COHOMOLOGY OF NUMBER FIELDS AND ANALYTIC PRO-p-GROUPS

by

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Abstract. — In this work, we are interested in the tame version of the Fontaine-Mazur conjecture. By viewing the pro-p-proup $\mathcal{G}_S$ as a quotient of a Galois extension ramified at p and S, we obtain a connection between the conjecture studied here and a question of Galois structure. Moreover, following a recent work of A. Schmidt, we give some evidence of links between this conjecture, the étale cohomology and the computation of the cohomological dimension of the pro-p-groups $\mathcal{G}_S$ that appear.

Contents

1. Introduction

Let us fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$.

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I thank Farshid Hajir for his help and his interest in this work. I thank Denis Vogel for having sent me his preprint [**26**].

Let $\mathbf{K}$ be a number field and let p be a prime number. Let S be a finite set of places of $\mathbf{K}$. Denote by $\mathbf{K}_S$ the maximal pro-p-extension of $\mathbf{K}$ unramified outside S. Put $\mathcal{G}_S = \mathrm{Gal}(\mathbf{K}_S/\mathbf{K})$.
For a place v of $\mathbf{K}$, denote by $\mathbf{K}_v$ the completion of $\mathbf{K}$ at v. Let $\mathcal{G}_v := \mathrm{Gal}(\overline{\mathbf{K}_v}/\mathbf{K}_v)$ be the absolute Galois group of $\mathbf{K}_v$ and let $I_v \subset \mathcal{G}_v$ be the absolute inertia group of $\mathbf{K}_v$.

In this work, we study the structure of the group $\mathcal{G}_S$. This is related to a famous conjecture due to Fontaine and Mazur [**6**]. This conjecture predicts which p-adic extensions of number fields come from algebraic geometry. More precisely. Let $\rho : G_{\mathbf{K}} \rightarrow \mathrm{Gl_n}(\mathbb{Q}_\mathrm{p})$ be a continuous p-adic representation of $\mathcal{G}_{\mathbf{K}}$. Suppose: (i) ρ is unramified outside a finite set S of places of $\mathbf{K}$; (ii) for all $v|p$, the representation $\rho_v := \rho_{|\mathcal{G}_v}$ is potentially semi-stable. Then ρ should come from "geometry": ρ should correspond to the action of $\mathcal{G}_{\mathbf{K}}$ on a subquotient of an étale cohomology group of an algebraic variety over $\mathbf{K}$, twisted by $\mathbb{Q}_p(j)$.

Here, we are interested in the particular case where the ramification is potentially tame. In this case, Fontaine and Mazur conjecture the following as "a consequence" of their philosophy:

Conjecture 1.1 **(Fontaine and Mazur, [6], conjecture 5a)**
Suppose that the characteristic of the residue field k_v of each place v of S is different from p. Then, every p-adic analytic quotient of $\mathcal{G}_S$ is finite.

Many authors (Boston [**2**], [**3**], Hajir [**9**], Wingberg [**27**] ...) have contributed to the study of conjecture 1.1.

In section 2, we develop, in the context of the conjecture 1.1, the p-adic analytic point of view. In particular, we show how the question of the knowledge of the cohomological dimension of the groups $\mathcal{G}_S$ is connected to conjecture 1.1. In section 3, we go back to the recent results of Labute [**13**] and Schmidt [**19**], [**20**], [**22**], [**23**]. Recall that Labute gave for the first time some examples of groups $\mathcal{G}_S$ of cohomological dimension 2, for S prime to p. Schmidt extended the work of Labute by showing that the cohomology of the pro-p-groups in question coincide with some étale cohomology. We then give some connections linking the point of view of Schmidt and the conjecture 1.1.
In section 4, we compare the groups $\mathcal{G}_S$ with the groups $\mathcal{G}_{S'}$, where $S' = S \cup S_p$, S_p being the set of all places of $\mathbf{K}$ above p. More precisely, let

$\mathcal{H} = \mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K}_S)$ be the closed normal subgroup of $\mathcal{G}_{S'}$ generated by all inertia groups of all places of $S' \setminus S$. The quotient $\mathcal{X} := \mathcal{H}^{ab} = \mathcal{H}/\mathcal{H}^{ab}$ is a finitely generated $\mathbb{Z}_p[[\mathcal{G}_S]]$-module. We then prove the following:

Theorem 1.2. — *Assume Leopoldt's conjecture for the prime p. If the $\mathbb{Z}_p[[\mathcal{G}_S]]$-module $\mathcal{X}$ is free, then the cohomological dimension of $\mathcal{G}_S$ is at most* 2.

The proof is inspired by some works of Brumer [**4**], Nguyen [**18**], Neukirch-Wingberg-Schmidt [**17**] ....
Then, by using some observation about the structure of analytic pro-p-groups of section 2, we obtain a connection between the conjecture 1.1 and the question of freeness of the $\mathbb{Z}_p[[\mathcal{G}_S]]$-module $\mathcal{X}$:

Corollary 1.3. — *Assume $S \cap S_p = \emptyset$. If the $\mathbb{Z}_p[[\mathcal{G}_S]]$-module $\mathcal{X}$ is free, then $\mathcal{G}_S$ is not p-adic analytic.*

To finish, we look at the examples of Labute in [**13**] and Vogel in [**26**]. In these cases, the groups $\mathcal{G}_S$ are of cohomological dimension 2 and then are not p-adic analytic. We prove here that for theses examples, the module $\mathcal{X}$ is $\mathbb{Z}_p[[\mathcal{G}_S]]$-free.

Notations. If M is a module over $\mathbb{Z}_p$, we denote by d_pM the dimension over $\mathbb{F}_p$ of $\mathbb{F}_p \otimes_{\mathbb{Z}_p} M$, and by $\mathrm{rk}_{\mathbb{Z}_p} M$ the dimension over $\mathbb{Q}_p$ of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$.

2. $\mathcal{G}_S$ and analytic structure

2.1. On analytic pro-p-groups. — The main references for this part are: Lazard[**14**], Dixon, Du Sautoy, Segal, Mann [**5**], ...

Definition 2.1. — A topological group $\mathcal{G}$ is p-adic analytic if $\mathcal{G}$ has a structure of p-adic analytic manifold for which the morphism $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (x, y) \mapsto xy^{-1}$ is analytic.

Example 2.2. — A p-adic analytic group $\mathcal{G}$ is of dimension 0 if and only if $\mathcal{G}$ is finite.
A p-adic analytic group $\mathcal{G}$ is of dimension 1 if and only if there exists an open subgroup $\mathfrak{U}$ of $\mathcal{G}$ isomorphic to $\mathbb{Z}_p$.

Now let $\mathcal{G}$ be a pro-p-group of finite type. Denote by $P_i(\mathcal{G})$ the subgroups of the p-lower central series of $\mathcal{G}$:

$$P_1(\mathcal{G}) = \mathcal{G}, \quad P_{i+1}(\mathcal{G}) = P_i(\mathcal{G})^p[\mathcal{G}, P_i(\mathcal{G})],$$

which is a sequence of closed subgroups of $\mathcal{G}$. Denote by $\mathcal{G}_1$ the quotient $\mathcal{G}/\mathcal{G}^p[\mathcal{G},\mathcal{G}] = \mathcal{G}/P_2(\mathcal{G})$.

