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# SOME NEW EVIDENCE FOR THE FONTAINE-MAZUR CONJECTURE

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Abstract. In this paper, thanks to the work of M. Lazard, we obtain some new evidence for the conjecture of Fontaine-Mazur in the spirit of Boston's results.

Let $\mathbf{K}$ be a number field, $p$ a prime number and $S$ a finite set of places of $\mathbf{K}$. Assume that $S$ is prime to $p$: for every finite place $v \in S$, the absolute norm of $v$ is prime to $p$. Denote by $\mathbf{K}_S$ the maximal pro-$p$-extension of $\mathbf{K}$ unramified outside $S$ and put $\mathcal{G}_S = \mathrm{Gal}(\mathbf{K}_S/\mathbf{K})$.

A consequence of a conjecture of Fontaine and Mazur [5] asserts that there is no infinite analytic quotient of $\mathcal{G}_S$. The philosophy of this conjecture is that every analytic quotient of $\mathcal{G}_S$ should come from geometry, and then, thanks to the above conditions on $S$, this quotient should be finite.

The study of analytic pro-$p$-groups was initiated by Michel Lazard in a famous paper [9]. In particular, in this paper, he describes relationships between various important invariants of these groups such as: the dimension of the group $\mathcal{G}$ as $\mathbb{Q}_p$-manifold, the structure of the Lie algebra of $\mathcal{G}$, and the virtual cohomological dimension of $\mathcal{G}$.

Using some of these tools, Boston in [1] and in [2] gives for the first time arithmetical criteria for verifying special cases of Fontaine-Mazur conjecture. In [12], in the same spirit, Wingberg considers the CM-case. Here, one obtains:

**Theorem 1.** *Let $\mathbf{K}/\mathbf{F}$ be an extension of number fields with finite group $\Delta$ of order prime to $p$. Let $\mathbf{L}/\mathbf{K}$ be a Galois pro-$p$-extension with group $\mathcal{G}$ such that: (i) $\mathbf{L} \subset \mathbf{K}_S$; (ii) $\mathbf{L}/\mathbf{F}$ is Galois. Put $\mathcal{G}_1 = \mathcal{G}/\mathcal{G}^p[\mathcal{G},\mathcal{G}]$.*

*If the $\Delta$-representation of $\bigwedge^2(\mathcal{G}_1)$ does not contain the $\Delta$-representation of $\mathcal{G}_1$, then $\mathcal{G}$ is not uniformly powerful.*

From this result, we deduce some new evidence for the Fontaine-Mazur conjecture in the non-abelian case and in the cyclic case.

For the CM-case, by using a result of Gras [6], one obtains a new proof of a result of Wingberg [12]:

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**Corollary 2.** *Assume $p > 2$. Let $\mathbf{K}$ be a CM-field with maximal totally real subfield $\mathbf{F}$, and let $d(\mathcal{G}_\emptyset) = d^+ + d^-$ be the decomposition of the generator rank of $\mathcal{G}_\emptyset$ under the action of $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$. Put $\delta_p = 1$ if $\mathbf{K}$ contains the pth roots of unity, 0 otherwise. If $d^+d^- > d^- + \delta_p$, then $\mathcal{G}_\emptyset$ is not uniformly powerful.*

## 1. Uniformly powerful pro-$p$-groups

**Definition 3.** *A pro-p-group $\mathcal{G}$ is called powerful if the quotient $\mathcal{G}/\overline{\mathcal{G}^p}$ (for $p = 2$, $\mathcal{G}/\overline{\mathcal{G}^4}$) is abelian.*

Before defining the notion of uniformly powerful pro-$p$-group, let us give some more notation. For a pro-$p$-group $\mathcal{G}$, define $P_i(\mathcal{G})$ to be its central descending series:

$$P_1(\mathcal{G}) = \mathcal{G}, \quad P_{i+1}(\mathcal{G}) = P_i(\mathcal{G})^p[\mathcal{G}, P_i(\mathcal{G})].$$

In what follows, we will denote by $\mathcal{G}_1$ the quotient $\mathcal{G}/\mathcal{G}^p[\mathcal{G},\mathcal{G}] = \mathcal{G}/P_2(\mathcal{G})$ and by $d(\mathcal{G})$ the $p$-rank of $\mathcal{G}$, that is the dimension of $\mathcal{G}_1$ over $\mathbb{F}_p$.

