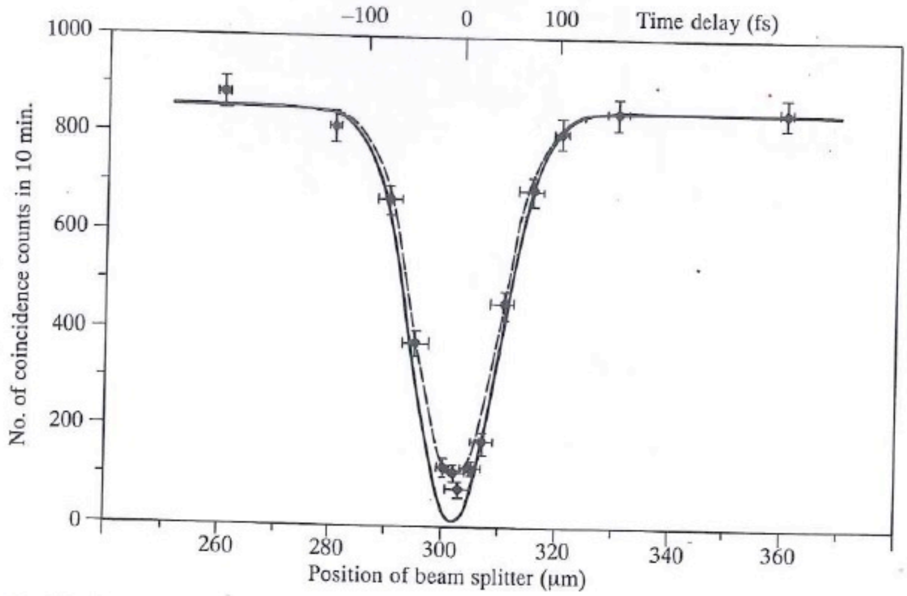


Fig. 22.6 Results of two-photon coincidence measurements as a function of differential time delay between the two photons superimposed on the theoretical (full) curve. (Reproduced from Hong, Ou and Mandel, 1987.)

MW p. 1087

Chapter 2: Quantification of the electromagnetic field: photons
(reference: "Photons et atomes", p.9 to 37 : **CC)**

I) Position of the problem

Maxwell's equations are partial differential: all points in space are connected. In contrast, in the spatial frequency domain, the field can be expressed as a superposition of independent plane waves. Moreover, a monochromatic plane wave is the equivalent of a harmonic oscillator.

If we show that a plane wave obeys the same classical equations as a harmonic oscillator, we can quantify it in a totally similar way and the energy levels of the harmonic oscillator, separated each other by hν, will correspond to a number of photons.

II) Modal decomposition of the electromagnetic field, equations of motion

II.1) Equations of motion

In an isotropic environment without charge and currents, the electromagnetic field is transverse and there are two Maxwell equations left:

$$\begin{aligned}\nabla \wedge \vec{E}(\vec{r},t) &= -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)\\ \nabla \wedge \vec{B}(\vec{r},t) &= \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t)\end{aligned} \qquad (8)$$

These nonlocal equations (because of the gradient operator) become local after a Fourier Transform on the space variables $\vec{r}$: $\quad \nabla \rightarrow T.F \rightarrow .j\vec{k}$

$$j\vec{k} \wedge \vec{E}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{k},t) \qquad (9)$$

$$jc^2\vec{k} \wedge \vec{B}(\vec{k},t) = \frac{\partial}{\partial t}\vec{E}(\vec{k},t) \qquad (10)$$

Let be $\vec{\kappa} \triangleq \vec{k}/\left\|\vec{k}\right\|$. Then (10) ± $c\vec{\kappa} \wedge (9)$ writes, by using $\vec{\kappa} \wedge (\vec{\kappa} \wedge \vec{E}) = -\left\|\vec{\kappa}\right\|^2 \vec{E}$ (since $\vec{\kappa} \perp \vec{E}$) and ω=k .c:

$$\frac{\partial}{\partial t}\left(\vec{E}(\vec{k},t) \pm c\vec{\kappa} \wedge \vec{B}(\vec{k},t)\right) = \pm j\omega \left(\vec{E}(\vec{k},t) \pm c\vec{\kappa} \wedge \vec{B}(\vec{k},t)\right) \qquad (11)$$

(11) is a system of two uncoupled equations, unlike (9) and (10). For each $\vec{k}$, two polarization directions can be defined in the orthogonal plane to $\vec{k}$. Without loss of generality, we will assume $\vec{E}(\vec{k},t)$ polarized along ӯ (we will see later that the two polarization directions correspond to two distinct modes), and we define the normal variables:

$$\alpha(\vec{k},t)\vec{y} = -\frac{j}{2\mathbb{N}(k)}\left(\vec{E}(\vec{k},t) - c\vec{\kappa}\wedge\vec{B}(\vec{k},t)\right)$$
$$\beta(\vec{k},t)\vec{y} = -\frac{j}{2\mathbb{N}(k)}\left(\vec{E}(\vec{k},t) + c\vec{\kappa}\wedge\vec{B}(\vec{k},t)\right) \qquad (12)$$

where $\mathbb{N}(k)$ is a normalization factor that will be defined later. $\alpha(\vec{k},t)$ obeys the equation of a harmonic oscillator with α=x+j(p/mω), giving:

$$\frac{\partial}{\partial t}\left(\alpha(\vec{k},t)\right) + j\omega\left(\alpha(\vec{k},t)\right) = 0 \qquad (13)$$

