GENUS THEORY, GOVERNING FIELD, RAMIFICATION AND FROBENIUS

by

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Abstract. — In this work we develop, through a governing field, genus theory for a number field K with tame ramification in T and splitting in S, where T and S are finite disjoint sets of primes of K. This approach extends that initiated by the second author in the case of the class group. It allows expressing the S-T genus number of a cyclic extension L/K of degree p in terms of the rank of a matrix constructed from the Frobenius elements of the primes ramified in L/K, in the Galois group of the underlying governing extension. For quadratic extensions $\mathrm{L}/\mathbb{Q}$, the matrices in question are constructed from the Legendre symbols between the primes ramified in $\mathrm{L}/\mathbb{Q}$ and the primes of S.

1. Introduction

Let K be a number field, and let S and T be two finite and disjoint sets of places of K. We assume that T contains only finite places. Let K_T^S denote the maximal abelian extension of K, totally decomposed at all places in S (or even S-split), unramified outside of T, and with at most tame ramification at the places $v \in T$ (or even T-tamely ramified). This is a finite extension, and the Artin map allows us to identify the Galois group $\mathrm{Gal}(\mathrm{K}_T^S/\mathrm{K})$ with the S-ray class group of K modulo $\mathfrak{m} := \prod_{v\in T} v$, which we denote by $\mathrm{Cl}_{\mathrm{K},\mathfrak{m}}^S$. For more details, see Section §1.1.1.

Now let L/K be an extension of number fields with ramification set Σ. The genus theory provides information about the class group $\mathrm{Cl}_{\mathrm{L},\mathfrak{m}_\mathrm{L}}^{S_\mathrm{L}}$ in terms of Σ and the behavior of the S-units of K in L/K. See Theorem 2.1.

The first remarkable result in genus theory dates back to Gauss, concerning the 2-Sylow subgroup of the class group of quadratic extensions of $\mathbb{Q}$ (see [**8**, Chapter 1, §1] and [**4**, Chapter IV, §4, Exercise 4.2.10]). The phenomenon described by Gauss has been studied,

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developed, and generalized by many authors, including the works of Hasse [**6**], Leopoldt [**9**], Furuta [**3**], and others. For more details, see [**4**, III.4].
The introduction of the sets T and S was initiated by Jaulent [**7**], Federer [**1**], and others. A very good overview of all this can be found in [**7**, Chapter II, 2.4, Chapter III, 2.1].

The work presented here is inspired by [**11**]. We develop the S-T genus theory via a governing extension denoted by $\mathrm{F}_T^S/\mathrm{K}$, where the usual ramification conditions are interpreted through relations between Frobenius elements. As a consequence, and similar to [**11**, Theorem 1.3], questions in genus theory can be translated into questions about the behavior of Frobenius elements in a governing field, for which the Chebotarev density theorem becomes central.

When the base field K is given and the Galois group of L/K is a fixed abelian group, Frei, Loughran, and Newton [**2**] studied the asymptotic behavior of the genus number of L/K (for the class group) with respect to the discriminant of L. It would be interesting to revisit their results in light of our work.

Before presenting our results, let us begin by specifying the context.

1.1. The context. —

1.1.1. Ray class groups. — Let K be a number field, T a finite set of finite places of K, and S a finite set of places of K, disjoint from T. Let us denote $S = S_0 \cup S_\infty$, where S_0 contains only non-archimedean places and S_∞ contains archimedean places, which we assume to be contained in the set $Pl^{re}_{\mathrm{K},\infty}$ of real places of K.
For a place v of K, let ι_v denote the embedding of K into its completion K_v.
Set

- $I_{\mathrm{K},T}$ the group of nonzero fractional ideals of K prime to T,
- $\mathfrak{m} = \prod_{v\in T} v$ the ray modulus of K associated to T,
- $P^{S_\infty}_{\mathrm{K},\mathfrak{m}}$ the subgroup of principal ideals (x) of $I_{\mathrm{K},T}$, $x \equiv 1\,(\mathfrak{m})$, and $\iota_v(x) > 0$ for all $v \in pl^{re}_{\mathrm{K},\infty}\backslash S_\infty$,
- $\langle S_0\rangle$ the subgroup of $I_{\mathrm{K},T}$ builted over places of S_0,
- $R^S_{\mathrm{K},\mathfrak{m}}$ the subgroup $P^{S_\infty}_{\mathrm{K},\mathfrak{m}}\langle S_0\rangle$ of $I_{\mathrm{K},T}$.

Let $\mathrm{Cl}^S_{\mathrm{K},\mathfrak{m}}$ be the S-ray-class group modulo $\mathfrak{m}$, *i.e.*

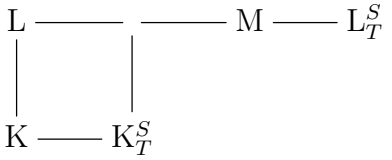
$$\mathrm{Cl}^S_{\mathrm{K},\mathfrak{m}} := I_{\mathrm{K},T}/R^S_{\mathrm{K},\mathfrak{m}}.$$

By class field theory, $\mathrm{Cl}^S_{\mathrm{K},\mathfrak{m}}$ is isomorphic to the Galois group of $\mathrm{K}^S_T/\mathrm{K}$, where K^S_T/K is the maximal abelian extension K, T-tamely ramified and S-split, see [**4**, Chapter II, §5].

1.1.2. Genus fields and genus numbers. — Let p be a prime number and let L/K be a cyclic extension of degree p. Denote by Σ the set of ramification of L/K; for an infinite place v, we will speak of decomposition versus non-decomposition, but not of ramification.

Let T_L (respectively S_L) denote the places of L lying above those of T (resp. of S), and consider $\mathrm{Cl}^S_{\mathrm{L},\mathfrak{m}_\mathrm{L}}$, where $\mathfrak{m}_\mathrm{L} := \prod_{w\in T_\mathrm{L}} w$.

Let M/K be the maximal abelian extension K contained in $\mathrm{L}^S_T/\mathrm{K}$.



The field M is the *S-T genus field* associated with L/K, and the quantity $(g_T^S)^* = [\mathrm{M} : \mathrm{L}]$ is the *S-T genus number*. Set $g_T^S = [\mathrm{M} : \mathrm{K}_T^S]$: this is the quantity g_T^S that we are studying. It can be observed that it is easy to pass from g_T^S to $(g_T^S)^*$ as soon as $\#\mathrm{Cl}_{\mathrm{K},\mathfrak{m}}^S$ is known, and the knowledge of g_T^S provides information about $\mathrm{Cl}_{\mathrm{L},\mathfrak{m}_\mathrm{L}}^S$. Of course, the genus theory makes sense when the field L is not contained in K_T^S, because otherwise $\mathrm{M} = \mathrm{K}_T^S$. The extension M/K being abelian, its Galois group can be approached through class field theory, which allows expressing [M : K] in terms of the ramification in L/K and the S-units of K, thus leading to a non-trivial lower bound for $\#\mathrm{Cl}_{\mathrm{L},\mathfrak{m}_\mathrm{L}}^{S_\mathrm{L}}$.

Since L/K is cyclic of degree p, the Galois group $\mathrm{Gal}(\mathrm{M}/\mathrm{K}_T^S)$ is abelian of exponent p (see Theorem 2.1). Thus, when $p > 2$, the infinite places play no role. Consequently, we assume that $S_\infty = Pl_{\mathrm{K},\infty}^{re}$ in this case.