**Definition 4.** *A finitely generated pro-p-group $\mathcal{G}$ is called uniformly powerful if*

*(1) $\mathcal{G}$ is powerful*
*(2) for all $i$, $\#P_i(\mathcal{G})/P_{i+1}(\mathcal{G}) = \#\mathcal{G}/P_2(\mathcal{G})$.*

Note that, as shown in [3], a finitely generated powerful pro-$p$-group $\mathcal{G}$ is uniformly powerful if and only if $\mathcal{G}$ is torsion free.

**Definition 5.** *A topological group $\mathcal{G}$ is p-adic analytic if $\mathcal{G}$ has a structure of a p-adic manifold such that the map $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (x, y) \mapsto xy^{-1}$ is analytic.*

As explained in [3], Chapter 8, a uniformly powerful pro-$p$-group $\mathcal{G}$ is analytic. But in fact:

**Theorem 6** ([3], Theorem 8.32)**.** *Let $\mathcal{G}$ be an analytic pro-p-group. Then $\mathcal{G}$ admits an open uniformly powerful sub-group $\mathfrak{U}$.*

As noted by the authors of [3], the class of uniformly powerful pro-$p$-groups is in the class of “groupes $p$-saturables”, following Lazard’s terminology. The great interest of the groups of the last class is that their cohomology is the same as the cohomology of their Lie algebra (see [9], chapter V).

**Theorem 7** (Lazard [9]). *Let $\mathcal{G}$ be a uniformy powerful pro-p-group. Then, for $i \geq 1$*

$$H^i(\mathcal{G}, \mathbb{F}_p) \simeq \bigwedge^i (\mathcal{G}_1^*)\,,$$

*where $\bigwedge^i (\mathcal{G}_1^*)$ denotes the i-th exterior power of the dual $\mathcal{G}_1^*$ of $\mathcal{G}_1$.*

One can find a proof of this in [11]. Note that if $\mathcal{G}$ is uniformly powerful, its cohomological dimension is exactly its $p$-rank.

Now one can follow the strategy of Boston. Suppose that we have the following exact sequence of Galois groups:

$$1 \longrightarrow \mathcal{G} \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 1,$$

where $\Delta$ is a finite group of order prime to $p$ and $\mathcal{G}$ is uniformly powerful. The group $\Delta$ acts on the groups $H^i(\mathcal{G}, \mathbb{F}_p)$. Hence thanks to Theorem 7, one can compute exactly the characters of the $\Delta$-module $H^2(\mathcal{G}, \mathbb{F}_p)$. Comparing this with the arithmetic of $\mathcal{G}$, one obtains in some cases a contradiction.

## 2. Theorem 1 and its consequences

**2.1. Basic properties on representations.** First recall some well-know properties about representations in the semi-simple case (see [10], chapters 14, 15 and 16).

Let $\Delta$ be a finite group of order prime to $p$. In this situtation, every finitely generated $\mathbb{F}_p[\Delta]$-module $M$ is projective. Then, to each such $M$, one can associate a unique (up to isomorphism) $\mathbb{Z}_p[\Delta]$-projective module $N$ such that $N/pN$ is isomorphic to $M$. Now one has to note that two $\mathbb{Z}_p[\Delta]$-projective modules $N$ and $N'$ are isomorphic if and only if $\mathbb{Q}_p \otimes N$ and $\mathbb{Q}_p \otimes N'$ are $\Delta$-isomorphic. Hence, taking an $\mathbb{F}_p[\Delta]$-module $M$, to study the action of $\Delta$ on it is equivalent to look at the action of $\Delta$ on $\mathbb{Q}_p \otimes N$ and then finally to find the decomposition of the character of $\mathbb{Q}_p \otimes N$ in terms of absolutely irreducible characters $\varphi$. In other words, the irreducible representations of $\Delta$ over $\mathbb{F}_p$ are in bijection with the irreducible representations of $\Delta$ over $\mathbb{Q}_p$.

Next, if $M$ is an $\mathbb{F}_p[\Delta]$-module, we will identify $M$ with its associated $\mathbb{Q}_p[\Delta]$-module $N$. In particular, if $M$ is an $\mathbb{F}_p[\Delta]$-module, we will denote by

$$\chi(M) := \chi(N) = \sum_{\varphi} a_\varphi \varphi$$

the decomposition of the character $\chi(N)$ in terms of absolutely irreducible characters $\varphi$, where $N$ is the associated $\mathbb{Q}_p[\Delta]$-module of $M$.