The variables α and β of equations (12) are not independent. Indeed:

$$\vec{E}(\vec{r},t) \ et \ \vec{B}(r,t) \text{ réels } \Rightarrow \vec{E}(\vec{k},t) = \vec{E}^*(-\vec{k},t) \Rightarrow \beta(\vec{k},t) = -\alpha^*(-\vec{k},t) \qquad (14).$$

It is therefore possible to reconstruct the fields from only one of these variables. For example, for $\vec{E}$: $\vec{E}(\vec{k},t) = j\mathbb{N}(k)\left(\alpha(\vec{k},t) - \alpha^*(-\vec{k},t)\right)\vec{y}$.

In order to express the energy in number of photons, we assume $\mathbb{N}(k) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}}$, which allows the energy of the transverse field H to be written in the form:

$$H = \frac{\varepsilon_0}{2}\int d^3k\left[\left\|\vec{E}(\vec{k},t)\right\|^2 + c^2\left\|\vec{B}(\vec{k},t)\right\|^2\right] = \int d^3k\frac{\hbar\varpi}{2}\left[\alpha^*(\vec{k},t)\alpha(\vec{k},t) + \alpha(\vec{k},t)\alpha^*(\vec{k},t)\right]$$
(15)

<u>N.B. 1):</u> in order that the integral takes into account all possible modes, the two possible polarizations should be considered for each mode.
<u>N.B 2):</u> The order of α and α* has, of course, no importance in this classical expression. A symmetrical form (15) has been written in order to prepare the quantification.

<u>II.2) Modal decomposition</u>

Continuous integrals (15) can be directly quantified by using commutators equal to Dirac pulses. The simpler solution of decomposition in discrete modes will be used here. We plunge the field into a cube of side L. The limiting conditions lead to L=$n_{x,y,z}$ λ, n integer, i.e. $k_{x,y,z} = \frac{2\pi}{L}n_{x,y,z}$. The normal variables become:
$\alpha(\vec{k},t) \rightarrow \alpha_i(t),\ i = f(n_x,n_y,n_z)$ integer numbering the mode.

<u>**III) Quantification**</u>

The electric field has just been decomposed into a sum of harmonic oscillators. It is quantified in the same way:

$\alpha_i \rightarrow$ opérateur a_i
$\alpha_i^* \rightarrow$ opérateur $a_i^\dagger$ and $\left[a_i, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = 0, \; \left[a_i, a_j^\dagger\right] = \delta_{ij}$

where [] designates a commutator.

The field becomes an operator, which can therefore be expressed as:

$$E(r,t) = E^+(r,t) + E^-(r,t) = \sum_i c_i \left[a_i e^{j\vec{k}_i.\vec{r}} e^{-j\omega t} - a_i^\dagger e^{-j\vec{k}_i.\vec{r}} e^{j\omega t} \right] \tag{16}$$

The terms with a positive frequency, in $e^{-j\omega t}$ with operators a_i, appear in E^+, while those with a negative frequency corresponding to operators $a_i^\dagger$ appear in E^-.

The Hamiltonian writes: $H = \frac{1}{2}\sum_i \hbar\omega_i (a_i^\dagger a_i + a_i a_i^\dagger) = \sum_i \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right)$ (17)

<u>IV) Eigenstates of energy, photons</u>

Let us define, as for the harmonic oscillator, the number state n_i called here a Fock state with n_i photons, eigenvector of $a_i^\dagger a_i$: $a_i^\dagger a_i \left|n_i\right\rangle = n_i \left|n_i\right\rangle$, n_i positive or zero integer, and:

$$a_i^\dagger \left|n_i\right\rangle = \sqrt{n_i+1}\left|n_i+1\right\rangle, \quad a_i\left|n_i\right\rangle = \sqrt{n_i}\left|n_i-1\right\rangle, \quad a_i\left|0\right\rangle = 0 \tag{18}$$

We can thus fill the different modes with photons by repeated applications of the operator created from vacuum:

$$\left|n_1, \ldots\ldots, n_i, \ldots\ldots\right\rangle = \frac{\left(a_1^\dagger\right)^{n_1}}{\sqrt{n_1!}} \cdots\cdots \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_1!}} \quad \left|0\right\rangle \tag{19}$$

we are thus led to the definition of the photon:

photon ≙ elementary excitation of a mode of the quantized electromagnetic field

Just like the harmonic oscillator, the Hamiltonian applied to an empty mode gives an energy $\frac{1}{2}\hbar\omega_i$. While this has consequences, there is no question of directly detecting the energy of the vacuum (in obvious contradiction with energy conservation). It can be noted in this regard that a measurable quantity such that the number of photons corresponds to what will be called the normal order: the operators $a_i^\dagger$ to the left of the operators. a_i.