Definition 1. — *The pro-p-group $\mathcal{G}$ is uniformly powerful if:*
(i) $\mathcal{G}/\overline{\mathcal{G}^p}$ (for $p=2$, $\mathcal{G}/\overline{\mathcal{G}^4}$) is abelian.
(ii) for all i, $\#P_i(\mathcal{G})/P_{i+1}(\mathcal{G}) = \#\mathcal{G}/P_2(\mathcal{G})$

An uniformly powerful pro-p-group $\mathcal{G}$ is p-adic analytic and the study of p-adic analytic group can be done via this family of pro-p-groups:

Theorem 2.3 **([5])**. — *Let $\mathcal{G}$ be a p-adic analytic group. Then $\mathcal{G}$ has an open uniformly powerful subgroup.*

If $\mathcal{G}$ is uniformly powerful, the cohomological dimension of $\mathcal{G}$ is equal to its analytic manifold dimension. Moreover, the cohomology groups $H^i(\mathcal{G},\mathbf{F}_p)$ are isomorphic to the exterior product $\bigwedge^i H^1(\mathcal{G},\mathbf{F}_p)$. Hence, for a uniformly powerful pro-p-group $\mathcal{G}$, the p-rank is equal to the cohomological dimension $cd(\mathcal{G})$ of $\mathcal{G}$ and the Euler-Poincaré characteristic $\chi(\mathcal{G}) = \sum_i (-1)^i d_p H^i(\mathcal{G},\mathbf{F}_p)$ of $\mathcal{G}$ is equal to zero.

Proposition 2.4. — *Let $\mathcal{G}$ be a uniformly powerful group of dimension 2. Then, there exists a surjective morphism: $\mathcal{G}^{ab} \twoheadrightarrow \mathbb{Z}_p$.*

Proof. — Indeed, in this case, one has: $d_p H^1(\mathcal{G},\mathbf{F}_p) = 2 d_p H^2(\mathcal{G},\mathbf{F}_p) = 2$. The short exact sequence

$$1 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 1,$$

gives

$$1 = d_p H^1(\mathcal{G},\mathbb{F}_p) - d_p H^2(\mathcal{G},\mathbb{F}_p) \leq \mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}^{ab}.$$

$\Diamond$

Corollary 2.5. — *Let $\mathcal{G}$ be a p-adic analytic group of dimension 2. There exists an open subgroup $\mathfrak{U}$ of $\mathcal{G}$, for which the $\mathbb{Z}_p$-rank of $\mathfrak{U}^{ab}$ is not trivial.*

2.2. A reformulation of the conjecture 1.1. — We can give a reformulation of Conjecture 1.1 in terms of dimension of the analytic manifold.

Conjecture 2.6 **(C_r)**. — *Let $\mathbf{K}$ be a number field and S be a finite set of places of $\mathbf{K}$. Let $\mathbf{K}_S$ be the maximal pro-p-extension of $\mathbf{K}$ unramified outside S. Put $\mathcal{G}_S = \mathrm{Gal}(\mathbf{K}_S/\mathbf{K})$. Assume that for all places v in S, k_v*

is of characteristic different from p. Let $r > 0$ be an integer. Then there is no p-adic analytic quotient of $\mathcal{G}_S$ of dimension r.

2.3. Conjecture C_1. — The abelian version of the conjecture 1.1 is controlled by Class field theory. More precisely, some classical calculations allow to show the following:

Proposition 2.7. — *(i) The prop-p-group $\mathcal{G}_S$ is finitely generated.*
(ii) If $S \cap S_p = \emptyset$, then $\mathcal{G}_S^{ab} := \mathcal{G}_S/[\mathcal{G}_S, \mathcal{G}_S]$ is finite.
(iii) If S contains all places of $\mathbf{K}$ above p, then the $\mathbb{Z}_p$-rank of $\mathcal{G}_S^{ab}$ is at least $r_2 + 1$, where (r_1, r_2) is the signature of $\mathbf{K}$. Leopoldt's conjecture asserts the equality.

We immediately obtain:

Proposition 2.8. — *Conjecture C_1 is true.*

Proof. — A p-adic analytic group $\mathcal{G}$ of dimension 1 has an open subgroup $\mathfrak{U}$ with $\mathfrak{U} \simeq \mathbb{Z}_p$. Suppose that $\mathcal{G}$ is the Galois group of an extension $\mathbf{L}/\mathbf{K}$, unramified outside S, with $S \cap S_p = \emptyset$. Put $\mathbf{F} = \mathbf{L}^{\mathfrak{U}}$. Then $\mathbf{F}_S = \mathbf{K}_S$, where $\mathbf{K}_S$ is the maximal pro-p-extension of $\mathbf{F}$ unramified oustide $S_{\mathbf{F}}$, and where $S_{\mathbf{F}}$ is the set of places of $\mathbf{K}$ above S. Then $\mathrm{Gal}(\mathbf{F}_S/\mathbf{F})^{ab} \twoheadrightarrow \mathbb{Z}_p$ which contredicts (ii) of Proposition 2.7. $\Diamond$

2.4. The importance of C_1. — In the context of conjecture 1.1, proposition 2.8 is important for an another reason. It allows one to prove the following fundamental example issuing from geometry:
Let $X/\mathbf{K}$ be a smooth projective variety over $\mathbf{K}$. Let

$$H^i_{et}(\overline{X}, \mathbb{Q}_p) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lim_{\substack{\leftarrow \\ n}} H^i_{et}(\overline{X}, \mathbb{Z}/p^n\mathbb{Z}),$$

be the ith etale cohomology group of $\overline{X} := X/\overline{\mathbf{K}}$ with coefficients in $\mathbb{Q}_p$. Put:

$$h_{i,j}(X) := H^i_{et}(\overline{X}, \mathbb{Q}_p)(j),$$

the twist of $H^i_{et}(\overline{X}, \mathbb{Q}_p)$ by $\mathbb{Q}_p(j)$.
This group $h_{i,j}(X)$ is a finite-dimensional $\mathbb{Q}_p$-vector space on which acts the absolute Galois group $\mathcal{G}_{\mathbf{K}}$ of $\mathbf{K}$. This action gives a Galois representation of $\mathcal{G}_{\mathbf{K}}$:

$$\rho^{i,j} : \mathcal{G}_{\mathbf{K}} \to \mathrm{Gl}_m(\mathbb{Q}_p) = \mathrm{Aut}(h_{i,j}(X)).$$

Definition 2.9. — A Galois represention ρ of $\mathcal{G}_{\mathbf{K}}$ is potentially unramified at p, if for all $v|p$, $\rho(I_v)$ is finite.

One has the following result from a paper of Kisin-Wortmann [**11**]:

Theorem 2.10. — *Suppose that the Galois representation*

$$\rho : \mathcal{G}_{\mathbf{K}} \to \mathrm{Aut}(H^i_{et}(X, \mathbb{Q}_p)(j))$$

is potentially unramified at p. Then, the image $\rho(G_{\mathbf{K}})$ is finite.

Let us to show why C_1 (in fact Proposition 2.7) appears in the proof. The crucial fact in the proof of Theorem 2.10 is a variation of places v with ℓ-adic cohomology, p-adic cohomology and isomorphisms of comparison of the different types of cohomology. Denote by S the set of places of $\mathbf{K}$ at which: ρ is ramified; or X has bad reduction. We add to S the places above p and infinite places. This final set S is finite. Let $v \notin S$ and denote by σ_v the geometric Frobenius at v. Let us consider the action of σ_v on $h_{i,j}(X)$ and let $\lambda_{k,v}$ be the eigenvalues of σ_v. Thanks to the facts that the Hodge-Tate weights of $h_{i,j}(X)$ are zero, the $\lambda_{k,v}$ are integers such for all isomorphisms $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we get that $|\iota(\lambda_{k,v})| = 1$. Hence, the $\lambda_{k,v}$ are some roots of the unity and then by a scalar extension $\mathbf{L}/\mathbf{K}$, the image of $\rho_{|\mathbf{L}}$ is unramified at p and unipotent. To conclude, an easy lemma issues from Proposition 2.7:

Lemma 2.11. — *Suppose S finite and such $S \cap S_p = \emptyset$. Let H be a quotient of $\mathcal{G}_S$ with $H \subset \mathrm{Gl_m}(\mathbb{Q}_\mathrm{p})$ and such that H is unipotent. Then H is finite.*

Proof. — The group H is solvable. Moreover, the commutator series is a sequence of closed subgroups of H. The successive quotients are abelian and then finite by Proposition 2.7. One deduces that H is finite. $\Diamond$

For more details, we refer to the papers of Kisin-Wortmann [**11**] and Kisin-Lehrer [**10**].