1.1.3. Governing fields. — We continue with a fixed prime number p. We then assume that for $v \in T$, we have $N_v = 1 \bmod p$, where N_v is the cardinality of the residue field of the completion K_v of K at v. Note that without this condition, the p-part contributed by the places v of T in $\mathrm{Cl}_{\mathrm{K},\mathfrak{m}}^S$, would be trivial.

Let E_T^S be the group of S-units of K congruent to $1 \,(\mathrm{mod}\, \mathfrak{m})$, that is,

$$E_T^S = \{x \in \mathrm{K}^\times \mid x \equiv 1 \,(\mathrm{mod}\, \mathfrak{m}),\ v(x) = 0\ \forall v \notin S\}.$$

Here, let us clarify the meaning $v(x) = 0$. If $v \in S_0$, we identify the place v with its valuation; if $v \in Pl_{\mathrm{K},\infty}^{re}$, $v(x) = 0$ means $\iota_v(x) > 0$; for v archimedean, $v \notin Pl_{\mathrm{K},\infty}^{re}$, we always have $v(x) = 0$.

Let $\mathrm{K}' = \mathrm{K}(\zeta_p)$, where ζ_p is a primitive pth root of unity.
The governing field F_T^S associated with the triplet (K, T, S) is defined as

$$\mathrm{F}_T^S := \mathrm{K}'(\sqrt[p]{E_T^S}).$$

We then define $\Gamma_T^S := \mathrm{Gal}(\mathrm{F}_T^S/\mathrm{K}')$: it is an abelian p-elementary group.

When $T = \emptyset$ and $S = Pl_{\mathrm{K},\infty}^{re}$, by Dirichlet's theorem the p-rank of $\Gamma^S := \Gamma_\emptyset^S$ is $r_1 + r_2 - 1 + \delta_{\mathrm{K},p} + \#S_0$, where (r_1, r_2) is the signature of K, and where $\delta_{\mathrm{K},p} = 1$ if $\zeta_p \in \mathrm{K}$, and 0 otherwise.

1.2. Our result. — To simplify the presentation of our result, we assume that the set Σ does not contain any places above p; in other words, the extension L/K is tamely ramified. To further simplify the presentation, we also assume that the places in S split in L/K.

For each place $v \in \Sigma$, we choose a place w of K′ above v and set $\sigma_v := \sigma_w$, the Frobenius element associated with w in $\Gamma_T^S := \mathrm{Gal}(\mathrm{F}_T^S/\mathrm{K})$; of course, this element depends on the choice of w, but we will see that the conditions involving it are independent of this choice.

Let $m = \#\Sigma$, and let $\{e_{v_1}, \ldots, e_{v_m}\}$ be a basis of $(\mathbb{F}_p)^m$ indexed by the places v of Σ.

We then consider the linear map $\Theta^S_{\Sigma,T}$ defined by

$$\begin{array}{rccc} \Theta^S_{\Sigma,T} : & (\mathbb{F}_p)^m & \longrightarrow & \Gamma^S_T \\ & e_v & \longmapsto & \sigma_v. \end{array}$$

We have the following result

Theorem 1.1. — *Under the previous conditions, we have*

$$g^S_T = \#\ker(\Theta^S_{\Sigma,T}).$$

Remark 1.2. — Taking $T = \emptyset$ and $S = Pl^{re}_{\mathrm{K},\infty}$ we find Theorem 1.1 of [**11**].

The essence of our work is to translate the ramification conditions through Frobenius elements in a governing field. Therefore, if we ensure that the Frobenius elements associated with the places of T form a linearly independent set in $\Gamma^S := \mathrm{Gal}(\mathrm{F}^S/\mathrm{K}')$, then we can express quite easily the Galois group Γ^S_T.
Set $H_T := \sum_{v\in T} \mathbb{F}_p\sigma_v \subset \Gamma^S$.

Proposition 1.3. — *Suppose that the family* $\{\sigma_v,\, v \in T\}$ *forms a linear independent set over* $\mathbb{F}_p$ *in* Γ^S. *Then* $\Gamma^S_T \simeq \Gamma^S/H_T$.

The condition of linear independence has an interpretation. Indeed, according to the Gras-Munnier theorem (see [**4**, Chapter V, Corollary 2.4.2], [**5**]), a relation between the Frobenius elements σ_v, $v \in T$, implies the existence of a cyclic extension of degree p of K, T-ramified and S-split, and consequently contributes "trivially" to g^S_T. Thus, the condition of linear independence forces the context to avoid this situation.

Theorem 1.1 becomes interesting when we have a good understanding of the governing field F^S, especially when we know about the units. Typically, this occurs for $\mathrm{K} = \mathbb{Q}$, but also, as noted in [**11**, §3.5.3], for $p = 3$ and for base field $\mathrm{K} = \mathbb{Q}(\zeta_3)$.
By introducing S-places, the units become S-units, and when the field K is principal, the governing field is relatively easy to describe. A remarkable situation arises when $p = 2$ and $\mathrm{L}/\mathbb{Q}$ is a quadratic extension. The quantity g^S_T corresponds to the kernel of a matrix constructed using Legendre symbols. Let's develop a very specific situation.
Let $\Sigma = \{p_1, \cdots, p_m\}$ and $S_0 = \{\ell_1, \cdots, \ell_{s'}\}$. We assume that $\Sigma \cap (\{S_0\} \cup \{2\}) = \emptyset$. When S_∞ contains the unique infinite place v_∞, let' set $\ell_0 = -1$ (we identify ℓ_0 with v_∞). Set $s := \#S$.
Here, we don't impose any condition on the behavior of $v \in S$ in $\mathrm{L}/\mathbb{Q}$.
Let $A = (a_{i,j})$ be the matrix of size $s \times m$ defined by

$$a_{i,j} = \left(\frac{\ell_j}{p_i}\right),$$

where $\left(\dfrac{\ell_i}{p_j}\right) \in \mathbb{F}_2$ is the additive Legendre symbol.
Next, consider the diagonal matrix $D = (d_{i,j})$ size $s \times s$ defined by

$$d_{j,j} = \begin{cases} 1 \text{ if } \ell_j \text{ is inert in } \mathrm{L}/\mathbb{Q}, \\ 0 \text{ if } \ell_j \text{ splits in } \mathrm{L}/\mathbb{Q}. \end{cases}$$

Finally, let $M = (AD)$ be the matrix of size $(m + s) \times s$.

Corollary 1.4. — *Under the previous conditions, we have:*

$$g_{\emptyset}^{S} = \#\ker(A).$$

The rest of our work consists of four sections. In Section 2, we introduce and develop the elements of genus theory that are useful for ours results. Section 3 is dedicated to the governing field. It is also in this section that we prove Proposition 1.3. Section 4 focuses on the main theorem and its proof. In the final section, we provide two applications.

2. Elements of genus theory

2.1. S-T genus formula. — For this part, we refer, for example, to [**4**, Chapter IV, §4], [**7**, Chapter III, §2], or [**10**].
Let's go back to the framework of Section §1.1. Let T and S be two finite disjoint sets of places of K, non-archimedean for T and arbitrary for $S = S_0 \cup S_\infty$.
Let L/K be a cylic extension of degree p.
We denote by

$$E_T^S \cap \mathscr{N}_{\mathrm{L/K}} := E_T^S \cap \mathrm{N}_{\mathrm{L/K}}(U_{\mathrm{L},\mathfrak{m}}^S),$$

the elements of E_T^S that are locally norms everywhere in L/K.