Chapter 3: Coherent states and quantum treatment of the beamsplitter

<u>I) Definition of a coherent state (MQ complement GV)</u>

We want to find the quantum state $|\alpha_i\rangle$ that reproduces at best the properties of the plane wave of amplitude α_i in mode i. The mean of the operator field applied to this mode must give the classical amplitude, and the mean number of photons must correspond to the classical intensity (expressed as a number of photons per spatio-temporal mode). So we look for α_i such as:

$$\langle\alpha_i|a_i|\alpha_i\rangle = \alpha_i$$
$$\langle\alpha_i|a_i^\dagger a_i|\alpha_i\rangle = \alpha_i^*\alpha_i = \langle N\rangle$$

where $\langle N\rangle$ is the mean number of photons per mode.

(20)

The eigenvector of a_i with the eigenvalue α_i is solution, i.e. the state verifying $a_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$. It will be the definition of the coherent state in mode i.

II) Properties

II.1) Photon number

The index i will be omitted in the following. From (20) and the hermitian conjugate of (18), it comes:

$$\left.\begin{cases} a\;|\alpha\rangle = \alpha|\alpha\rangle \\ \langle n|a^\dagger = \sqrt{n}\,\langle n_i - 1| \end{cases}\right\} \Rightarrow \langle n|a^\dagger a\;|\alpha\rangle = n\langle n|\alpha\rangle = \sqrt{n}\alpha\langle n-1|\alpha\rangle$$
$$\Rightarrow \langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle \tag{20}$$

If we know the amplitude of $|\alpha\rangle$ projected on $|n-1\rangle$, we can deduce the amplitude of $|\alpha\rangle$ projected on $|n\rangle$. The recurrence relationship is initiated by noting that the vacuum state $|0\rangle$ is both a coherent and a Fock state. By applying n times (20), we find the amplitude of $|\alpha\rangle$ on $|n\rangle$, which gives:

$$|\alpha\rangle = K\left[|0\rangle + \frac{\alpha}{\sqrt{1}}|1\rangle + ... + \frac{\alpha^n}{\sqrt{n!}}|n\rangle + ...\right] \tag{21}$$

K ensures the normalization of $|\alpha\rangle$:

$$\langle\alpha|\alpha\rangle = 1 = K^2\sum_{n=0}^{\infty}\left(\frac{|\alpha|^n}{\sqrt{n!}}\right)^2 = K^2 e^{|\alpha|^2} \Rightarrow \qquad |\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{22}$$

The probability of measuring n photons is:

$$\mathrm{P(n)} = |\langle\alpha|n\rangle|^2 = e^{-|\alpha|^2}\frac{\left(|\alpha|^2\right)^n}{n!} = e^{-\langle N\rangle}\frac{\langle N\rangle^n}{n!} \tag{23}$$

giving, as expected, a Poisson distribution.

II.2) Variance

Since the number of photons is Poissonian, the variance is equal to the mean. This property can also be established directly:

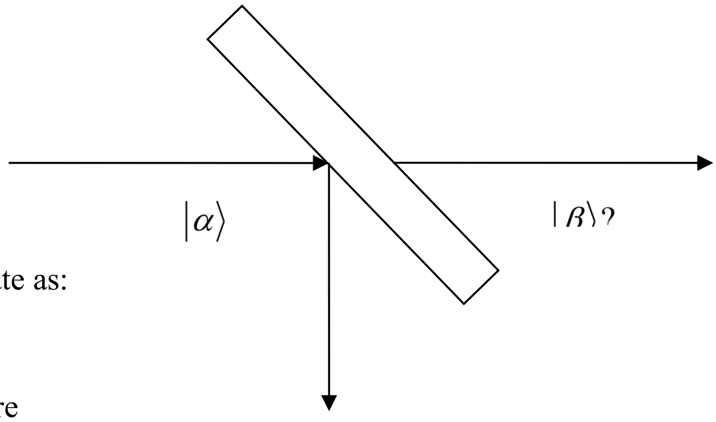
$$\langle N \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = \alpha^* \alpha$$

$$\langle \Delta N^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \langle \alpha | a^\dagger a a^\dagger a | \alpha \rangle - (\alpha^* \alpha)^2 = \langle \alpha | a^\dagger a^\dagger a a | \alpha \rangle + \langle \alpha | a^\dagger a | \alpha \rangle - (\alpha^* \alpha)^2 = \alpha^* \alpha = \langle N \rangle$$

(24)