2.5. Conjecture C_2. — As for C_1, the conjecture C_2 is deduced from class field theory.

Proposition 2.12. — *Conjecture C_2 is true.*

Proof. — Let $\mathbf{L}/\mathbf{K}$ be a Galois extension in $\mathbf{K}_S/\mathbf{K}$, where S is such that $S \cap S_p = \emptyset$. Suppose that the group $\mathcal{G} := \mathrm{Gal}(\mathbf{L}/\mathbf{K})$ is p-adic analytic of dimension 2. By corollary 2.5, there exists an open subgroup $\mathfrak{U}$ of $\mathcal{G}$ such that $\mathfrak{U}^{ab}$ has a non trivial $\mathbb{Z}_p$-rank which is impossible by Proposition 2.7. $\Diamond$

2.6. Conjecture C_3 and just infinite pro-p-groups.—

Definition 2.13. — A pro-p-group $\mathcal{G}$ is just infinite if it is infinite but every proper quotient of $\mathcal{G}$ is finite.

By Zorn's lemma, every infinite finitely generated pro-p-group has a just infinite quotient.

Proposition 2.14. — *Suppose that all places in S are prime to p. Let $\mathcal{G}$ be a quotient of $\mathcal{G}_S$. Suppose that $\mathcal{G}$ is infinite of cohomological dimension at most 3. If $\mathcal{G}$ is not just infinite, then $\mathcal{G}$ is not p-adic analytic.*

Proof. — Suppose that $\mathcal{G}$ is p-adic analytic. The conjectures (C_1) and (C_2) being true, one can assume that $\mathrm{cd}(\mathcal{G}) = 3$. The pro-$p$-group $\mathcal{G}$ being not just infinite, there exists a proper, closed and distinguished subgroup N such that $\mathcal{G}/N$ is infinite. As $\mathcal{G}$ is torsion-free, the analytic subgroup N is of cohomological dimension $k \in \{1, 2, 3\}$. The analytic quotient $\mathcal{G}/N$ has dimension $3 - k \leq 2$, i.e. 1 or 2 (because $\mathcal{G}/N$ is infinite), which contradicts (C_1) and (C_2). $\diamondsuit$

Corollary 2.15. — *Let $T \subsetneq S$ be two finite sets of places of $\mathbf{K}$, with $(S, p) = 1$. Suppose: (i) $\#\mathcal{G}_T^{ab} < \#\mathcal{G}_S^{ab}$; (ii) $\mathcal{G}_T$ is infinite; (iii) the cohomological dimension of $\mathcal{G}_S$ is at most 3. Then $\mathcal{G}_S$ is not p-adic analytic.*

Proof. — The inequality $\#\mathcal{G}_T^{ab} < \#\mathcal{G}_S^{ab}$ shows that at least one place of $S \backslash T$ is ramified in $\mathbf{K}_S/\mathbf{K}$. So $\mathbf{K}_S \neq \mathbf{K}_T$ and consequently $\mathcal{G}_T$ is a proper quotient of $\mathcal{G}_S$. In particular, $\mathcal{G}_S$ is not just infinite. By Proposition 2.14, $\mathcal{G}_S$ is not p-adic analytic. $\diamondsuit$

3. $\mathcal{G}_S$ and étale cohomology

Let S be a finite set of places of $\mathbf{K}$. Recall that $\mathbf{K}_S$ is the maximal pro-p-extension of $\mathbf{K}$ unramified outside S; put $\mathcal{G}_S = \mathrm{Gal}(\mathbf{K}_S/\mathbf{K})$.
Recall that analytic pro-p-group are virtually of finite cohomological dimension. Then, in the context of the conjecture 1.1, it seems natural to ask the following question:

Question 3.1. — *Suppose $S \cap S_p = \emptyset$. Is the pro-p-group $\mathcal{G}_S$ virtually of finite cohomological dimension ?*

When $S_p \subset S$, the following is well-known (see for example [**8**], [**17**] ...) :

Theorem 3.2. — *Let > 2 and suppose that S contains all the places above p. Then the cohomological dimension of $\mathcal{G}_S$ is 1 or 2.*

For $p = 2$, see a work of Schmidt [**21**].
When S contains no places above p, Labute, in [**13**], has recently given some examples for which the cohomological dimension of $\mathcal{G}_S$ is 2. For the mixed case (i.e $S \cap S_p \neq \emptyset$) and a different approach, see for example [**16**] ...

Remark 3.3. —

(i) Suppose S prime to p. Then $\mathcal{G}_S$ can not be free and if $cd(\mathcal{G}_S) = 2$, then the Principal Ideal Theorem shows that the strict cohomological dimension of $\mathcal{G}_S$ is 3.
(ii) If $S_p \subset S$, under the Leopoldt's conjecture along $\mathbf{K}_S/\mathbf{K}$, the strict cohomological dimension of $\mathcal{G}_S$ is 2 (see for example [**17**], chapter III).
(iii) Recently, in the direction of question 3.1, Schmidt shows the following [**22**]: given a finite set S, there exists a finite set T of places of $\mathbf{K}$, $T \cap S_p = \emptyset$, such that $cd(\mathcal{G}_{S\cup T}) = 2$.

3.1. The results of Schmidt. — In [**19**], Schmidt expands on the work of Labute [**13**]. He proves that the examples of the groups $\mathcal{G}_S$ given by Labute have a cohomology related to the étale cohomology of certain schemes.

Let $\mathfrak{U}$ be an open subgroup of $\mathcal{G}_S$. Denote by $\mathbf{K}_S^{\mathfrak{U}}$ the maximal subfield of $\mathbf{K}_S$ fixed by $\mathfrak{U}$. Put Spec $\mathcal{O}_{\mathfrak{U}} \setminus S :=$ Spec$(\mathcal{O}_{\mathbf{K}_{\mathfrak{U}}}) \setminus S_{\mathbf{K}_S^{\mathfrak{U}}}$, where $S_{\mathbf{K}_S^{\mathfrak{U}}}$ is the set of places of $\mathbf{K}_S^{\mathfrak{U}}$ above the places of S.
Let M be a torsion $\mathcal{G}_S$-module. We consider M as the constant sheaf over Spec $\mathcal{O}_{\mathfrak{U}} \setminus S$. Denote by $H^i_{et}(\text{Spec } \mathcal{O}_{\mathfrak{U}} \setminus S, M)$ the ith étale cohomology group of the sheaf M and put

$$H^i_{et}(X(\mathbf{K}_S), M) := \lim_{\substack{\rightarrow \\ \mathfrak{U}}} H^i_{et}(\text{Spec } \mathcal{O}_{\mathfrak{U}} \setminus S, M),$$

the inductive limit being on open subgroups $\mathfrak{U}$ of $\mathcal{G}_S$ with $\bigcap_{\mathfrak{U}} \mathfrak{U} = \{1\}$.