The following theorem can be formulated in a more general context (see [**4**, Chapter IV]), but we will focus on the case of cyclic extension of degree p.

Theorem 2.1. — *Let* L/K *be a cylic extension of degree p with ramification set Σ. Then* $\mathrm{Gal}(\mathrm{M/K}_T^S)$ *is an abelian group of exponent p. In particular, g_T^S is a power of p, and*

$$\log_p\left(g_T^S\right) = \#S^{ns} + \#\Sigma\backslash\Sigma \cap (S \cup T) - \log_p\left(E_T^S : E_T^S \cap \mathscr{N}_{\mathrm{L/K}}\right),$$

where S^{ns} denotes the set of places in S that are not split in L/K.

Thus, the study of g_T^S is closely related to the quantity $E_T^S \cap \mathscr{N}_{\mathrm{L/K}}$. This will be done through the governing field F_T^S. To achieve this, Proposition 2.4 from the upcoming section is central.

We set $\Sigma' := \Sigma\backslash\Sigma \cap (S \cup T)$.

2.2. Genus fields and ray class fields. — Let L/K be an abelian extension. For $v \in Pl_{\mathrm{K}}$, we denote by $D_v := D_v(\mathrm{L/K})$ its decomposition group in L/K and by $I_v := I_v(\mathrm{L/K})$ its inertia group. It is worth mentioning that for an archimedean place v, we do not speak of ramification but rather of non-decomposition.
We now make the choice of a place $w|v$, and we set $\mathrm{L}_v := \mathrm{L}_w$. Recall that the local reciprocity map provides the following isomorphisms:

$$\frac{\mathrm{K}_v^\times}{\mathrm{N}_{\mathrm{L}_v/\mathrm{K}_v}\mathrm{L}_v^\times} \simeq D_v, \quad \frac{U_v}{\mathrm{N}_{\mathrm{L}_v/\mathrm{K}_v}U_{\mathrm{L}_v}} \simeq I_v.$$

Here, $U_v \subset \mathrm{K}_v^\times$ denotes the group of local units at v, and $\mathrm{N}_{\mathrm{L}_v/\mathrm{K}_v}$ denotes the norm map of the local extension $\mathrm{L}_v/\mathrm{K}_v$. For a real infinite place v, we adopt the convention $U_v = (\mathbb{R}^\times)^2$, and for a complex place $U_v = \mathbb{C}^\times$.
Thus,

- for places $v \in \Sigma' := \Sigma\backslash\Sigma \cap (S \cup T)$, the local reciprocity map induces a surjective morphism from U_v to I_v, with kernel $W_v := \mathrm{N}_{\mathrm{L}_v/\mathrm{K}_v}U_{\mathrm{L}_v}$,

– for places $v \in S$, the local reciprocity map induces a surjective morphism from $\mathrm{K}_v^\times$ to D_v, with kernel $W_v := \mathrm{N}_{\mathrm{L}_v/\mathrm{K}_v}\mathrm{L}_v^\times$. Note that $W_v = \mathrm{K}_v^\times$ if and only if v splits in L/K.

Set

$$W = \prod_{v\in(S\cup\Sigma)\setminus(T\cap\Sigma)} W_v = \prod_{v\in\Sigma'} W_v \prod_{v\in S} W_v.$$

Remark 2.2. — Observe that if L/K s cyclic of degree p, then for any place $v \in \Sigma' \cup S$ we have $\iota_v((E_T^S)^p) \subset W_v$.

Definition 2.3. — We denote by $\mathrm{K}_{\Sigma,S,T}$ the abelian extension of K corresponding, via the global reciprocity map, to the idèle group V:

$$V := W\left(\prod_{v\notin\Sigma'\cup S\cup T} U_v\right)\left(\prod_{v\in T} U_v^1\right) = \left(\prod_{v\in\Sigma'\cup S} W_v\right)\left(\prod_{v\notin\Sigma'\cup S\cup T} U_v\right)\left(\prod_{v\in T} U_v^1\right).$$

Here, U_v^1 is the subgroup of principal units of U_v.

The following proposition is central.

Proposition 2.4. — *We have* $\mathrm{M} = \mathrm{K}_{\Sigma,S,T}$. *Moreover*

$$\mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}_T^S) \simeq \frac{U_{\mathrm{K},\Sigma'}^S}{\nu(E_T^S)W},$$

where $U_{\mathrm{K},\Sigma'}^S = \prod_{v\in S}\mathrm{K}_v^\times \prod_{v\in\Sigma'} U_v$, *and where* $\nu : E_T^S \longrightarrow U_{\mathrm{K},\Sigma'}^S$ *is the diagonal embedding.*

Proof. — Let's note that:

– a *finite* place $v \notin \Sigma' \cup S \cup T$ of K is unramified in M/K,
– a place $v \in T$ is tamely ramified in M/K.

Therefore, the global reciprocity map for the extension M/K is trivial on

$$\left(\prod_{v\notin\Sigma'\cup S\cup T} U_v\right)\left(\prod_{v\in T} U_v^1\right).$$

Now let's look at the part W.
For $v \in S$, since v splits totally in $\mathrm{L}_S^T/\mathrm{L}$ and thus in M/L, then $\mathrm{M}_v = \mathrm{L}_v$. Consequently, every element ε of W_v is also locally norm in M/K. In other words, the local symbol at v in the extension M/K vanishes on W_v.

For $v \in \Sigma'$, let $\varepsilon \in W_v$. Then, by definition of W_v, there exists $z \in U_v$ such that $\varepsilon = N_{\mathrm{L}_v/\mathrm{K}_v}(z)$. But since the extension $\mathrm{M}_v/\mathrm{K}_v$ is unramified at v, the element z is norm in $\mathrm{M}_v/\mathrm{L}_v$, and thus ε is norm in $\mathrm{M}_v/\mathrm{K}_v$. In other words, here too, the local symbol at v in the extension M/K vanishes on W_v.

In conclusion, the global reciprocity map for the extension M/K is trivial on V. Therefore, by maximality of $\mathrm{K}_{\Sigma,S,T}$, we have $\mathrm{M} \subset \mathrm{K}_{\Sigma,S,T}$.

Let's show the reverse inclusion. For that, observe that $\mathrm{K}_{\Sigma,S,T}/\mathrm{L}$ is an abelian extension such that:

– every place $v \in T$ is tamely ramified (possibly unramified);

– for every place $v \in S$, the following commutative diagram holds:

$$\begin{array}{ccc} \mathrm{K}_v^\times/W_v & \twoheadrightarrow & D_v(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}) \\ & \searrow^{\simeq} & \downarrow \\ & & D_v(\mathrm{L}/\mathrm{K}) \end{array}$$

showing that $D_v(\mathrm{K}_{\Sigma,S,T}/\mathrm{L})$ is trivial, hence $\mathrm{K}_{\Sigma,S,T}/\mathrm{L}$ is decomposed at every place $v \in S$;

– similary, every place $v \in \Sigma'$ is unramified in $\mathrm{K}_{\Sigma,S,T}/\mathrm{L}$.

Thus, $\mathrm{K}_{\Sigma,S,T}$ is contained in L_S^T, and by maximality of M, we deduce that $\mathrm{K}_{\Sigma,S,T} \subset \mathrm{M}$. Consequently, $\mathrm{M} = \mathrm{K}_{\Sigma,S,T}$.