III) states transmitted by a beamsplitter

III.1) One incident coherent beam



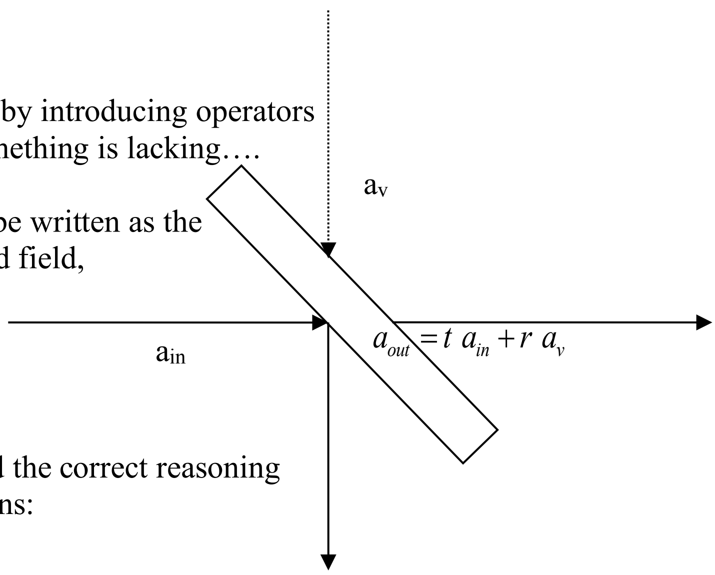
Are we allowed to simply write the output state as:
$|\beta\rangle = t|\alpha\rangle$ (or $r|\alpha\rangle$ for reflection)?

We'll see that the answer is almost correct here (there is however an obvious problem of normalization), but only because the system is linear. Such simple reasoning can be dangerous in quantum optics. On the other hand, quantification has transformed the fields into operators. It is therefore often easier and safer to deduce from the classical field propagation equations the operators' equivalent propagation equations, and therefore to adopt the Heisenberg's point of view, by ultimately having the output operators act on the input wave functions. The operators' propagation equations differ from the classical equations only in one respect......

The classical equations lead here to $a_{out} = t\ a_{in}$
When calculating the commutator, we realize that this result is not correct: $\left[a_{out} a_{out}^\dagger \right] = tt^* < 1$

However, the field quantification was made by introducing operators a_i on the mode i with unity commutator. Something is lacking….

Actually, the field in the output mode must be written as the superposition of a transmitted and a reflected field, even if this field acts on the vacuum.



It is easy to verify that the commutator of $a_{out} = t\ a_{in} + r\ a_v$ is unitary.
The reasoning that "forgets" the vacuum and the correct reasoning give the same mean number of output photons:

$$\langle N_{out}\rangle = \langle \alpha_{in}|\langle 0|a^{\dagger}_{out}a_{out}|0\rangle|\alpha_{in}\rangle = t^*_{in}t_{in}\alpha^*_{in}\alpha_{in} + r^*_v r_v.0 = |t_{in}|^2|\alpha_{in}|^2 \quad (25)$$

On the other hand, a computation like (24) shows that $\langle \Delta N^2_{out}\rangle = \langle N_{out}\rangle$ only if we use the correct form of a_{out}. This computation is rapid by using the fact that the commutator of a_{out} is unitary, or a bit more tedious by starting from $a_{out} = t\ a_{in} + r\ a_v$. However, this second way (do it!) evidences the variance terms which come from vacuum fluctuations. In the terms of the intuitive representations of chapter 1, this is the vacuum that ensures flip coin of photons at the beamsplitter!

III.2) An incident entangled state

A) Correspondence Schrödinger's point of view Heisenberg's point of view (**BR p.80**)

While classical optics can determine how operators propagate, the same cannot be said for wave functions. Very often, determining $|\psi_{out}\rangle$ is difficult, especially if the number of photons varies (amplifiers). In other words, we can easily find the mean values $\langle\psi_{out}|A_{in}|\psi_{out}\rangle = \langle\psi_{in}|A_{out}|\psi_{in}\rangle$ because A_{out} is easily calculated by correspondence with the classical equations, but $|\psi_{out}\rangle$ can be of a frightening complexity. Let us take the case of a matrix of 8x8 photodetectors, on which can arrive from 0 to 5 photons. $|\psi_{out}\rangle$ is the superposition of the probability amplitudes for each of the 6^{64} possibilities.
However, there is one exception, when the system only includes linear elements, which do not create photons (an absorption is modelled by a beamsplitter). Indeed:

$$a_{out} = U^{\dagger}\ a_{in}\ U, \text{ où U est l'opérateur d'évolution : } \mathrm{U}|\psi_{\mathrm{in}}\rangle = |\psi_{\mathrm{out}}\rangle \quad (26)$$