The spectral sequence $H^i(\mathcal{G}_S, (H^j_{et}(X(\mathbf{K}_S), M)) \implies H^{i+j}_{et}(\text{Spec } \mathcal{O}_{\mathbf{K}} \setminus S, M)$, shows the existence of a morphism

$$\phi_i(M) : \; H^i(\mathcal{G}_S, M) \to H^i_{et}(\text{Spec } \mathcal{O}_{\mathbf{K}} \setminus S, M).$$

As shown in [**19**], the study of this morphism is related to the case where $M = \mathbb{F}_p$. Hence we define:

Definition 3.4. — The pro-p-group $\mathcal{G}_S$ is cohomologically étale if for all $i \geq 0$, the morphism $\phi_i := \phi_i(\mathbb{F}_p)$ is an isomorphism.

Remark 3.5. — When ϕ_i is an isomorphism for all i, following Schmidt in [**20**], the scheme Spec $\mathcal{O}_{\mathbf{K}} \setminus S$ is called $K(\pi, 1)$.

Question 3.6. — *Suppose the pro-p-group $\mathcal{G}_S$ infinite. In which cases, $\mathcal{G}_S$ is cohomologically étale?*

Questions 3.1 and 3.6 are related by the following Theorem:

Theorem 3.7 **([1] or [20]).** — *Suppose $\mathcal{G}_S$ cohomologically étale. For $p = 2$, suppose moreover that $\mathbf{K}$ is totally imaginary. Then the cohomological dimension of $\mathcal{G}_S$ is at most 3. Moreover, $\mathcal{G}_S$ is of cohomological dimension at most 2 when S is not empty or when $\mathbf{K}_S/\mathbf{K}$ does not contain the pth roots of unity.*

Proof. — The first part of this Theorem can be found in SGA [**1**] (Proposition 6.1). The second part is a calculation of Schmidt in [**20**]. ◊

From now on, we assume $p > 2$ or K totally imaginary.

3.2. Some consequences of a spectral sequence. — First recall two lemmas:

Lemma 3.8. — *One has:*

$$H^1_{et}(X(\mathbf{K}_S), \mathbb{F}_p) = 1 \ .$$

Proof. — The group $H^1_{et}(X(\mathbf{K}_S), \mathbb{F}_p)$ classifies the Galois étale covers of degree p of $X(\mathbf{K}_S)$. By maximality of $X(\mathbf{K}_S)$, this group is trivial. ◊

Hence, the morphism ϕ_1 is an isomorphism. When $\mathcal{G}_S$ is infinite, one has more:

Lemma 3.9 **(Schmidt, [20]).** — *If $\mathcal{G}_S$ is infinite, then*

$$H^3_{et}(X(\mathbf{K}_S), \mathbb{F}_p) = 1.$$

These lemmas applied to the spectral sequence

$$E_2^{i,j} = H^i(\mathcal{G}_S, (H^j_{et}(X(\mathbf{K}_S), \mathbb{F}_p)) \Longrightarrow E^{i+j} = H^{i+j}_{et}(\text{Spec } \mathcal{O}_K \setminus S, \mathbb{F}_p)$$

allow us to show:

Proposition 3.10. — *Assume $\mathcal{G}_S$ infinite. Then, the following long exact sequence holds:*

$$\begin{array}{ccccc} H^2(\mathcal{G}_S,\mathbb{F}_p) & \hookrightarrow & H^2_{et}(\mathrm{Spec}\mathcal{O}_{\mathbf{K}}\setminus S,\mathbb{F}_p) & \longrightarrow & H^2_{et}(X(\mathbf{K}_S),\mathbb{F}_p)^{\mathcal{G}_S} \\ & & & & \downarrow \\ H^1(\mathcal{G}_S,H^2_{et}(X(\mathbf{K}_S),\mathbb{F}_p)) & \longleftarrow & H^3_{et}(\mathrm{Spec}\mathcal{O}_{\mathbf{K}}\setminus S,\mathbb{F}_p) & \longleftarrow & H^3(\mathcal{G}_S,\mathbb{F}_p) \\ \downarrow & & & & \\ H^4(\mathcal{G}_S,\mathbb{F}_p) & & & & \end{array}$$

Proof. — For the surjective map $H^1(\mathcal{G}_S,H^2_{et}(X(\mathbf{K}_S),\mathbb{F}_p)) \twoheadrightarrow H^4(\mathcal{G}_S,\mathbb{F}_p)$, thanks to Lemma 3.9, it suffices to note that

$$\begin{array}{rcl} E_4^{4,0} & = & E_5^{5,0} \\ & = & E_\infty^{4,0} \\ & = & E_4^4 \\ & \subset & E^4 = 1 \quad \text{(Theorem 3.7).} \end{array}$$

$\Diamond$

One could introduce the notion of pro-p-group virtually cohomologically étale. But in fact, it is not necessary:

Corollary 3.11. — *(i) Suppose that the pro-p-group $\mathcal{G}_S$ is virtually cohomologically étale. Then $\mathcal{G}_S$ is cohomologically étale.*
(ii) If $\mathcal{G}_S$ is cohomologically étale, then for all open subgroup $\mathfrak{U}$ of $\mathcal{G}_S$, $\mathfrak{U}$ is cohomologically étale.

Proof. — (i) The pro-p-group $\mathcal{G}_S$ being virtually cohomologically étale, it means that there exists an open subgroup $\mathfrak{U}$ of $\mathcal{G}_S$ such that for all i, the morphisms $\phi_{i,\mathfrak{U}}$ are isomorphisms, where

$$\phi_{i,\mathfrak{U}} : H^i(\mathfrak{U},\mathbb{F}_p) \to H^i_{et}(\mathrm{Spec}\ \mathcal{O}_{\mathbf{K}_S^{\mathfrak{U}}}\setminus S,\mathbb{F}_p).$$

Then, by using the long exact sequence of Proposition 3.10, one deduces the triviality of $H^2_{et}(X(\mathbf{K}_S),\mathbb{F}_p)^{\mathfrak{U}}$, i.e. $H^2_{et}(X(\mathbf{K}_S),\mathbb{F}_p) = 1$ (remark that $\mathfrak{U}$ is not trivial !). It suffices to take the same exact sequence with $\mathcal{G}_S$ instead of $\mathfrak{U}$. Remark that the cohomological dimension of $\mathcal{G}_S$ is then at most 3.
(ii) - clear. $\Diamond$

3.3. The Euler-Poincaré characteristic. — Let $\mathfrak{U}$ be an open subgroup of $\mathcal{G}_S$. Denote by $\chi_n(\mathfrak{U})$ the Euler-Poincaré characteristic truncated at the order n, associated to the Galois cohomology groups $H^i(\mathfrak{U}, \mathbb{F}_p)$ of $\mathfrak{U}$ and denote by $\chi_{et}(\mathfrak{U})$ the Euler-Poincaré characteristic associated to the étale cohomology groups $H^i_{et}(\text{Spec } \mathcal{O}_{\mathfrak{U}} \setminus S, \mathbb{F}_p)$:

$$\chi_n(\mathfrak{U}) = \sum_{i=0}^{n} (-1)^i d_p H^i(\mathfrak{U}, \mathbb{F}_p),$$

$$\chi_{et}(\mathfrak{U}) = \sum_{i \geq 0} (-1)^i d_p H^i_{et}(\text{Spec } \mathcal{O}_{\mathbf{K}_S^{\mathfrak{U}}} \setminus S, \mathbb{F}_p).$$