In summary, if we denote by $\mathscr{J}_\mathrm{K}$ the idèle group of K, and by $\mathscr{U}_{\mathrm{K},T}^S$ the idèle subgroup given by

$$\mathscr{U}_{\mathrm{K},T}^S := \prod_{v\in S} \mathrm{K}_v^\times \prod_{v\in T} U_v^1 \prod_{v\notin T\cup S} U_v,$$

we have

$$\mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}) \simeq \mathscr{J}_\mathrm{K}/V\mathrm{K}^\times \text{ and } \mathrm{Gal}(\mathrm{K}_T^S/\mathrm{K}) \simeq \mathscr{J}_\mathrm{K}/\mathscr{U}_{\mathrm{K},T}^S\mathrm{K}^\times.$$

Therefore,

$$\mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}_T^S) \simeq \mathscr{U}_{\mathrm{K},T}^S\mathrm{K}^\times/V\mathrm{K}^\times \simeq \mathscr{U}_{\mathrm{K},T}^S/\left(V\mathrm{K}^\times\right) \cap \mathscr{U}_{\mathrm{K},T}^S \simeq \mathscr{U}_{\mathrm{K},T}^S/VE_T^S.$$

We conclude by noticing that $\mathscr{U}_{\mathrm{K},T}^S/V \simeq U_{\mathrm{K},\Sigma'}^S/W$. □

3. Governing fields

Set $\mathrm{K}' = \mathrm{K}(\mu_p)$. We fix a generator ζ_p of μ_p. If B is an $\mathbb{F}_p$-module, let $B^\vee := \hom(B, \mu_p)$. By Kummer duality, recall that for a subgroup of A of $\mathrm{K}'^\times$, one has $A(\mathrm{K}'^\times)^p/(\mathrm{K}'^\times)^p \simeq \mathrm{Gal}(\mathrm{K}'(\sqrt[p]{A})/\mathrm{K}')^\vee$. Moreover, if $A \subset \mathrm{K}^\times$, then

$$\mathrm{Gal}(\mathrm{K}'(\sqrt[p]{A})/\mathrm{K}')^\vee \simeq A(\mathrm{K}'^\times)^p/(\mathrm{K}'^\times)^p \simeq A/A\cap(\mathrm{K}'^\times)^p \simeq A/A\cap(\mathrm{K}^\times)^p \simeq A(\mathrm{K}^\times)^p/(\mathrm{K}^\times)^p,$$

because $[\mathrm{K}' : \mathrm{K}]$ is coprime to p.

3.1. Frobenius. — For any place v of K, let's define

$$\mathscr{E}_{T,v}^S = \{\varepsilon \in E_T^S,\ \varepsilon \in (\mathrm{K}_v^\times)^p\}.$$

This group of S-units fits into the exact sequence

$$1 \longrightarrow \mathscr{E}_{T,v}^S(\mathrm{K}^\times)^p/(\mathrm{K}^\times)^p \longrightarrow E_T^S(\mathrm{K}^\times)^p/(\mathrm{K}^\times)^p \longrightarrow i_v(E_T^S) \longrightarrow 1,$$

where $i_v : E_T^S \longrightarrow \mathrm{K}_v^\times/(\mathrm{K}_v^\times)^p$ is induced by the embedding ι_v of K into K_v.
By Kummer duality we have

$$i_v(E_T^S)^\vee \simeq (E_T^S(\mathrm{K}^\times)^p/\mathscr{E}_{T,v}^S(\mathrm{K}^\times)^p)^\vee \simeq \mathrm{Gal}(\mathrm{K}'(\sqrt[p]{E_T^S})/\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})).$$

This latter Galois group is easily to interpret:

Lemma 3.1. — *One has* $\mathrm{Gal}(\mathrm{K}'(\sqrt[p]{E_T^S})/\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})) = D_v(\mathrm{F}_T^S/\mathrm{K}')$.

(We will see latter that it does not depend on the choice of a place $w|v$ of K'.)

Proof. — Let's denote by N the subfield of $\mathrm{F}_T^S/\mathrm{K}'$ corresponding, via Galois theory, to $D_v(\mathrm{F}_T^S/\mathrm{K}')$. Clearly, $\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S}) \subset \mathrm{N}$. For the reverse inclusion, note that if there exists an intermediate subfield N' of degree p over N, then, as $\mathrm{Gal}(\mathrm{K}'(\sqrt[p]{E_T^S})/\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})$ is an abelian p-elementary group, N' arises from the compositum with a cyclic extension N_0/K' of degree p: there exists $x \in E_T^S$ such that $\mathrm{N}_0 = \mathrm{K}'(\sqrt[p]{x})$. Now, since v splits in N/K', it follows that $x \in (\mathrm{K}'_v)^p$, hence $x \in \mathrm{K}_v^p$ because $[\mathrm{K}'_v : \mathrm{K}_v]$ is coprime to p; thus $\mathrm{N}_0 \subset \mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})$, which leads to a contradiction. □

When v is unramified in $\mathrm{F}_T^S/\mathrm{K}'$, the Galois group of $\mathrm{K}'(\sqrt[p]{E_T^S})/\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})$ is generated by the Frobenius element associated to the choice of a place $w|v$ of K'.

For now on, let'us fix $w|v$ and set $\sigma_v := \sigma_w$, where σ_w is the Frobenius at w in $\mathrm{Gal}(\mathrm{K}'(\sqrt[p]{\mathscr{E}^S})/\mathrm{K}'$.

Next, let D_v be the decomposition group of v in the extension $\mathrm{F}_T^S/\mathrm{K}'$.

Let

$$\Phi_v \,:\, \left(E_T^S(\mathrm{K}^\times)^p/\mathscr{E}_{T,v}^S(\mathrm{K}^\times)^p\right)^\vee \longrightarrow \mathrm{Gal}(\mathrm{F}_T^S/\mathrm{K}'(\sqrt[p]{\mathscr{E}_{T,v}^S})) = D_v$$

be the isomorphism arising from Kummer duality. Recall how Φ_v is defined: for $\chi \in \left(E_T^S(\mathrm{K}^\times)^p/\mathscr{E}_{T,v}^S(\mathrm{K}^\times)^p\right)^\vee$, we associate the element $g_\chi := \Phi_v(\chi)$ defined as follows:

$$g_\chi(\sqrt[p]{\varepsilon}) = \chi(\varepsilon) \cdot \sqrt[p]{\varepsilon},$$

for any $\varepsilon \in E_T^S$.

For $v \in \Sigma' \cup S$, consider the local map φ_v also derived from Kummer duality:

$$\varphi_v \,:\, (A_v/W_v)^\vee \hookrightarrow (A_v/A_v^p)^\vee \twoheadrightarrow i_v(E_T^S)^\vee \xrightarrow{\simeq} D_v,$$

where $A_v = U_v$ (respectively $A_v = \mathrm{K}_v^\times$) for $v \in \Sigma'$ (resp. $v \in S$).

When $(A_v/W_v)^\vee$ is non-trivial, it is generated by a certain character $\chi_v = \chi_w$. Now observe that if we choose another place $w'|v$ of K', then $w' = hw$ for some $h \in \mathrm{Gal}(\mathrm{K}'/\mathrm{K})$. Let $\chi_{w'} := \chi_{hw} := \chi_w(h^{-1}(.))$; this is a non-trivial character of $(A_{w'}/W_{w'})^\vee$.

Lemma 3.2. — *Set $g_w := \varphi_w(\chi_w)$ and $g_{w'} := \varphi_{w'}(\chi_{w'})$. Then $\langle g_w \rangle = \langle g_{w'} \rangle$.*

Proof. — This is a consequence of Kummer theory where we have $g_{w'} = g_w^a$ for some $a \in \mathbb{F}_p^\times$ (see, for example, [**4**, Chapter I, §6, Theorem 6.2]). □

Thus, all the subsequent results do not depend on the choice of $w|v$.