Let us write the input wavefunction on the Fock basis:

$$|\psi_{in}\rangle = c_0|0\rangle + c_1|1\rangle + \ldots + c_n|n\rangle + \ldots = \left[c_0 + c_1 a^{\dagger} + \ldots + c_n\frac{(a^{\dagger})^n}{\sqrt{n!}}\right]|0\rangle \quad (27)$$

then the output wavefunction writes:

$$|\psi_{out}\rangle = U|\psi_{in}\rangle = U\left[c_0 + c_1 a^{\dagger} + \ldots + c_n\frac{(a^{\dagger})^n}{\sqrt{n!}}\right]U^{\dagger}U|0\rangle = \left[c_0 + c_1 a'^{\dagger} + \ldots + c_n\frac{(a'^{\dagger})^n}{\sqrt{n!}}\right]|0\rangle$$

(28)

with $a' = UaU^{\dagger}$, inverse relation of that relying a_{out} to a_{in}: $a'_{out} = a_{in} = Ua_{out}U^{\dagger}$. To obtain (28), we have used the fact that the system does not create photons, $\mathrm{U}|0\rangle = |0\rangle$ and added as many identity operators $U^{\dagger}U$ as needed. $|\psi_{out}\rangle$ is therefore obtained by applying to the output modes the input operators used in $|\psi_{in}\rangle$, expressed as linear combinations of the output operators.

Examples :

beamsplitter (out1 is the name of the output where in is transmitted and out2 is the output where in is reflected. t and r are assumed real):

$$a_{out1} = t\ a_{in} + r\ a_v \text{ et } a_{out2} = t\ a_v - r\ a_{in} \Rightarrow$$
$$a_{in} = t\ a_{out1} - r\ a_{out2} \text{ et } a_v = r\ a_{out1} + t\ a_{out2} \tag{29}$$

(29) can be directly established, by solving the system, or by applying the principle of reversibility of light

example 1:

$$|\psi_{in}\rangle = |1_{in}, 0_v\rangle = a_{in}^{\dagger}|0_{in}, 0_v\rangle$$
$$|\psi_{out}\rangle = a_{in}^{\dagger}|0_{out1}, 0_{out2}\rangle = (t\ a_{out1}^{\dagger} - r\ a_{out2}^{\dagger})|0_{out1}, 0_{out2}\rangle = t|1_{out1}, 0_{out2}\rangle - r|0_{out1}, 1_{out2}\rangle \tag{30}$$

example 2: entangled state

$$|\psi_{in}\rangle = |1_{in}, 1_v\rangle = a_{in}^{\dagger}a_v^{\dagger}|0_{in}, 0_v\rangle, \text{ état intriqué}$$
$$|\psi_{out}\rangle = a_{in}^{\dagger}a_v^{\dagger}|0_{out1}, 0_{out2}\rangle = \left[tr(a_{out1}^{\dagger}a_{out1}^{\dagger} - a_{out2}^{\dagger}a_{out2}^{\dagger}) + (t^2 - r^2)a_{out1}^{\dagger}a_{out2}^{\dagger}\right]|0_{out1}, 0_{out2}\rangle \tag{31}$$
$$= \sqrt{2}\ tr\left[|2_{out1}, 0_{out2}\rangle - |0_{out1}, 2_{out2}\rangle\right] + (t^2 - r^2)|1_{out1}, 1_{out2}\rangle$$

(7) of chapter 1 is retrieved, but this time with a correct normalization factor!

Chapter 4: Squeezing

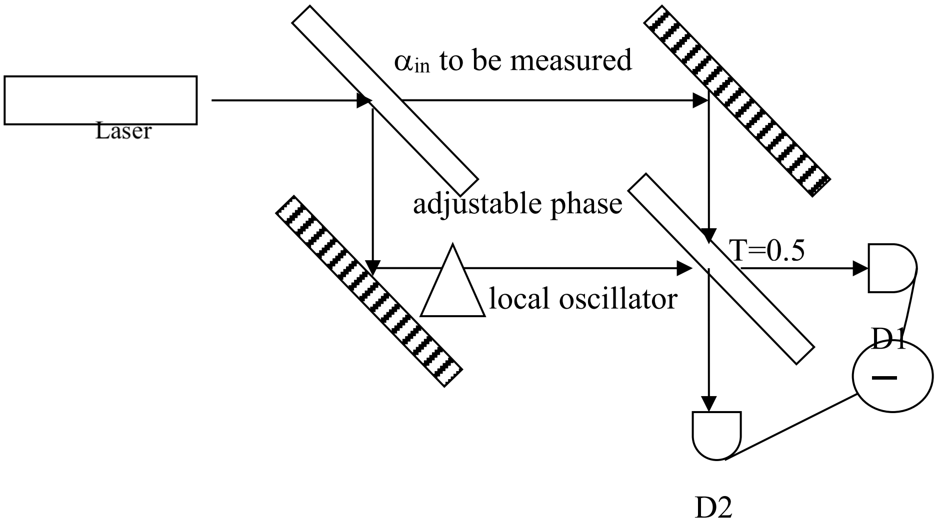
I) Introduction: classical measurement of phase and amplitude fluctuations (BR p.206-207).