By Class Field Theory, one knows an upper bound for the p-rank of $H^2(\mathcal{G}_S, \mathbf{F}_p)$. When S contains no places above p, an exact calculation of this rank is at the heart of the construction of asymptotically exact extensions. In all cases, one has:

$$\chi_2(\mathcal{G}_S) \leq -\delta_S + r_1 + r_2 + \delta_{p,S},$$

where $\delta_S = \sum_{v \in S \cap S_p} [\mathbf{K}_v : \mathbb{Q}_p]$, and $\delta_{p,S}$ is equals to 1 when $\mathbf{K}$ contains the p-roots of the unity and S is empty, 0 otherwise (see for example [**7**], Appendix).
Recall a result that appears in [**20**]:

Proposition 3.12 **(Schmidt)**. — *Suppose $p > 2$ or $\mathbf{K}$ be a totally imaginary field . One has:*

$$\chi_{et}(\mathcal{G}_S) = -\delta_S + r_1 + r_2.$$

In particular, for $\mathfrak{U} \subset_O \mathcal{G}_S$, the following holds:

$$\chi_{et}(\mathfrak{U}) = (\mathcal{G}_S : \mathfrak{U})\chi_{et}(\mathcal{G}_S).$$

By comparing the Euler-Poincaré characteristics, one obtains a criteria for $\mathcal{G}_S$ to be cohomologically étale:

Proposition 3.13. — *One has (for $\mathcal{G}_S \neq 1$):*

$$\chi_3(\mathcal{G}_S) \leq \chi_{et}(\mathcal{G}_S).$$

When $cd(\mathcal{G}_S) \leq 2$ and $\delta_{p,S} = 0$, the equality holds if and only if $\mathcal{G}_S$ is cohomologically étale.

Proof. — One knows:

$$\begin{array}{rcl}\chi_3(\mathcal{G}_S) & = & \chi_2(\mathcal{G}_S) - d_pH^3(\mathcal{G}_S, \mathbf{F}_p)\\ & \leq & (-\delta_S + r_1 + r_2) + \delta_{p,S} - d_pH^3(\mathcal{G}_S, \mathbb{F}_p)\\ & = & \chi_{et}(\mathcal{G}_S) + \delta_{p,S} - d_pH^3(\mathcal{G}_S, \mathbb{F}_p)\end{array}$$

Then $\chi_3(\mathcal{G}_S) \leq \chi_{et}(\mathcal{G}_S) + 1$ and $\chi_3(\mathcal{G}_S) \leq \chi_{et}(\mathcal{G}_S)$ with the eventual exception: $H^3(\mathcal{G}_S, \mathbb{F}_p) = 1$ and $\delta_{p,S} = 1$. In this last case, the cohomological dimension of $\mathcal{G}_S$ is at most 2 and this one of Spec $\mathcal{O}_{\mathbf{K}} \setminus S$ is at most 3.
Suppose: $\chi_2(\mathcal{G}_S) = \chi_{et}(\mathcal{G}_S) + 1$. Then for all open subgroup $\mathfrak{U}$ of $\mathcal{G}_S$, one has:

$$\begin{array}{rcl}\chi_3(\mathfrak{U}) & = & \chi_2(\mathfrak{U})\\ & = & [\mathcal{G}_S : \mathfrak{U}]\ \chi_2(\mathcal{G}_S)\\ & = & [\mathcal{G}_S : \mathfrak{U}]\,(\chi_{et}(\mathcal{G}_S) + 1)\\ & = & \chi_{et}(\mathfrak{U}) + [\mathcal{G}_S : \mathfrak{U}]\end{array}$$

which contradicts the inequality $\chi_3(\mathfrak{U}) \leq \chi_{et}(\mathfrak{U}) + 1$.
If $\delta_{p,S} = 0$, then $H^3_{et}(\text{Spec } \mathcal{O}_{\mathfrak{U}} \backslash S, \mathbb{F}_p) = 0$. Hence, when the cohomological dimension of $\mathcal{G}_S$ is at most 2, by comparing the dimensions, the morphism ϕ_2 is an isomorphism. $\diamondsuit$

The next proposition shows that, except in one situation, $\mathcal{G}_S$ can not be cohomologically étale and p-adic analytic:

Proposition 3.14. — *When $\delta_S \neq r_1 + r_2$, a cohomologically étale pro-p-group $\mathcal{G}_S$ is never p-adic analytic. In particular, when $S \cap S_p = \emptyset$.*

Proof. — Suppose that $\mathcal{G}_S$ is cohomologically étale. Then the cohomological dimension of $\mathcal{G}_S$ is at most 3 and, by proposition 3.12, $\chi_{et}(\mathcal{G}_S) = \delta_S - (r_1 + r_2)$. But, if $\mathcal{G}_S$ is p-adic analytic, then for any open uniformly powerful subgroup $\mathfrak{U}$ of $\mathcal{G}_S$, $\chi(\mathcal{G}_S) = [\mathcal{G}_S : \mathfrak{U}]\chi(\mathfrak{U}) = 0$. $\diamondsuit$

3.4. The mixed case. — To finish this section, we want to find some examples that illustrate proposition 3.13.
Let $S \subset S_p$ be a *non-empty* subset of S_p. Denote by φ_S the morphism:

$$\varphi_S : \mathbb{Z}_p \otimes E_{\mathbf{K}} \to \prod_{v \in S} U_v^1,$$

U_v^1 being the principal local units at v. When $S = S_p$, Leopoldt's conjecture predicts the injectivity of the map φ_S.
Suppose now that S is sufficently "large" i.e. such that $\mathbf{K}_S/\mathbf{K}$ contains a $\mathbb{Z}_p$-extension $\mathbf{K}_\infty/\mathbf{K}$. Put $\mathcal{H}_S := \text{Gal}(\mathbf{K}_S/\mathbf{K}_\infty)$ and $\mathcal{Y}_S := \mathcal{H}_S^{ab}$. The

pro-p-group $\mathcal{Y}_S$ is a $\mathbb{Z}_p[[T]]$-module. Let $(\rho_S, \mu_S, \lambda_S)$ be the Iwasawa invariants of $\mathcal{Y}_S$.
Then, let us give some explicit conditions which imply that the cohomological dimension of $\mathcal{G}_S$ is at most 2 (see for example, [**17**], [**16**]...):

Proposition 3.15. — *If φ_S is injective and if $\mu_S = 0$, then the cohomological dimension of $\mathcal{G}_S$ is at most* 2.

Now, when φ_S is injective, the value of $\chi_2(\mathcal{G}_S)$ is known (see for example [**16**], section 3):

Proposition 3.16. — *If φ_S is injective, then $\chi_2(\mathcal{G}_S) = \delta_S - (r_1 + r_2)$.*

Let us give some examples.

Corollary 3.17. — *(i) Let $\mathbf{K} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, where d is square free. Let p be a splitting prime in $\mathbf{K}/\mathbb{Q}$ and put $S = \{\mathfrak{p}\}$, where $\mathfrak{p}$ is a place of $\mathbf{K}$ above p. Then $\mathcal{G}_S$ is cohomologically étale.*
(ii) The examples of [**16**]. *Let $\mathbf{K} = \mathbb{Q}(\theta)$, where θ is a root of $x^4 + x^3 + 8x^2 - 4x + 2 = 0$. The signature of this field is $(0, 2)$. Put $p = 2$ and $S = \{(\theta)\}$. Here (θ) is a prime ideal above 2 with index of ramification over $\frac{1}{2}\mathbb{Q}$ equals to 3. Note that $\mathbf{K}$ has two primes above 2. Then the cohomological dimension of $\mathcal{G}_S$ is 2, $\chi_2(\mathcal{G}_S) = -1$, and $\mathcal{G}_S$ is cohomologically étale.*

Proof. — In these two cases, $\mu_S = 0$ and φ_S is injective. Hence $\chi_2(\mathcal{G}_S) = r_1 + r_2 - \delta_S = \chi_{et}(\mathcal{G}_S)$. The groups $\mathcal{G}_S$ being of cohomological dimension at most 2, one concludes with Proposition 3.13. $\lozenge$

4. $\mathcal{G}_S$ and wild ramification

For all this section, we assume that $\mathcal{G}_S$ is not trivial.