We will now describe φ_v more precisely.

(i) This is the most important case. Let $v \in \Sigma'$. Recall that $U_v/W_v \simeq \mathbb{Z}/p$, hence $U_v^p \subset W_v$. There exists a non-trivial element χ_v of $(U_v/W_v)^\vee$ such that

$$\langle \chi_v \rangle = (U_v/W_v)^\vee \hookrightarrow (U_v/U_v^p)^\vee \cdot$$

Then $\varphi_v(\chi_v)$ is an element $g_v := g_{\chi_v}$ of D_v, defined by

$$g_v(\sqrt[p]{\varepsilon}) = \chi_v(i_v(\varepsilon)) \cdot \sqrt[p]{\varepsilon},$$

for all $\varepsilon \in E_T^S$.

Let $Pl_{\mathrm{K},p} = \{v \in Pl_\mathrm{K}, v|p\}$ be the set of p-adic places of K.

Observe that if $v \notin Pl_{\mathrm{K},p} \cup S_0$, then v is unramified in $\mathrm{F}_T^S/\mathrm{K}'$, and $U_v^p = W_v$. In particular, D_v is a cylic group generated by the Frobenius σ_v at v. Thus

$$\varphi_v : \langle \chi_v \rangle \twoheadrightarrow \langle \sigma_v \rangle.$$

By taking a suitable power of χ_v, we obtain $\varphi_v(\chi_v) = \sigma_v$.

(ii) Let $v \in S_0 \backslash S_0 \cap \Sigma$. Then v is unramified in L/K.
First, note that if v splits in L/K, then $W_v = \mathrm{K}_v^\times$ and thus φ_v is the trivial map.
Now, suppose v is inert in L/K. Then $W_v = U_v \langle \pi_v^p \rangle$ and thus

$$\mathrm{K}_v^\times / W_v \simeq \mathrm{K}_v^\times / U_v \langle \pi_v^p \rangle \simeq \langle \pi_v \rangle / \langle \pi_v^p \rangle.$$

Let χ_v be the generator of $(\langle \pi_v \rangle / \langle \pi_v^p \rangle)^\vee$ defined by $\chi_v(\pi_v^i) = \zeta_p^i$. Then $g_v := \varphi(\chi_v)$ satisfies: for all $\varepsilon \in E_T^S$,

$$g_v(\sqrt[p]{\varepsilon}) = \chi_v(\iota_v(\varepsilon)) \cdot \sqrt[p]{\varepsilon}.$$

Thus $\chi_v(\iota_v(\varepsilon)) = 1$ if and only if the valuation $v(\varepsilon)$ of ε is zero modulo p.

(iii) Let $v \in S_0 \cap \Sigma$. This is analogous to (i), noting that $A_v = \mathrm{K}_v^\times$.

(iv) Here $p = 2$ and v is a real place in S.
As in (iii), if v is splitting in L/K, then $W_v = \mathrm{K}_v^\times$ and φ_v is the trivial map. Otherwise for $\varepsilon \in E_T^S$

$$g_v(\sqrt{\varepsilon}) = \mathrm{sign}(\iota_v(\varepsilon)) \cdot \sqrt{\varepsilon},$$

where $\mathrm{sign}(\iota_v(\varepsilon))$ is the sign of the embedding $\iota_v(\varepsilon)$ of ε in K_v.

3.2. A restriction. — Let $T = \{v_1, \ldots, v_t\}$, and for $i = 1, \cdots, t$, let σ_{v_i} be the Frobenius at v_i in Γ^S; set $\mathrm{H}_T := \langle \sigma_v,\ v \in T \rangle$.

Proposition 3.3. — *Suppose that the set $\{\sigma_{v_1}, \ldots, \sigma_{v_t}\}$ forms a $\mathbb{F}_p$-free family in Γ^S. Then*

$$\Gamma_T^S := \mathrm{Gal}(\mathrm{F}_T^S/\mathrm{K}') \simeq \Gamma^S / \mathrm{H}_T.$$

Proof. — Let's give a proof by induction on the cardinality of T. Recall that for $v \in T$, one has $N_v = 1 \bmod p$.
• Suppose $T = \{v\}$. Let $E_{\{v\}}^S = \{\varepsilon \in E^S,\ \varepsilon \equiv 1\,(v)\}$. Define $\mathscr{E}_v^S = \{\varepsilon \in E^S,\ \varepsilon \in (\mathrm{K}_v^\times)^p\}$. By Hensel's lemma, we have $E_{\{v\}}^S \subset \mathscr{E}_v^S$. Moreover, $E^S/E_{\{v\}}^S \hookrightarrow \mathbb{F}_v^\times$, where $\mathbb{F}_v$ is the residue field at v. Thus, $E^S/E_{\{v\}}^S$ is cyclic of order dividing p. Since $E_{\{v\}}^S \subset \mathscr{E}_v^S$, it follows

$$\mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \frac{E^S(\mathrm{K}^\times)^p}{E_{\{v\}}^S(\mathrm{K}^\times)^p} \twoheadrightarrow \frac{E^S(\mathrm{K}^\times)^p}{\mathscr{E}_v^S(\mathrm{K}^\times)^p}. \tag{1}$$

By Lemma 3.1, we have:

$$\left(\frac{E^S(\mathrm{K}^\times)^p}{\mathscr{E}_v^S(\mathrm{K}^\times)^p}\right)^\vee = \langle \sigma_v \rangle \subset \Gamma^S.$$

Now, since $\sigma_v \neq 0$ by assumption, it follows that $\dfrac{E^S(\mathrm{K}^\times)^p}{\mathscr{E}_v^S(\mathrm{K}^\times)^p} \simeq \mathbb{Z}/p\mathbb{Z}$. Thus, from (1) we have $\dfrac{E^S(\mathrm{K}^\times)^p}{E_{\{v\}}^S(\mathrm{K}^\times)^p} = \langle \sigma_v \rangle^\vee$, or equivalently $\mathrm{Gal}(\mathrm{F}^S/\mathrm{F}_T^S) = \langle \sigma_v \rangle$. This concludes this case.

• Let's suppose $T = T_0 \cup \{v\}$, and that the proposition is true for T_0. Define $\mathscr{E}_{T_0,v}^S = \{\varepsilon \in E^S,\ \varepsilon \equiv 1\,(v'),\ v' \in T_0,\ \varepsilon \in U_v^p\}$. By Hensel's lemma, we have $E_T^S \subset \mathscr{E}_{T_0,v}^S$. Similary as

before, $\dfrac{E^S_{T_0}}{E^S_T} \hookrightarrow \mathbb{F}_v^\times$, implying that $\dfrac{E^S_{T_0}}{E^S_T(E^S_{T_0})^p}$ is a cyclic group of order dividing p, so we have

$$\mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \frac{E^S_{T_0}}{E^S_T(E^S_{T_0})^p} \twoheadrightarrow \frac{E^S_{T_0}(\mathrm{K}^\times)^p}{E^S_T(\mathrm{K}^\times)^p} \twoheadrightarrow \frac{E^S_{T_0}(\mathrm{K}^\times)^p}{\mathscr{E}^S_{T_0,v}(\mathrm{K}^\times)^p}. \tag{2}$$