The electric field, as a quantity measurable by means of an interferometric device, has an uncertain value. For example, for a coherent state, uncertainty on amplitude is due to photon noise, but uncertainty on phase can also be measured

Classically, we write $E(t)=(E_0 + \Delta E_0(t)\cos(\omega t - \Delta\varphi(t))$, i.e. if $\Delta E_0(t) << E_0$ et $\Delta\varphi(t) << 2\pi$: $E(t)=E_0\cos(\omega t) + \Delta E_0(t)\cos(\omega t) + \Delta\varphi(t)\sin(\omega t)$ (32)

or, in complex notations: $E=E_0 + \Delta X_1 + j\Delta X_2$ (we will see later the purpose of this new notation)

It is possible of measuring the phase uncertainty by homodyne detection:



The amplitude of the local oscillator writes: $\alpha_{LO}=E_0+\Delta X_{1LO}+j\Delta X_{2LO}$, while the field to be measured is given by: $\alpha_{in}=\alpha+\Delta X_{1in}+j\Delta X_{2in}$.

The field on the photodiodes is given by:

$$\alpha_{D1}=\frac{1}{\sqrt{2}}(\alpha_{LO}+\alpha_{in}),\quad \alpha_{D2}=\frac{1}{\sqrt{2}}(\alpha_{LO}-\alpha_{in})$$

leading to an intensity difference $I_-=\dfrac{|\alpha_{D1}|^2-|\alpha_{D2}|^2}{2}=\dfrac{\left(\alpha_{LO}\alpha_{in}^*+\alpha_{LO}^*\alpha_{in}\right)}{2}$ (33)

The key point of the following is that the fluctuation terms on the field and on the local oscillator have the same order of magnitude. Quantum computation would be immediate (see TD2). A more traditional, although not entirely, way of seeing it is to notice that the standard deviation on intensity is equal to $|\Delta X|\,|\alpha|$, giving a signal-to-noise ratio of $|\alpha|^2/(|\Delta X|\,|\alpha|)=|\alpha|/|\Delta X|$. However, this signal-to-noise ratio is proportional, for a coherent state, to the root square of the photon number, i.e. proportional to $|\alpha|$. In other words, two coherent states have field fluctuations of equal amplitude, even if these states have very different amplitudes (and therefore intensity fluctuations of different magnitude).

Hence: $\alpha_{LO}>>\alpha_{in}\Rightarrow \Delta X_{in}.\alpha_{LO}>>\Delta X_{LO}\alpha_{in}$ (34)

The phase with respect to the measured field of the local oscillator, φ_{LO}, is adjustable. Hence (33) writes, by taking in account (34):

$$I_-\approx|E_0\alpha_{in}|\cos(\varphi_{LO})+E_0\left(\Delta X_{1in}(t)\cos(\varphi_{LO})+\Delta X_{2in}(t)\sin(\varphi_{LO})\right) \quad (35)$$

In (35), the time dependence of fluctuations is explicitly indicated to emphasize that the first term is a continuous background term: it is easily eliminated by filtering. You can then access to ΔX_{1in}(t) or $\Delta X_{2in}(t)$ or any combination of the two by adjusting the phase of the local oscillator.

The variance of this intensity difference writes:

$$\left\langle \Delta I_-^{\ 2} \right\rangle = E_0^{\ 2} \left(\left\langle \Delta X_{1in}^{\ \ 2} \right\rangle \cos^2(\varphi_{LO}) + \left\langle \Delta X_{2in}^{\ \ 2} \right\rangle \sin^2(\varphi_{LO}) \right) \tag{36}$$

We will see in TD 2 that all the above reasoning can be extended to the quantum case. A key point is that all the ports of the second beamsplitter are occupied: an empty port has not been forgotten. Also (36) allows measuring the quantum properties of in (it can be quantum vacuum!), see TD2.
The following page presents the first experimental realization of vacuum squeezing: we will see in TD2 that, if the squeezed vacuum is not vacuum at all (it is composed of pairs of photons with a calculable mean, last question of TD2), it fluctuates "less" than vacuum, in the sense that the fluctuations on the appropriate quadrature, measured as in (36), are lower than those obtained for $\left|\alpha_{in}\right\rangle = \left|0\right\rangle$.

$\left|\alpha_{in}\right\rangle = \left|0\right\rangle$ allows the interferometer to be calibrated to the "shot noise limit" (SNL) (trace (i) in Figure 9.29 (a) on the following page). By varying the phase of the local oscillator (abscissa of this same figure), we obtain with squeezed vacuum the curve (ii): squeezing is obtained for the phases where this curve passes below the SNL.