To finish this part, we recall a classical convention. Let $N$ and $N'$ be two $\mathbb{Q}_p[\Delta]$-modules and let $\chi(N) = \sum_{\varphi} a_\varphi \varphi$ and $\chi(N') = \sum_{\varphi} a'_\varphi \varphi$ be the decomposition of the characters of these modules into absolutely irreducible characters $\varphi$. Then $\chi(N) \leq \chi(N')$ means that for every absolutely irreducible character $\varphi$, $a_\varphi \leq a'_\varphi$, and $\chi(N) < \chi(N')$ means that $\chi(N) \leq \chi(N')$ and there exists $\varphi$ such that $a_\varphi < a'_\varphi$.

**2.2. The proof of Theorem 1.** The exact sequence of $\mathcal{G}$-modules

$$1 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 1$$

gives

$$\cdots \longrightarrow H^1(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{p} H^1(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^2(\mathcal{G}, \mathbb{F}_p) \longrightarrow \cdots$$

By passing to the dual, one obtains:

$$H^2(\mathcal{G}, \mathbb{F}_p)^* \twoheadrightarrow \mathcal{G}^{ab}[p].$$

Now if the group $\mathcal{G}$ is uniformly powerful, the $\Delta$-module $H^2(\mathcal{G}, \mathbb{F}_p)$ is exactly $\bigwedge^2(\mathcal{G}_1{}^*)$. Hence, as $\Delta$-module:

$$H^2(\mathcal{G}, \mathbb{F}_p)^* \simeq \bigwedge^2(\mathcal{G}_1{}^*)^* \simeq \bigwedge^2(\mathcal{G}_1).$$

Moreover, by class field theory, the finiteness of ray class groups and the fact that $S$ is prime to $p$ imply that the pro-$p$-group $\mathcal{G}^{ab}$ is finite. The Theorem will be established once we take note of the following well-known lemma:

**Lemma 8.** *Let $M$ be a finite $\Delta$-module of order a power of $p$, where $\Delta$ is a finite group of order prime to $p$. Then:*

$$M[p] \simeq_{\mathbb{F}_p[\Delta]} M/M^p.$$

$\diamond$

**2.3. First applications.** When $\Delta$ is abelian, the character of $\bigwedge^2(\mathcal{G}_1)$ is easy to estimate (thanks to section 2.1):

**Corollary 9.** *Let $\mathbf{K}/\mathbf{F}$ be a finite abelian extension of number fields of degree prime to $p$ and with group $\Delta$. Let $\mathbf{L}/\mathbf{K}$ be a Galois pro-$p$-extension with group $\mathcal{G}$ such that: (i) $\mathbf{L} \subset \mathbf{K}_S$; (ii) $\mathbf{L}/\mathbf{F}$ is Galois.*

*Let $\chi(\mathcal{G}_1) = \sum_\varphi \alpha_\varphi \varphi$ be the decomposition of $\mathcal{G}_1$ as $\Delta$-module.*

*If*

$$\sum_\varphi \alpha_\varphi(\alpha_\varphi - 1)/2 \cdot \varphi^2 + 1/2 \sum_{\varphi \neq \varphi'} \alpha_\varphi \alpha_{\varphi'} \varphi\varphi' \;<\; \sum_\varphi \alpha_\varphi \varphi,$$

*then $\mathcal{G}$ is not uniformly powerful.*

**Example 10.** *Suppose that $p > 2$ and that $\Delta$ is the Klein 4-group. Let $\varphi_i$, $i = 1, 2, 3$ be its three non-trivial irreducible characters of order 2. If $\chi(\mathcal{G}_1) = \alpha\varphi_1 + \beta\varphi_2$, then $\mathcal{G}$ is not uniformly powerful.*

More generally, the computation of the character of $\bigwedge^2(\mathcal{G}_1)$ can be done thanks to the following formula: Let $N$ be a $\mathbb{Q}_p[\Delta]$-module. Fix a $\mathbb{Q}_p$-basis of $N$ and let $\rho$ be the representation of $\Delta$ associated to $N$. Then, for all $g \in \Delta$,

$$\chi(\bigwedge^2(N))(g) = \frac{1}{2}\left(\mathrm{Tr}(\rho^2(g)) - (\mathrm{Tr}(\rho(g)))^2\right).$$

In fact, thanks to the isomorphism $\bigwedge^2(N \oplus N') \simeq \bigwedge^2 N \oplus N \otimes N' \oplus \bigwedge^2 N'$, one needs to know only the computation of $\bigwedge^2 N$ for an absolutely irreducible $\Delta$-module $N$.