Then we define

$$\left(\frac{E^S_{T_0}(\mathrm{K}^\times)^p}{\mathscr{E}^S_{T_0,v}(\mathrm{K}^\times)^p}\right)^\vee = \langle \overline{\sigma}_v \rangle,$$

where $\overline{\sigma}_v$ is the restriction of the Frobenius $\sigma_v \in \Gamma^S$ to $\mathrm{F}^S_{T_0}$. By the induction hypothesis,

$$\left(\frac{E^S(\mathrm{K}^\times)^p}{E^S_{T_0}(\mathrm{K}^\times)^p}\right)^\vee = \langle \sigma_{v'},\, v' \in T_0 \rangle.$$

But $\overline{\sigma}_v = 1$ would imply $\sigma_v \in \langle \sigma_{v'},\, v' \in T_0 \rangle$ which contradicts the assumption. Therefore, the surjections in (2) are isomorphisms, and $\mathrm{Gal}(\mathrm{F}^S/\mathrm{F}^S_T)$ is generated by the Frobenius elements σ_v and $\sigma_{v'}$, $v' \in T_0$. This concludes the proof. □

Remark 3.4. — The Galois group $\mathrm{Gal}\left(\mathrm{F}^S/\mathrm{F}^S_{\{v\}}\right)$ may not be generated by the Frobenius at v. Let's give an example.
Take $\mathrm{K} = \mathbb{Q}$ and $p = 2$. Choose $T = \{\ell\}$, where $\ell \equiv 1 (\mathrm{mod}\ 4)$ is a prime number.
Let $S = S_\infty = \{v_\infty\}$. We have $E^S = \langle \pm 1 \rangle$ and $E^S_{\{\ell\}} = \langle 1 \rangle$. Thus

$$\mathrm{F}^S = \mathbb{Q}(\sqrt{E^S}) = \mathbb{Q}(\sqrt{-1}), \text{ and } \mathrm{F}^S_{\{\ell\}} = \mathbb{Q}(\sqrt{E^S_{\{\ell\}}}) = \mathbb{Q}\,.$$

However, since ℓ is splitting in $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$, it follows that $\sigma_\ell = 1$. Consequently, $\mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q})$ is not generated by the Frobenius at ℓ.

When $p = 2$ we can handle the archimedean places in the same way. Set $\overline{S} := S_0 \cup Pl^{re}_{\mathrm{K},\infty}$. Observe that $E^{\overline{S}}$ is the group of S_0-units in the classical sense (with no sign condition).

Proposition 3.5. — *Take $p = 2$. Set $H^{S_\infty} = \langle \sigma_v;\, v \in pl^{re}_{\mathrm{K},\infty} \backslash S_\infty \rangle \subset \Gamma^{\overline{S}}$ and identify H_{S_∞} with its restriction to $\Gamma^{\overline{S}}_T$. Then we have*

$$\Gamma^S_T := \mathrm{Gal}(\mathrm{K}(\sqrt{E^S_T})/\mathrm{K}) \simeq \Gamma^{\overline{S}}_T / H_{S_\infty}.$$

Moreover, if the set $\{\sigma_v,\, v \in T\}$ forms a linearly independent family in $\Gamma^{\overline{S}}$, then

$$\mathrm{Gal}(\mathrm{K}(\sqrt{E^S_T})/\mathrm{K}) \simeq \Gamma^{\overline{S}} / \left(H_{S_\infty} + H_T\right).$$

Proof. — Similar to Lemma 3.1, we can show that $\mathrm{K}(\sqrt{E^S_T})$ corresponds, by Galois theory, to the subgroup H_{S_∞} of $\Gamma^{\overline{S}}_T$.
As for the second part, from Proposition 3.3 we know that $\Gamma^{\overline{S}}_T \simeq \Gamma^{\overline{S}}/H_T$; thus, we conclude with the first point. □

4. Main result

We keep the notations from the previous sections. In particular, $\Sigma' = \Sigma \backslash \Sigma \cap (S \cup T)$.

For $v \in \Sigma' \cup S$, let's consider the elements $g_v := \varphi_v(\chi_v) \in \Gamma_T^S$ defined in $(i) - (iv)$ of §3.1. Recall that if v is unramified in $\mathrm{F}_S^T/\mathrm{K}'$, then $g_v = \sigma_v$ is the Frobenius of v in Γ_T^S.
Let $\Theta_{\Sigma,T}^S$ be the following linear map:

$$\Theta_{\Sigma,T}^S \ : \ \left(\frac{U_{\mathrm{K},\Sigma'}^S}{W}\right)^\vee \longrightarrow \Gamma_T^S.$$

defined by $\Theta_{\Sigma,T}^S(\chi_v) = g_v$.

Theorem 4.1. — *The Artin map induces the following isomorphism:*

$$\ker(\Theta_{\Sigma,T}^S) \simeq \mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}_T^S)^\vee.$$

Proof. — First, let's observe that

$$E^S \cap (\mathrm{K}^\times)^p = (E^{\overline{S}})^p,$$

where $E^{\overline{S}}$ is the group of S_0-units in the classical sense. Then

$$E_T^S(E^{\overline{S}})^p/(E^{\overline{S}})^p \simeq E_T^S(\mathrm{K}^\times)^p/(\mathrm{K}^\times)^p.$$

Now, consider the exact sequence obtained from Proposition 2.4 and Remark 2.2:

$$1 \longrightarrow \nu(E_T^S(E^{\overline{S}})^p/(E^{\overline{S}})^p) \longrightarrow U_{\mathrm{K},\Sigma'}^S/W \longrightarrow \mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}_T^S) \longrightarrow 1.$$

By Kummer duality, we have:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(\mathrm{K}_{\Sigma\cup S}/\mathrm{K}_T^S)^\vee & \longrightarrow & \left(U_{\mathrm{K},\Sigma'}^S/W\right)^\vee & \longrightarrow & \left(\nu(E_T^S(E^{\overline{S}})^p/(E^{\overline{S}})^p)\right)^\vee & \longrightarrow & 1 \\ & & & & \vdots & & \downarrow & & \\ & & & & \Gamma_T^S & \xleftarrow[\simeq]{\Psi} & \left(E_T^S(E^{\overline{S}})^p/(E^{\overline{S}})^p\right)^\vee & & \end{array}$$

Now,

$$\left(U_{\mathrm{K},\Sigma'}^S/W\right)^\vee \simeq \prod_{v\in\Sigma'}(U_v/W_v)^\vee \prod_{v\in S}(U_v/W_v)^\vee.$$

Then, it suffices to observe that the induced map from $\left(U_{\mathrm{K},\Sigma'}^S/W\right)^\vee$ to Γ_T^S corresponds to $\Theta_{\Sigma,T}^S$. Therefore, we finally obtain:

$$\mathrm{Gal}(\mathrm{K}_{\Sigma,S,T}/\mathrm{K}_T^S)^\vee \simeq \ker\left(\left(U_{\mathrm{K},\Sigma'}^S/W\right)^\vee \xrightarrow{\Theta_{\Sigma,T}^S} \Gamma_T^S\right).$$

Hence the result. □

Therefore, it follows that

Corollary 4.2. — *We have* $g_T^S = \#\ker(\Theta_{\Sigma,T}^S)$.