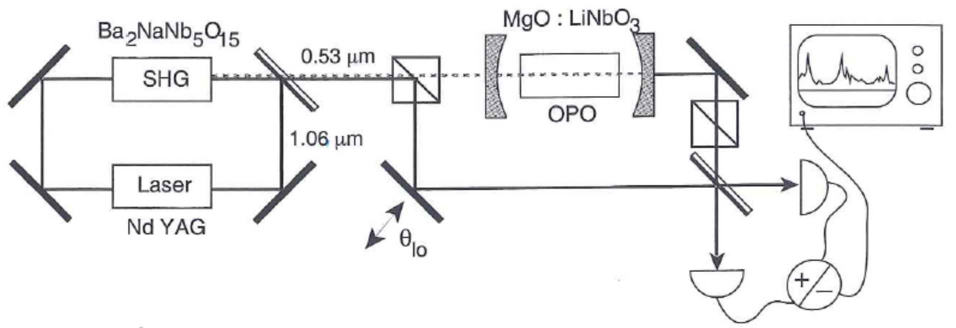


Figure 9.28: The schematic layout of the experiment by Wu et al. [Wu87] generating a squeezed vacuum state with a sub threshold OPO.

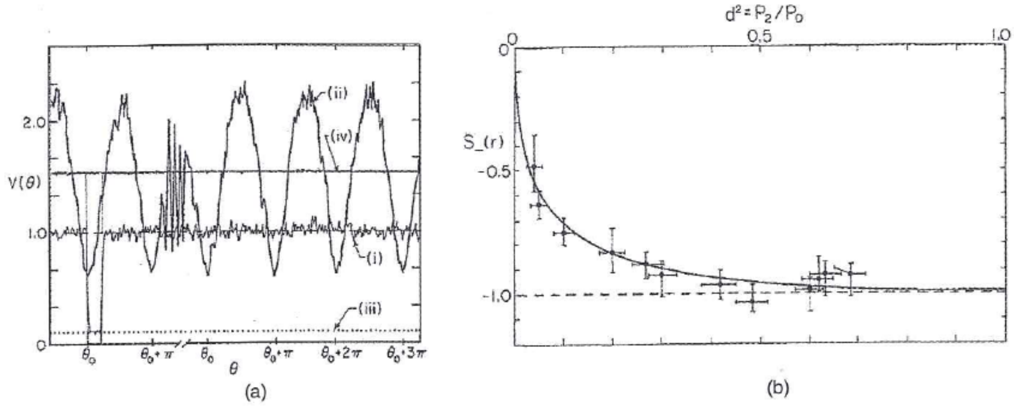


Figure 9.29

BR p. 266-267

TD 1 Quantum optics master 1 Polarization entanglement. (MW p.648)

A process, like a radiative cascade radiative of an atom or spontaneous down conversion in a χ^2 crystal, produces entangled pairs of photons with a non separable wavefunction:

$$|\psi> \ = \frac{1}{\sqrt{2}}(|1_{1x},0_{1y},0_{2x},1_{2y}> - |0_{1x},1_{1y},1_{2x},0_{2y}>$$

1x et 1y (2x et 2y) stand for two directions of polarization, orthogonal each other and orthogonal to the propagation direction. The photons 1 and 2 can be far away, because of propagation in two different directions after their simultaneous emission. Polarizers and detectors are supposed to be of unity quantum efficiency.

1) What is the probability P(1x:+) of detecting the photon 1 behind a polarizer along x ?

2) A photon 1 is detected along x. What is the probability P(2y:+ | 1x:+) of detecting a photon 2 along y ? (N.B. | means here "given")

3) The polarizer 1 is rotated of an angle θ_1 with respect to x. What is the probability $P(1\theta_1:+)$ of detecting the photon 1?

Suggestion: associate to the projected classical field $E_1 = E_x\cos(\theta_1) + E_y\sin(\theta_1)$ an annihilation operator a_1 of the same form (what is its commutator?) and compute the mean number of photons $<\psi|a_1^\dagger a_1|\psi>$.

4) Show that the joint probability of detecting 1 along θ_1 and 2 along θ_2 is given by:

$$P(1\theta_1:+ \text{ et } 2\theta_2:+) = \frac{1}{2}\left(\sin^2(\theta_1)\cos^2(\theta_2) + \sin^2(\theta_2)\cos^2(\theta_1) - 2\sin(\theta_1)\cos(\theta_1)\sin(\theta_2)\cos(\theta_2)\right)$$

$$= \frac{1}{2}\left(\sin^2(\theta_1 - \theta_2)\right)$$

Suggestion: compute $<\psi|a_1^\dagger a_2^\dagger a_2 a_1|\psi>$.

5) Deduce the probability $P(2\theta_2:+\,|\,1\theta_1:+)$

Recall: P(A and B) = P(A|B). P(B)= P(B|A). P(A)

6) The value of θ_1 is modified when the photons 1 and 2 are separated, in route to the polarizers. Is the probability $P(2\theta_2:+)$ modified? Give the answer in three ways (hoping to obtain three times the same result!):

- with the help of question 3.
- by using question 5 and (simply!) calculating $P(2\theta_2:+\,|\,1\theta_1:-)$.
- by applying the causality principle issued from special relativity.