4.1. The setup. — Let S_p be the set of all places of $\mathbf{K}$ above p. Put $S' = S \cup S_p$. (A priori, S can contain some places above p.) Let $\mathbf{K}_{S'}$ be the maximal pro-p-extension of $\mathbf{K}$ unramified outside S'. Put $\mathcal{G}_{S'} = \mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K})$. For the next, we will consider only the situation where $p > 2$ or where $\mathbf{K}$ is totally imaginary. In this case, the cohomological dimension of $\mathcal{G}_{S'}$ is 1 or 2.
The group $\mathcal{G}_S$ is a quotient of $\mathcal{G}_{S'}$. Let $\mathcal{H}$ be the closed and normal subgroup of $\mathcal{G}_{S'}$ generated by the inertia groups of all places $v \in S' \backslash S$ in

$\mathbf{K}_{S'}/\mathbf{K}$. Then, $\mathcal{G}_{S'}/\mathcal{H} \simeq \mathcal{G}$. Put $\mathcal{X} = \mathcal{H}^{ab}$. Then, it is well-known that $\mathcal{X}$ is a compact $\mathbb{Z}_p[[\mathcal{G}_S]]$-module, where $\mathbb{Z}_p[[\mathcal{G}_S]] := \varprojlim_{\mathfrak{U}} \mathbb{Z}_p[\mathcal{G}_S/\mathfrak{U}]$.

The ring $\mathbb{Z}_p[[\mathcal{G}_S]]$ is local (hence if M is a $\mathbb{Z}_p[[\mathcal{G}_S]]$-module of finite type which is projective then M is $\mathbb{Z}_p[[\mathcal{G}_S]]$-free), and its projective dimension is $1 + cd(\mathcal{G}_S)$.
As an application of Nakayama's lemma, one obtains:

Proposition 4.1. — *The $\mathbb{Z}_p[[\mathcal{G}_S]$-module $\mathcal{X}$ is finitely generated.*

Proof. — The Hochschild-Serre spectrale sequence applied to the short exact sequence

$$1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}_{S'} \longrightarrow \mathcal{G}_S \longrightarrow 1,$$

shows

$$\cdots \longrightarrow H^1(\mathcal{G}_{S'}, \mathbb{F}_p) \longrightarrow H^1(\mathcal{H}, \mathbb{F}_p)^{\mathcal{G}_S} \longrightarrow H^2(\mathcal{G}_S, \mathbb{F}_p) \longrightarrow ...$$

As $H^1(\mathcal{G}_{S'}, \mathbb{F}_p)$ and $H^2(\mathcal{G}_S, \mathbb{F}_p)$ are finite, one deduces that $\mathcal{X}_{\mathcal{G}_S}/p = (H^1(H, \mathbb{F}_p)^{\mathcal{G}_S})^*$ is finite. Then by Nakayama's lemma, one has the result. $\diamondsuit$

For the same reason, one also has

Proposition 4.2. — *Let $1 \longrightarrow \mathcal{R} \longrightarrow F \longrightarrow \mathcal{G}_S \longrightarrow 1$, be a presentation of $\mathcal{G}_S$ by a finitely generated free pro-p-group F. Then $R^{ab} := R/[R, R]$ is a finitely generated $\mathbb{Z}_p[[\mathcal{G}_S]]$-module.*

Next recall the following:

Proposition 4.3. — *Leopoldt's conjecture is equivalent to the triviality of $H^2(\mathcal{G}_{S'}, \mathbb{Q}_p/\mathbb{Z}_p)$.*

Proof. — See for example [**18**]. $\diamondsuit$

Hence, one has a weak form of Leopoldt's conjecture:

Proposition 4.4. — *If one assumes Leopoldt's conjecture for all number fields and the prime number p,*

$$H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

Proof. — Let $\mathbf{F}/\mathbf{K}$ be a finite sub-extension of $\mathbf{K}_S/\mathbf{K}$. The extension $\mathbf{K}_{S'}/\mathbf{F}$ is also the maximal pro-p-extension of $\mathbf{F}$ unramified outside S'. As one assumes Leopoldt's conjecture, the group $H^2(\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{F})$ is trivial. Then, thanks to **[24]**, chapter 1,

$$H^2(\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K}_S), \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_{\mathbf{K}\subset\mathbf{F}\subset\mathbf{K}_S} H^2(\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{F}, \mathbb{Q}_p/\mathbb{Z}_p)) = 0.$$

$\lozenge$

4.2. The cohomological dimension of $\mathcal{G}_S$ and the structure of $\mathcal{X}$. — From now on, one assumes Leopoldt's conjecture for all number fields and the prime number p.
We can prove Theorem 1.2:

Proposition 4.5. — *1) If $\mathcal{X}$ is $\mathbb{Z}_p[[\mathcal{G}_S]]$-free, then $cd(\mathcal{G}_S) \leq 2$.*
2) If $cd(\mathcal{G}_S) \leq 2$, then the projective dimension of the $\mathbb{Z}_p[[\mathcal{G}_S]]$-module $\mathcal{X}$ is at most 1.

Proof. — Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W & \longrightarrow & F & \longrightarrow & \mathcal{G}_{S'} & \longrightarrow & 1 \\ & & & & \| & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{R} & \longrightarrow & F & \longrightarrow & \mathcal{G}_S & \longrightarrow & 1 \end{array}$$

be two presentations of the groups $\mathcal{G}_{S'}$ and $\mathcal{G}_S$ where F is the free pro-p-groups on $d_p(\mathcal{G}_{S'})$ generators. This diagram induces an injective map $W \hookrightarrow \mathcal{R}$, and by the snake lemma, one gets the exact sequence:

$$1 \longrightarrow W \longrightarrow \mathcal{R} \longrightarrow \mathcal{H} \longrightarrow 1.$$

From the homology version of the Hochschild-Serre spectral sequence, one gets:

$$0 = H^2(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)^* \longrightarrow (W^{ab})_{\mathcal{H}} \longrightarrow \mathcal{R}^{ab} \longrightarrow \mathcal{X} \longrightarrow 0,$$

thanks to proposition 4.4. Now recall a result of Brumer:

***Proposition 4.6* (Brumer,[4], corollary 5.3).** — *Let $\mathcal{G}$ be a finitely generated pro-p-group (with $H^2(\mathcal{G}, \mathbb{F}_p)$ finite). Let*

$$1 \longrightarrow R \longrightarrow F \longrightarrow \mathcal{G} \longrightarrow 1,$$

be a presentation of $\mathcal{G}$, where F is a free pro-p-group on d generators. Then, the group $\mathcal{G}$ is of cohomological dimension at most 2 if and only if R^{ab} is a free $\mathbb{Z}_p[[\mathcal{G}]]$-module of finite type.