Proof. — This is a consequence of Theorem 4.1 and Proposition 2.4. □

It follows that if $v \in S$ splits in L/K, then the component at v in $\dfrac{U^S_{\mathrm{K},\Sigma'}}{W}$ is trivial. Set $S = S^{sp} \cup S^{ns}$, where S^{sp} is the set of places in S that split in L/K, and $S^{ns} = S \backslash S^{sp}$. Let $s^{ns} = \#S^{ns}$ and $m := \#\Sigma$.
Then $\left(\dfrac{U^S_{\mathrm{K},\Sigma'}}{W}\right)^\vee$ is isomorphic to $(\mathbb{Z}/p)^{s^{ns}+m}$.

Corollary 4.3. — *We have $m + s^{ns} - r_T^S \leq log_p(g_S^T) \leq m + s^{ns}$, where r_T^S is the p-rank of E_T^S.*

Proof. — Il suffices to observe that $\dim \Gamma_T^S = \dim E_T^S(\mathrm{K}^\times)^2/(\mathrm{K}^\times)^2 \leq r_T^S$. □

Remark 4.4. — We have $\dim \Gamma_T^S \leq r_{S_0}$, where $r_{S_0} = r_1 + r_2 + |S_0| - \delta_{\mathrm{K},p}$.
When the Frobenius elements of the places $v \in T$ are linearly independent in Γ^S, we also have $\dim \Gamma_T^S = \dim \Gamma^S - |T| \leq r_{S_0} - |T|$. (See Proposition 3.3.)

Corollary 4.5 **(Theorem 1.1)**. — *If $S^{ns} = \Sigma \cap Pl_{\mathrm{K},p} = \emptyset$, let $\{e_{v_1}, \ldots, e_{v_m}\}$ be a basis of $(\mathbb{F}_p)^m$ indexed by the places v in Σ, and let Θ be the linear map defined by:*

$$\begin{array}{rccc} \Theta : & (\mathbb{F}_p)^m & \longrightarrow & \Gamma_T^S \\ & e_v & \longmapsto & \sigma_v. \end{array}$$

Then $g_S^T = \#\ker(\Theta)$.

Proof. — In this case, $g_v = \sigma_v$. □

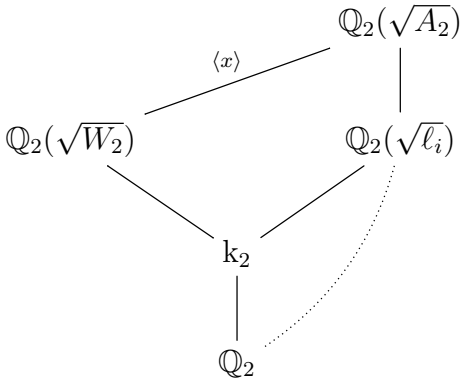
5. Examples

5.1. Quadratic extensions. — Let's take $p = 2$ and $\mathrm{K} = \mathbb{Q}$. Let $\mathrm{L}/\mathbb{Q}$ be a quadratic extension with set of ramification $\Sigma = \{p_1, \cdots, p_m\}$. Set $\mathrm{L} = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is square-free.
Let $S_0 = \{\ell_1, \ldots, \ell_s\}$. We assume that $\Sigma \cap S = \emptyset$.
We denote ℓ_∞ as the infinite place; then $S_\infty = \{\ell_\infty\}$ or $S_\infty = \emptyset$. In the spirit of Proposition 3.3, we assume $T = \emptyset$.
Let E^S be the group of S-units of $\mathbb{Q}$. We write $E^S = \langle \ell_0, \ldots, \ell_s \rangle$, with $\ell_0 = -1$ or 1 depending on whether $S_\infty = \{\ell_\infty\}$ or not.
In this context, the governing field is written as $\mathrm{F}^S = \mathbb{Q}(\sqrt{E^S}) = \mathbb{Q}(\sqrt{\ell_0}, \ldots, \sqrt{\ell_s})$. Its Galois group $\Gamma^S := \mathrm{Gal}(\mathrm{F}^S/\mathbb{Q})$ is isomorphic to $\prod_{j=0}^s \mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_j})/\mathbb{Q})$. Note that $\mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_0})/\mathbb{Q})$ may be trivial.
Let's revisit the morphism φ_v defined in Section §3.1 and consider its restriction to $\mathbb{Q}(\sqrt{\ell_j})$: its value is in $\{0, 1\}$. For what follows, the quadratic residue symbol is viewed additively, meaning it takes values in $\mathbb{F}_2$.

Lemma 5.1. — *The elements g_ℓ takes the following values:*
(a) For $\ell \in \Sigma'$ and ℓ odd, $g_\ell = \sigma_\ell$ restricts to $\mathbb{Q}(\sqrt{\ell_j})$ equals $\left(\dfrac{\ell_j}{\ell}\right)$.
(b) For $\ell \in S_0^{ns} \backslash S_0^{ns} \cap \Sigma$, g_ℓ restricts to $\mathbb{Q}(\sqrt{\ell_j})$ is trivial if and only if $\ell \neq \ell_j$.
(c) For $\ell = \ell_\infty$, the element g_{ℓ_∞} restricts to $\mathbb{Q}(\sqrt{\ell_j})$ is trivial unless $\ell_j = \ell_0 = -1$ and L *is imaginary.*

Proof. — (a) is (i) of §3.1, (b) is (ii) and (c) is (iv). □

It remains to describe g_2 when 2 is ramified in $\mathrm{L}/\mathbb{Q}$. So, suppose $2 \in \Sigma$. We identify g_2 with its restriction to $\mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$. We have the following extensions



Recall that $A_2 = U_2$ (respectively $A_2 = \mathbb{Q}_2^\times$) if $2 \notin S$ (resp. $2 \in S$).
The desired element g_2 is the image of the reduction of x in $\mathrm{Gal}(\mathbb{Q}_2(\sqrt{\ell_i})/\mathbb{Q}_2) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$. Therefore g_2 (restricted) is trivial if and only if $\ell_i \in W_2$ modulo $(A_2)^2$.
Let's take for example $\ell_i = 2$. Observe that for $2 \in S$, then $2 \in \mathrm{W}_2$, and thus $g_2 = 0$. On the other hand, if $2 \notin S$, then $W_2 \subset U_2$ and consequently $g_2 = 1$.
In general, everything relies on determining W_2, which is the conductor at 2 of L/K; see [**4**, Chapiter II, §1, Exercise 1.6.5] for calculations.
For example, suppose $d \equiv -1$ modulo 8. Then $W_2 = \langle 5 \rangle$. Hence, g_2 restricted to $\mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$ is trivial if and only if, $\ell_i \equiv 1$ modulo 4.

A particularly noteworthy situation arises when we are only dealing with cases (a) and (b) of Proposition 5.1 and $\mathrm{L}/\mathbb{Q}$ is unramified at 2. Let's detail this situation.