TD 2: Quadratures, quantum vacuum and noise squeezing

Let us define the quadrature operators: $X_1 = \frac{1}{\sqrt{2}}(a^\dagger + a),\ X_2 = \frac{i}{\sqrt{2}}(a^\dagger - a)$

1) calculate the commutator $[X_1, X_2]$

2) Classical correspondence: the operator a being associated to the envelope of a field with a time dependence in $e^{-j\omega t}$, to which time dependences are respectively associated X_1 et X_2?

3) Vacuum energy: Compute the energy operator $\frac{X_1^2 + X_2^2}{2}$ with respect to a et $a^\dagger$.

Deduce the value of $\left\langle 0 \left| \frac{X_1^2 + X_2^2}{2} \right| 0 \right\rangle$, sometimes called vacuum energy (in photons per mode).

4) Energy of the squeezed vacuum: A degenerate parametric amplifier obeys to the coupled equations:

$$\begin{cases} \dfrac{da(z)}{dz} = g\ a^\dagger(z) \\ \dfrac{da^\dagger(z)}{dz} = g\ a(z) \end{cases}$$

Find $X_1(z)$ et $X_2(z)$ versus $X_1(0)$ and $X_2(0)$. Calculate the mean number of photons obtained at the output of an amplifier of length z, in 'absence of injected signal.

5) Energy of a squeezed coherent state: same question for a coherent state $|\alpha\rangle$ injected in the amplifier. First take α real (which means an input field on the amplified quadrature), then α with an arbitrary phase: $\alpha = |\alpha|\ e^{i\varphi}$. In both cases, give first the mean value of the energy operator, at the input then at the output of the amplifier.

6) Variance of a coherent state: calculate the variances $\langle \Delta X_1^2 \rangle$ et $\langle \Delta X_2^2 \rangle$ of the quadrature operators applied to a coherent sate.

7) Extension 1 (easy): same question at the output of an amplifier with a coherent state at the input.

8) Extension 2: variance of the photon number (tedious.... or using cleverness): calculate $\langle \Delta N^2 \rangle$ at the output of an amplifier with a coherent state at the input.

Some useful formulas: $\cos^2(\varphi) = \frac{1+\cos(2\varphi)}{2}, \qquad \sin^2(\varphi) = \frac{1-\cos(2\varphi)}{2}$

$$\mathrm{sh}(2a) = 2\mathrm{sh}(a)\mathrm{ch}(a); \qquad sh^2(a) = \frac{\mathrm{c}h(2a)-1}{2}$$

TD3: Ramsey interferometer

A two-level atom |g> and |e> interacts in two cavities R1 and R2 at resonance with a classical microwave field issued from a single source. The cavities are both set so that the atom performs a Rabi oscillation corresponding to a quarter turn on the Bloch sphere. The purpose of this TD is to show that the probability of detecting one of the levels at the output oscillates sinusoidally as a function of the phase shift of the microwave field between the two cavities.

Cavity R1. The field E_1 is taken as real phase reference, which allows, at resonance, the interaction Hamiltonian to be written as: $H_1 = \begin{bmatrix} 0 & -i\Omega_1/2 \\ i\Omega_1/2 & 0 \end{bmatrix}$, where Ω_1 is proportional to E_1.

1) show that the eigenvectors are written as

$|\psi_+\rangle = \frac{\sqrt{2}}{2}\left(|g\rangle + i\,|e\rangle\right)$, $|\psi_-\rangle = \frac{\sqrt{2}}{2}\left(-|g\rangle + i\,|e\rangle\right)$, with which eigenvalues?

2) If the atom is in the state |g> at t=0, which is its state at $\Omega_1 t_1 = \pi/2$?

Cavity R2. By taking into account the phase corrections due to the atom move, the field E_2 in the second cavity writes $E_2 = E_1 \exp(i\phi)$, leading to an Hamiltonian $H_2 = \begin{bmatrix} 0 & -i\Omega_1/2\exp(-i\phi) \\ i\Omega_1/2\exp(i\phi) & 0 \end{bmatrix}$.

3) Show that the eigenvectors write:

$|\psi_+\rangle = \frac{\sqrt{2}}{2}\left(\exp(-i\phi/2)|g\rangle + i\exp(i\phi/2)|e\rangle\right)$, $|\psi_-\rangle = \frac{\sqrt{2}}{2}\left(-\exp(-i\phi/2)|g\rangle + i\exp(i\phi/2)|e\rangle\right)$

with which eigenvalues?

4) What does become the atom state, entering in R2 in the state obtained after R1 (the phase reference taken for E_2 allows ignoring the phase shift of the atom state between R1 and R2) after an interaction of duration t_1 in R_2? Show that the probability of finding the atom in the |g> state is: $P(|g\rangle) = \sin^2(\phi/2)$.