As the cohomological dimension of $\mathcal{G}_{S'}$ is at most 2, the module W^{ab} is free over $\mathbb{Z}_p[[\mathcal{G}_{S'}]]$ and so $(W^{ab})_{\mathcal{H}}$ is free as $\mathbb{Z}_p[[\mathcal{G}_S]]$-module: one has $W^{ab} \simeq \mathbb{Z}_p[[\mathcal{G}_{S'}]]^s$ and $(W^{ab})_{\mathcal{H}} \simeq \mathbb{Z}_p[[\mathcal{G}_S]]^s$.
One finally obtains the exact sequence (Nguyen, **[18]**, theorem 1.4):

$$0 \longrightarrow \mathbb{Z}_p[[\mathcal{G}_S]]^s \longrightarrow \mathcal{R}^{ab} \longrightarrow \mathcal{X} \longrightarrow 0.$$

Suppose first that $\mathcal{X} \simeq \mathbb{Z}_p[[\mathcal{G}_S]]^r$. It comes:

$$1 \longrightarrow \mathbb{Z}_p[[\mathcal{G}_S]]^t \longrightarrow \mathcal{R} \longrightarrow \mathbb{Z}_p[[\mathcal{G}_S]]^r \longrightarrow 1$$

and then the module $\mathcal{R}^{ab}$ is $\mathbb{Z}_p[[\mathcal{G}_S]]$-free. By proposition 4.6, 1) holds.
Now suppose that $cd(\mathcal{G}_S) \leq 2$. Then by proposition 4.6, 2) holds. $\Diamond$

Thanks to C_2 (see section 2.5), one obtains immediately corollary 1.3.

Now, we give a criterion for $\mathcal{X}$ to be free. First, we have:

Lemma 4.7. — *If the pro-p-group $\mathcal{G}_S$ is of cohomological dimension at most 2, then $H^1(\mathcal{G}_S, \mathcal{X}^*) = 1$.*

Proof. — Let $\mathbf{F}/\mathbf{K}$ be a finite sub-extension of $\mathbf{K}_S/\mathbf{K}$. The cohomological dimension of the group $\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{F})$ is at most 2.
For $i \geq 3$, $H^i(\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{F}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and then

$$H^i(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_{\mathbf{K} \subset \mathbf{F} \subset \mathbf{K}_S} H^i(\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{F}, \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

As one assumes Leopoldt's conjecture, thanks to Proposition 4.4, we finally get $H^i(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, for $i > 1$. Then the Hochschild-Serre spectrale sequence $H^i(\mathcal{G}_{S'}/\mathcal{H}, H^j(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) \Longrightarrow H^{i+j}(\mathcal{G}_{S'}, \mathbb{Q}_p/\mathbb{Z}_p)$ allow us to obtain the following long exact sequence

$$\begin{array}{ccccc}
H^1(\mathcal{G}_S, \mathbb{Q}_p/\mathbb{Z}_p) & \hookrightarrow & H^1(\mathcal{G}_{S'}, \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{G}_S} \\
 & & & & \downarrow \\
H^1(\mathcal{G}_S, H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) & \longleftarrow & H^2(\mathcal{G}_{S'}, \mathbb{Q}_p/\mathbb{Z}_p) & \longleftarrow & H^2(\mathcal{G}_S, \mathbb{Q}_p/\mathbb{Z}_p) \\
\downarrow & & \| & & \\
H^3(\mathcal{G}_S, \mathbb{Q}_p/\mathbb{Z}_p) & & 1 & &
\end{array}$$

and the result follows. $\Diamond$

One obtains the main result of this section (that contains Theorem 1.2).

Proposition 4.8. — *Assume $cd(\mathcal{G}_S) \leq 2$. Suppose that the natural morphism* $\mathrm{Tor}({\mathcal{G}_{S'}}^{ab}) \to {\mathcal{G}_S}^{ab}$*, where* $\mathrm{Tor}({\mathcal{G}_{S'}}^{ab})$ *is the torsion part of* ${\mathcal{G}_{S'}}^{ab}$*, is injective. Then the* $\mathbb{Z}_p[[\mathcal{G}_S]]$*-module* $\mathcal{X}$ *is free.*
Moreover, when $\mathcal{X} \simeq \mathbb{Z}_p[[\mathcal{G}_S]]^r$*, then* $r = r_2 + \chi_2(\mathcal{G}_S)$.

Proof. — Since $\mathcal{X}$ is finitely generated as $\mathbb{Z}_p[[\mathcal{G}_S]]$-module, let

$$(1) \qquad 1 \longrightarrow N \longrightarrow \mathcal{F} \longrightarrow \mathcal{X} \longrightarrow 1,$$

be a minimal presentation of $\mathcal{X}$, where $\mathcal{F} = \mathbb{Z}_p[[\mathcal{G}_S]]^r$. Hence, $\mathcal{X}$ is free if and only if, $N = 0$, i.e. if and only if $N_{\mathcal{G}_S} = 0$ (by Nakayama's lemma).
The homology version of the spectral sequence $H^i(\mathcal{G}_{S'}/\mathcal{H}, H^j(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p)) \Longrightarrow H^{i+j}(\mathcal{G}_{S'}, \mathbb{Q}_p/\mathbb{Z}_p)$ gives

$$(2) \qquad H_2(\mathcal{G}_S, \mathbb{Z}_p) \hookrightarrow \mathcal{X}_{\mathcal{G}_S} \longrightarrow \mathcal{G}_{S'}^{ab} \twoheadrightarrow \mathcal{G}_S^{ab}.$$

Suppose that $cd(\mathcal{G}_S) \leq 2$. As $H_3(\mathcal{G}_S, \mathbb{Z}_p) = 0$, one has

$$H_2(\mathcal{G}_S, \mathbb{Z}_p) \xrightarrow{p} H_2(\mathcal{G}_S, \mathbb{Z}_p) \longrightarrow H_2(\mathcal{G}_S, \mathbb{F}_p) \longrightarrow \cdots .$$

Hence, $H_2(\mathcal{G}_S, \mathbb{Z}_p)$ is $\mathbb{Z}_p$-free.
If moreover the natural morphism ${\mathcal{G}_{S'}}^{ab} \to {\mathcal{G}_S}^{ab}$ restricted to the torsion part is injective, then, thanks to (2), $\mathcal{X}_{\mathcal{G}_S}$ is $\mathbb{Z}_p$-free.
Now passing to the $\mathcal{G}_S$-homology of the sequence (1), one obtains:

$$1 \longrightarrow H_1(\mathcal{G}_S, \mathcal{X}) \longrightarrow N_{\mathcal{G}_S} \longrightarrow \mathbb{Z}_p^r \longrightarrow \mathcal{X}_{\mathcal{G}_S} \longrightarrow 1$$

Then, the map $\varphi : \mathbb{Z}_p^r \to \mathcal{X}_{\mathcal{G}_S}$ is an isomorphism.
But, by lemma 4.7, $H_1(\mathcal{G}_S, \mathcal{X}) = 1$, and then $N_{\mathcal{G}_S} = 1$.