**Corollary 11.** *Let $\mathbf{K}/\mathbf{F}$ be a finite extension of number fields of degree prime to $p$ and with group $\Delta$. Let $\mathbf{L}/\mathbf{K}$ be a Galois pro-$p$-extension with group $\mathcal{G}$ such that: (i) $\mathbf{L} \subset \mathbf{K}_S$; (ii) $\mathbf{L}/\mathbf{F}$ is Galois. Suppose that $\Delta = \mathrm{Sym}_d$ is the symmetric group on $d$ letters ($d > 3$, $(d!, p) = 1$). Suppose moreover that $\mathcal{G}_1$ is of $p$-rank $d$ and the action of $\Delta$ on $\mathcal{G}_1$ is the permutation action on a set of $d$ letters. In others words, $\chi(\mathcal{G}_1) = \mathbf{1} + \varphi$, where $\varphi$ is the character of the standard representation of $\mathrm{Sym}_d$ and where $\mathbf{1}$ is the trivial character. Then the pro-$p$-group $\mathcal{G}$ is not uniformly powerful.*

*Proof.* The representation $\varphi$ is absolutely irreducible, and moreover $\bigwedge^2(\varphi)$ is still irreducible (see for example [4], §3). Then $\bigwedge^2(\mathcal{G}_1) = \varphi + \bigwedge^2(\varphi)$ does not contain $\mathbf{1}$, and so, by Theorem 1, $\mathcal{G}$ is not uniformly powerful. $\diamond$

**2.4. The cyclic case.**

One has to compare all of our results with the following result due to Boston:

**Theorem 12** (Boston)**.** *Let $\mathbf{K}/\mathbf{F}$ be a cyclic extension of prime order $l \neq p$. Let $\mathbf{L}/\mathbf{F}$ be an unramified pro-$p$-extension such that $\mathbf{L}/\mathbf{F}$ is Galois. If the $p$-class group of $\mathbf{F}$ is trivial then $\mathcal{G}$ is not uniformly powerful.*

In other words, when $\Delta$ is cyclic of prime order $\neq p$, Boston has proved that the representation of $\mathcal{G}_1$ necessarily contains the trivial representation.

Suppose that $\Delta$ is generated by $\sigma$ of order $n$ prime to $p$. The absolutely irreducible characters $\varphi_i$ of $\mathbb{Q}_p[\Delta]$ are of degree 1 defined by: $\sigma \cdot \zeta_n = \zeta_n^{\varphi_i(\sigma)}$, where $\zeta_n$ is a primitive $n$th root of unity and where $\varphi_i(\sigma) \in \mathbb{Z}/n\mathbb{Z}$.

Let $M$ be an $\mathbb{F}_p[\Delta]$-module, and let

$$A(M) = \{\langle \varphi_i, \chi(M)\rangle \ \varphi_i(\sigma),\ i = 0, \cdots, n-1\}$$

be the multiset of elements of $\mathbb{Z}/n\mathbb{Z}$ corresponding to the decompositon of the $\Delta$-module $M$ (see §2.1).

**Definition 13.** *If $A = \{a_0, \cdots, a_s\}$ is a multiset of elements of $\mathbb{Z}/n\mathbb{Z}$, we denote the multiset $\{a_i + a_j, i < j\}$ of $\mathbb{Z}/n\mathbb{Z}$ by $\bigwedge^2 A$.*

With the properties developed in section §2.1, one deduces easily the following proposition:

**Proposition 14.** *Suppose $M$ is an $\mathbb{F}_p[\Delta]$-module and let $A(M)$ be the multiset associated to the $\Delta$-action on $M$. Then $\bigwedge^2 A(M)$ is the multiset associated to the $\Delta$-module $\bigwedge^2(M)$.*

Hence, as corollary of Theorem 1, one obtains:

**Corollary 15.** *Let $A(\mathcal{G}_1)$ be the multiset of $\mathbb{Z}/n\mathbb{Z}$ associated to the $\Delta$-decomposition of $\mathcal{G}_1$. If the multiset $A(\mathcal{G}_1)$ is not contained in $\bigwedge^2 A(\mathcal{G}_1)$, then $\mathcal{G}$ is not uniformly powerful.*

When $n$ is a prime number $l$, the condition of this corollary does not necessarily imply that 0 is in $A(\mathcal{G}_1)$. (I thank Y. Bilu for this remark). In other words, it does not imply Boston's theorem 12.