Let the canonical basis $\mathscr{B} := \{e_{p_1}, \ldots, e_{p_m}, e_{\ell_0}, e_{\ell_1}, \ldots, e_{\ell_s}\}$ of $\mathbb{F}_2^{m+s+1}$ be indexed by the places of $\Sigma \cup \{\ell_\infty\} \cup S_0$; here we take $S_\infty = \{\ell_\infty\}$ and set $e_{\ell_0} := e_{\ell_\infty}$.
The map $\Theta := \Theta_\Sigma^S$ on the basis $\mathscr{B}$, taking values in $\prod_{j=0}^n \mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_j})/\mathbb{Q})$, is defined by

$$\Theta(e_{p_i})_{|\mathbb{Q}(\sqrt{\ell_j})} = \left(\frac{\ell_j}{p_i}\right), \quad \Theta(e_{\ell_i})_{|\mathbb{Q}(\sqrt{\ell_j})} = \delta^*_{i,j},$$

where for $0 \leq i \leq s$, $\delta_{i,j} = 0$ when $i \neq j$; for $j > 0$,

$$\delta^*_{j,j} = \begin{cases} 1 \text{ if } \ell_j \text{ is inert in } \mathrm{L}/\mathbb{Q} \\ 0 \text{ if } \ell_j \text{ splits in } \mathrm{L}/\mathbb{Q} \end{cases}$$

and $\delta^*_{0,0} = 1$ if L is imaginary, 0 otherwise.
The matrix Θ is then written as follows

$$\begin{pmatrix} \left(\frac{-1}{p_1}\right) & \left(\frac{-1}{p_2}\right) & \cdots & \left(\frac{-1}{p_m}\right) & \delta^*_{0,0} & 0 & \cdots & 0 \\ \left(\frac{\ell_1}{p_1}\right) & \left(\frac{\ell_1}{p_2}\right) & \cdots & \left(\frac{\ell_1}{p_m}\right) & 0 & \delta^*_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\ell_s}{p_1}\right) & \left(\frac{\ell_s}{p_2}\right) & \cdots & \left(\frac{\ell_s}{p_m}\right) & 0 & 0 & \cdots & \delta^*_{s,s} \end{pmatrix}.$$

Observe that if we take $S_\infty = \emptyset$, then to obtain the matrix, simply remove the first row and the $(m+1)$th column of the above matrix.

5.2. A result of existence. — In this subsection, we make explicit Corollary 4.3 by adapting Theorem 1.3 from [**11**]. To simplify, we only consider extensions that are splitting at infinity. Thus, we assume that $Pl^{re}_{\mathrm{K},\infty} \subset S$. Note that for $p > 2$, the archimedean places play no role in the calculations.
Recall that $r_S = \dim E^S = r_1 + r_2 + \#S_0 - 1 + \delta_{\mathrm{K}}$. Let $s = \#S_0$ and define $A_S := A_{S_0} := \prod_{v \in S_0} N_v$.

Theorem 5.2. — *Let* K *be a number field, and let* $S = S_0 \cup Pl^{re}_{\mathrm{K},\infty}$ *be a set of places of* K. *Let* $k, m \geq 1$, *such that* $m - r_S \leq k \leq m$. *Let* p *be a prime number. Then there exist infinitely many sets* Σ *of finite places of* K *of size* m, *such that there exists an extension* L/K *cyclic of degree* p, Σ*-(totally) ramified,* S*-split, and with* $g^S := g^S_\emptyset = p^k$.
Moreover, assuming GRH, when m *is fixed, such a set* Σ *can be chosen such that the absolute norm* N_v *of each of its elements* v *is smaller than* $p^{2r_S+2}(c_1 \log p + c_2 \log A_S)^2$, *where* c_1 *and* c_2 *are constants depending on* K *and* m.

Proof. — The proof relies on [**11**, Section 4, Proof of Theorem 1.3], with two modifications due to S. The first modification concerns the "existence" part, and the second concerns the "quantitative" part.

• We need the existence of a cyclic extension L/K of degree p with specific properties. For the first part, we utilize the Gras-Munnier theorem with splitting (see [**4**, Chapter V, Section 2, Corollary 2.4.2] and [**5**]).

Let $V_S := \{x \in \mathrm{K}^\times, v(x) = 0,\ \forall v \notin S\}$ be the S-Selmer group of K. Set $\widetilde{\mathrm{F}}^S := \mathrm{K}'(\sqrt[p]{V_S})$. The exact sequence

$$1 \longrightarrow E^S/(E^S)^p \longrightarrow V_S/(\mathrm{K}^\times)^p \longrightarrow \mathrm{Cl}^S_{\mathrm{K}}[p] \longrightarrow 1$$

shows that $[\widetilde{\mathrm{F}}^S : \mathrm{F}^S] = O(1)$, uniformly for p and S.

Let k and m be non-negative integers such that $m - r_S \leq k \leq m$. Set $r = m - k \leq r_S$. Let p^l be the degree of $\widetilde{\mathrm{F}}^S/\mathrm{F}^S$.

Let us take an $\mathbb{F}_p$-basis $(e_i)_{i=1,\dots,r_S}$ of $\mathrm{Gal}(\mathrm{F}^S/\mathrm{K}')$ and complete it to an $\mathbb{F}_p$-basis $(e_i)_{i=1,\dots,r_S+l}$ of $\mathrm{Gal}(\widetilde{\mathrm{F}}^S/\mathrm{K}')$. By Chebotarev's density theorem, let $\Sigma = \{v_1, \dots, v_m\}$ be places of K prime to p such that their Frobenius elements, denoted $\sigma_{v_i} \in \mathrm{Gal}(\widetilde{\mathrm{F}}_S/\mathrm{K}') \subset \mathrm{Gal}(\widetilde{\mathrm{F}}_S/\mathrm{K})$, satisfy:

(*a*) $\sigma_{v_1} = -(e_1 + \cdots + e_r)$;
(*b*) for $i = 2, \cdots, r+1$, $\sigma_{v_i} = e_{i-1}$;
(*c*) for $i = r+2, \cdots, m$, $\sigma_{v_i} = 0$,

when $r \geq 1$. When $r = 0$, choose the v_i's such that $\sigma_{v_i} = 0,\ i = 1, \cdots, m$.

One has $\sum_{i=1}^{m} \sigma_{v_i} = 0$: By the Gras-Munnier theorem referenced earlier, there exists a cyclic extension L/K of degree p, Σ-totally ramified and S-split. Furthemore by the choice of the e_i's and the v_i's, the map Θ of Corollary 4.5 has rank r. Hence, $\mathrm{Gal}(\mathrm{M}/\mathrm{K}^S) \simeq (\mathbb{F}_p)^{m-r} = (\mathbb{F}_p)^k$.

• We now need a second modification: it concerns the discriminant $\mathrm{disc}(\widetilde{\mathrm{F}}^S)$ of $\widetilde{\mathrm{F}}^S$. The introduction of S can affect $\mathrm{disc}(\widetilde{\mathrm{F}}^S)$ of $\widetilde{\mathrm{F}}^S$ compared to $\mathrm{disc}(\widetilde{\mathrm{F}}^\emptyset)$ in two different ways.

Firstly, the places prime to p contained in S, may become ramified, and for the places w of K' in S lying above p, the valuation of the local conductor $\mathfrak{f}_w$ in a cyclic extension $\mathrm{L}_w/\mathrm{K}'_w$ of degree p may increase. Let e_w be the index of ramification of w in $\mathrm{K}'/\mathbb{Q}$. Observe now that

- for $w|p$, $w(\mathfrak{f}_w) \leq 1 + \frac{p}{p-1}e_w$,
- for $w \nmid p$, $w(\mathfrak{f}_w) \leq p-1$.

See for example [**12**, Chapter III, §6, Remarks & Chapter V, §3, Lemma 3].
If we follow the calculations from [**11**, proof of Proposition 3.2], we obtain:

$$|\mathrm{disc}(\widetilde{\mathrm{F}}^S)| \leq \left(|\mathrm{disc}(\mathrm{K})| \cdot A_S \cdot p^{4[\mathrm{K}:\mathbb{Q}]}\right)^{p^{r_S+l+1}}.$$

By using the fact that $[\widetilde{\mathrm{F}}^S : \mathrm{F}^S] = O(1)$, Lemma 4.4 of [**10**] provides the result. □

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