When $\mathcal{X} \simeq \mathbb{Z}_p[[\mathcal{G}_S]]^r$, then $H_1(\mathcal{H}, \mathbb{Z}_p)_{\mathcal{G}_S} \simeq \mathbb{Z}_p^r$. Hence taking the $\mathbb{Z}_p$-rank of the dual of the exact sequence of lemma 4.7, one obtains

$$\mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}_S^{ab} - \mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}_{S'}^{ab} + r - \mathrm{rk}_{\mathbb{Z}_p}H_2(\mathcal{G}_S, \mathbb{Z}_p) = 0$$

As $cd(\mathcal{G}_S) \leq 2$, the group $H_2(\mathcal{G}_S, \mathbb{Q}_p/\mathbb{Z}_p)$ is $\mathbb{Z}_p$-free and thus one has

$$\mathrm{rk}_{\mathbb{Z}_p}H_2(\mathcal{G}_S, \mathbb{Z}_p) - \mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}_S^{ab} = \chi_2(\mathcal{G}_S) - 1.$$

Then, by assuming Leopoldt's conjecture ($\mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}_{S'}^{ab} = r_2 + 1$), one finally obtains:

$$r = r_2 + \chi_2(\mathcal{G}_S).$$

◊

4.3. Examples. — From now on, for a place v of $\mathbf{K}$, one identifies $k_v^\times$ with its p-part $k_v^\times \otimes \mathbb{Z}_p$, k_v being the residue field of $\mathbf{K}_v$
Before giving some examples, one has to compute the kernel of the map: $\mathrm{Tor}(\mathcal{G}_{S'}^{ab}) \to \mathcal{G}_S^{ab}$. Here, $\mathrm{Tor}(\mathcal{G}_{S'}^{ab})$ is the $\mathbb{Z}_p$-torsion part of $\mathcal{G}_{S'}^{ab}$. To do this, we use a result of Gras [**7**].

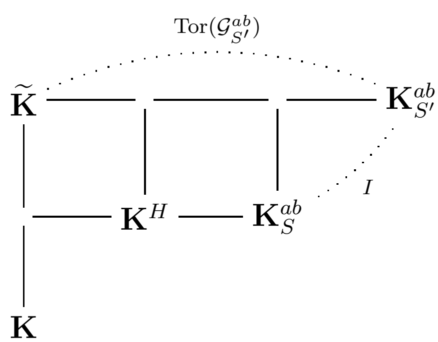
Proposition 4.9. — *Assume $S \cap S_p = \emptyset$. The map $\mathrm{Tor}(\mathcal{G}_{S'}^{ab}) \to \mathcal{G}_S^{ab}$ is injective if and only if*

$$\#\mathrm{Tor}\left(\oplus_{v|p} U_v^1 \oplus_{v\in S} k_v^\times/\varphi_{S'}(\mathbb{Z}_p \otimes E_{\mathbf{K}})\right) = \#\left(\oplus_{v\in S} k_v^\times/\varphi_S(\mathbb{Z}_p \otimes E_{\mathbf{K}})\right).$$

Proof. — Denote by I the subgroup of $\mathcal{G}_{S'}^{ab}$ generated by the inertia groups of all the places in S_p. Then: $1 \longrightarrow I \longrightarrow \mathcal{G}_{S'}^{ab} \longrightarrow \mathcal{G}_S^{ab} \longrightarrow 1$. To show that the map $\mathrm{Tor}(\mathcal{G}_{S'}^{ab}) \to \mathcal{G}_S^{ab}$ is injective is equivalent to showing that $I \cap \mathrm{Tor}(\mathcal{G}_{S'}^{ab}) = 1$. As $S \cap S_p = \emptyset$, the maximal abelian p-extension $\mathbf{K}_S^{ab}$ of $\mathbf{K}$ unramified outside S is finite, contains the p-Hilbert Class field $\mathbf{K}^H$ of $\mathbf{K}$ and moreover

$$\mathbf{K}_S^{ab} \cap \widetilde{\mathbf{K}} = \mathbf{K}^H \cap \widetilde{\mathbf{K}},$$

where $\widetilde{\mathbf{K}}$ is the compositum of all $\mathbb{Z}_p$-extension of $\mathbf{K}$.



Here $\mathbf{K}_{S'}^{ab}$ is the maximal abelian pro-p-extension of $\mathbf{K}$, unramified outside S'. Of course: $\mathcal{G}_S^{ab} = \mathrm{Gal}(\mathbf{K}_S^{ab}/\mathbf{K})$ and $\mathcal{G}_{S'}^{ab} = \mathrm{Gal}(\mathbf{K}_{S'}^{ab}/\mathbf{K})$.

Hence:

$$\begin{aligned} I \cap \mathrm{Tor}(\mathcal{G}_{S'}^{ab}) &= \mathrm{Tor}(I) \cap \mathrm{Tor}(\mathcal{G}_{S'}^{ab}) \\ &= [\mathbf{K}_{S'}^{ab} : \mathbf{K}_S^{ab}\widetilde{\mathbf{K}}] \\ &= [\mathbf{K}_{S'}^{ab} : \mathbf{K}^H\widetilde{\mathbf{K}}]/[\mathbf{K}_S^{ab} : \mathbf{K}^H] \end{aligned}$$

By Theorem 2.6, chapter III of [**7**], the first index $[\mathbf{K}_{S'}^{ab} : \mathbf{K}^H\widetilde{\mathbf{K}}]$ is exactly $\#\mathrm{Tor}\left(\oplus_{v|p} U_v^1 \oplus_{v\in S} k_v^\times/\varphi_{S'}(\mathbb{Z}_p \otimes E_{\mathbf{K}})\right)$, and the second index is well-konwn: it compares the order of a ray p-class group to the order of the p-class group. Here it is: $\#\left(\oplus_{v\in S} k_v^\times/\varphi_S(\mathbb{Z}_p\otimes E_{\mathbf{K}})\right)$. $\Diamond$

Corollary 4.10. — *Let us conserve the notations of section 4 ($p>2$). Let S be a finite set of places of $\mathbf{K}$, $S\cap S_p=\emptyset$, and such that the cohomological dimension of $\mathcal{G}_S$ is 2. Put $\mathcal{X}=\mathcal{H}^{ab}$, where $\mathcal{H}=\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K}_S)$. Then, in the two following situations, $\mathcal{X}$ is $\mathbb{Z}_p[[\mathcal{G}_S]]$-free.*

(i) $\mathbf{K}=\mathbb{Q}$.

(ii) $\mathbf{K}/\mathbb{Q}$ *is an imaginary quadratic field (If $p=3$, $\mathbf{K}_{v_3}$ is different from $\mathbb{Q}_3(j)$, for $v_3|3$).*

Proof. — In these situations the group $\mathbb{Z}_p\otimes E_{\mathbf{K}}$ is trivial. Hence, thanks to Proposition 4.9 the result is clear if we note that for all $v|p$, U_v^1 has no non-trivial p-torsion. $\Diamond$

Example 4.11. — We recall that we assume Lepoldt's conjecture.
(i) The examples of Labute (and of Schmidt) allow us to illustrate the previous corollary. For example, let's take $p=3$ and $S=\{7,19,61,163\}$. Then Labute proved that for this case, $cd(\mathcal{G}_S)=2$. Put $S'=S\cup\{3\}$, $\mathcal{H}=\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K}_S)$ and $\mathcal{X}:=\mathcal{H}^{ab}$. Then $\mathcal{X}$ is a $\mathbb{Z}_p[[\mathcal{G}_S]]$-free module of rank 1.
(ii) Thanks to the work of Vogel [**26**], one can give some examples with base field $\mathbf{K}$ an imaginary quadratic field. Let $\mathbf{K}=\mathbb{Q}(i)$. Take $p=3$ and let S be the set of all places above 229 and 241. Then Vogel proved in this case that $cd(\mathcal{G}_S)=2$. Put $S'=S\cup\{3\}$, $\mathcal{H}=\mathrm{Gal}(\mathbf{K}_{S'}/\mathbf{K}_S)$ and $\mathcal{X}:=\mathcal{H}^{ab}$. Then $\mathcal{X}$ is a $\mathbb{Z}_p[[\mathcal{G}_S]]$-free module of rank 2.

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