**Corollary 16.** *Suppose that $\Delta$ is of prime order $l \neq p$. Let $A(\mathcal{G}_1) = \{a_0, \cdots, a_s\}$ be the multiset associated to $\mathcal{G}_1$ coming from the action of $\Delta$. Suppose that: $\forall i \neq j$, $a_i + a_j \neq 0$. Then, $\mathcal{G}$ is not uniformly powerful.*

*Proof.* With these conditions, 0 is not in $\bigwedge^2 A(\mathcal{G}_1)$ and so Boston's theorem gives the result. $\diamond$

**Remark 17.** *The conditions of this corollary should be compared to the conditions of* [8].

**2.5. The CM-case.** In this section we assume $p > 2$. Before giving the result that we need, let us fix some notation. For a finite extension $\mathbf{L}/\mathbf{K}$ of number fields, put

$$\mathcal{E}(S, \mathbf{L}/\mathbf{K}) = \mathcal{E}_{\mathbf{K},S}/(\mathcal{E}_{\mathbf{K},S} \cap N_{\mathbf{L}/\mathbf{K}}(\mathcal{L}_S) \cdot \mathcal{N}(\mathbf{L}/\mathbf{K})^p)$$

where

- $\mathcal{E}_{\mathbf{K},S} = \{x \in \mathbb{Z}_p \otimes E_{\mathbf{K}}, x \equiv 1 \bmod v, \forall v \in S\}$, $E_{\mathbf{K}}$ being the group of units of $\mathbf{K}$;
- $\mathcal{L}_S = \{x \in \mathbb{Z}_p \otimes \mathbf{L}^\times, x \equiv 1 \bmod v, \forall v \in S\}$
- $\mathcal{N}(\mathbf{L}/\mathbf{K}) = \{x \in \mathbb{Z}_p \otimes \mathbf{K}^\times, x \equiv 1 \bmod v, \forall v \in S\} \cap \mathcal{N}_{loc}$, where $\mathcal{N}_{loc}$ is the group of elements of $\mathbf{K}^\times$ that are local norms at every place of $\mathbf{K}$
- $\mathrm{Tor}(\mathcal{E}_{\mathbf{K},S})$ the group of the $p$th roots of unity in $\mathcal{E}_{\mathbf{K},S}$. In particular, if $S$ is not empty, $\mathrm{Tor}(\mathcal{E}_{\mathbf{K},S}) = \{1\}$.

In the appendix of his book [6], Gras computes the relations of the group $\mathcal{G}_S$. His strategy is to study the inflation map Inf:

$$H^2(\mathcal{G}_S/H, \mathbb{F}_p) \xrightarrow{\mathrm{Inf}} H^2(\mathcal{G}_S, \mathbb{F}_p),$$

where $H$ is a small open subgroup of $\mathcal{G}_S$. He obtains:

**Proposition 18** (Gras, [6], Appendix, theorem 3.7)**.** *For a sufficiently large finite extension* $\mathbf{L}/\mathbf{K}$ *inside* $\mathbf{K}_S/\mathbf{K}$*, the following sequence is exact:*

$$1 \longrightarrow \mathcal{E}(S, \mathbf{L}/\mathbf{K}) \longrightarrow H_2(\mathcal{G}_S, \mathbb{F}_p) \longrightarrow \mathcal{G}_S^{ab}[p] \longrightarrow 1.$$

Now let $\mathbf{K}/\mathbf{F}$ be a finite Galois extension of degree prime to $p$ and with Galois group $\Delta$. Let $S$ be a finite set of places of $\mathbf{K}$ prime to $p$. Assume moreover that $S$ is $\Delta$-stable. Then $\mathbf{K}_S/\mathbf{F}$ is a Galois extension. Hence in the above proposition, we can assume that $\mathbf{L}/\mathbf{F}$ is a Galois extension. In this case, the short sequence of Proposition 18 is a sequence of $\Delta$-modules.

**Lemma 19.** *The character of the* $\Delta$*-module* $\mathcal{E}(S, \mathbf{L}/\mathbf{K})$ *is less than*

$$-\mathbf{1} + \chi(\mathrm{Tor}(\mathcal{E}_{\mathbf{K},S})) + \sum_{v \in Pl_\infty} \mathrm{Ind}_{D_v}^{G} \mathbf{1},$$

*where* $Pl_\infty$ *is the set of archimedean places of* $\mathbf{F}$*,* $D_v$ *is the decomposition group of* $v$ *in* $\mathbf{K}/\mathbf{F}$ *and* $\mathbf{1}$ *is the trivial character.*

*Proof.* First note that $\mathcal{E}(\mathbf{K}, S) \subset \mathcal{N}_{loc}$ (for the archimedean places, recall that we have assumed that $p > 2$). Then the group $\mathcal{E}(S, \mathbf{L}/\mathbf{K})$ is a quotient of $\mathcal{E}_{\mathbf{K},S}/\mathcal{E}_{\mathbf{K},S}^p$. Now, the group $\mathcal{E}_{\mathbf{K},S}$ is of finite index in $\mathbb{Z}_p \otimes E_{\mathbf{K}}$. Hence, in the semi-simple case, the character of $\mathcal{E}_{\mathbf{K},S}/\mathcal{E}_{\mathbf{K},S}^p$ can be computed by the Dirichlet map, and, up to $p$th roots of unity, it is the same as the group of units $E_{\mathbf{K}}$ of $\mathbf{K}$ (see [7] §6). ◇

Now we specialize to the situation where $\Delta$ is of order 2.

**Theorem 20.** *Assume* $p > 2$*. Let* $\mathbf{K}/\mathbf{F}$ *be a quadratic extension. Assume that* $S$ *is* $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$*-stable. Let* $t$ *be the number of archimedean places of* $\mathbf{F}$ *which split completely in* $\mathbf{K}/\mathbf{F}$*. If* $S$ *is empty and,* $\mathbf{K}$ *contains* $\mu_p$ *but* $\mathbf{F}$ *does not, let* $\delta_p(S) = 1$*, otherwise, let* $\delta_p(S) = 0$*. Let*

$$d(\mathcal{G}_S) = d^+ + d^-$$

*be the decomposition of the generator rank of* $\mathcal{G}_S$ *according to the action of* $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$*. If*

$$d^+ d^- > d^- + t + \delta_p(S),$$

*then* $\mathcal{G}_S$ *is not uniformly powerful.*

*Proof.* By Proposition 18, the rank of the minus part of $H_2(\mathcal{G}_S, \mathbb{F}_p)$ is less than $d^-$ plus the rank of the minus part of $\mathcal{E}(S, \mathbf{L}/\mathbf{K})$. By Lemma 19, the minus part of $\mathcal{E}(S, \mathbf{L}/\mathbf{K})$ is of $p$-rank less than $\delta_p(S) + t$. Then, the minus part of $H_2(\mathcal{G}_S, \mathbb{F}_p)$ is of $p$-rank less than $d^- + \delta_p(S) + t$.

Suppose now that $\mathcal{G}_S$ is uniformly powerful. Then $\bigwedge^2((\mathcal{G}_S)_1)$ is isomorphic to $H_2(\mathcal{G}_S,\mathbb{F}_p)$. Moreover, the minus part of $\bigwedge^2((\mathcal{G}_S)_1)$ is exactly of rank $d^+d^-$ and then in this case the minus part of $H_2(\mathcal{G}_S,\mathbb{F}_p)$ is of rank $d^+d^-$ which contradicts the initial inequality. $\diamond$

As corollary, one find the result of Wingberg [12] given in the introduction:

**Corollary 21** (The CM-case)**.** *Assume $p>2$. Let $\mathbf{K}$ be a CM-field with maximal totally real subfiel $\mathbf{F}$. Assume that $S$ is $\mathrm{Gal}(\mathbf{K}/\mathbf{F})$-stable. If $d^+d^- > d^- + \delta_p(S)$, then $\mathcal{G}_S$ is not uniformly powerful.*

*Proof.* In this case $t=0$. $\diamond$

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