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Tamely Ramified Towers and Discriminant Bounds for Number Fields—II

FARSHID HAJIR[†§] AND CHRISTIAN MAIRE[‡¶]

[†] *Department of Mathematics, California State University, San Marcos, CA 92096, U.S.A.*
[‡] *Laboratoire A2X, Université Bordeaux I, Cours de la Libération, 33405 Talence Cedex, France*

The root discriminant of a number field of degree n is the nth root of the absolute value of its discriminant. Let $R_0(2m)$ be the minimal root discriminant for totally complex number fields of degree $2m$, and put $\alpha_0 = \liminf_m R_0(2m)$. Define $R_1(m)$ to be the minimal root discriminant of totally real number fields of degree m and put $\alpha_1 = \liminf_m R_1(m)$. Assuming the Generalized Riemann Hypothesis, $\alpha_0 \geq 8\pi e^{\gamma} \approx 44.7$, and, $\alpha_1 \geq 8\pi e^{\gamma+\pi/2} \approx 215.3$. By constructing number fields of degree 12 with suitable properties, we give the best known upper estimates for α_0 and α_1: $\alpha_0 < 82.2, \alpha_1 < 954.3$.

1. Introduction

Let $I = \mathbb{Q} \cap [0, 1]$ be the rational unit interval. For a number field K of signature (r_1, r_2) and degree $n = r_1 + 2r_2$, let $\tau(K) = r_1/n \in I$ be the proportion of its embeddings which are real. Let us call $\tau(K)$ the *infinity type* of K. Number fields of degree $n \geq 1$ and infinity type $t \in I$ exist if and only if nt and $n(1-t)/2$ are integral (see, for example, Ankeny *et al.*, 1956). For such n and t, let $R_t(n)$ be the minimal root discriminant for number fields of degree n and infinity type t. (The root discriminant rd_K of K is defined by $rd_K = |d_K|^{1/n}$ where d_K is the discriminant of K.) Define a function α on I by

$$\alpha_t = \liminf_{n\to\infty} R_t(n),$$

with n tending to infinity under the condition that nt and $n(1-t)/2$ are integral.

Using his "geometry of numbers," Minkowski proved in 1891 that there exist constants $A, B > 1$ such that

$$\alpha_t \geq A^{1-t} B^t. \tag{1}$$

Minkowski's values $A = \pi e^2/4 \approx 5.8\ldots$ and $B = e^2 \approx 7.3\ldots$ were steadily improved over the years. The best current asymptotic bounds, dating from the mid-1970s, stem from variations on an analytic method of Stark by Odlyzko and Serre and give the values $A = 4\pi e^{\gamma} \approx 22.3, B = 4\pi e^{1+\gamma} \approx 60.8$, and on GRH, $A = 8\pi e^{\gamma} \approx 44.7, B = 8\pi e^{\gamma+\pi/2} \approx$

[§] Current address: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, U.S.A. E-mails: fhajir@csusm.edu; fhajir@unc.edu
[¶] Current address: DMA-Ecole Polytechnique Federale de Lausanne, 1015 Lausanne, Suisse.
E-mail: maire@math.u-bordeaux.fr

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215.3, where $\gamma = 0.577\ldots$ is Euler's constant (see the survey article of Odlyzko (1990) and references therein).

It is an important problem to determine the best possible constants A and B in (1). To this end, upper bounds for α can be given, thanks to the theory of class field towers; naturally, one is most interested in estimating α_t at $t = 0, 1$ (totally complex, totally real fields, respectively) since $A \le \alpha_0, B \le \alpha_1$. Traditionally, one uses unramified (Hilbert) class field towers, as in Martinet (1978), where the estimates $\alpha_0 < 92.3$ and $\alpha_1 < 1058.6$ were established. In Hajir and Maire (2001), a variation on Martinet's method using tamely ramified ray class field towers and base fields with large Galois closure was introduced and used to improve the first estimate to $\alpha_0 < 83.9$.

Starting with the PARI database for number fields of small discriminant and degree up to seven available from Batut *et al.* (2001), we implemented in PARI a search for quadratic extensions of these fields, looking at only those having the smallest relative discriminant of odd norm with a specified number of prime factors. The parameters were chosen so that according to the best known genus theory and Golod–Shafarevich bounds, the quadratic extensions admit infinite 2-extensions ramified at a small specified set of primes, giving upper bounds for α. As a result of this machine search, we prove here that $\alpha_1 < 954.3$ and $\alpha_0 < 82.2$. We remark that we now have several dozen towers improving Martinet's record for totally complex fields, but the totally real example we give here is the only one we have so far been able to find which improves Martinet's 22-year old totally real bound.

2. Golod–Shafarevich for Tame Towers

We recall briefly the ray class field construction of infinite towers with bounded root discriminant. First we introduce some notation.

In this section, we fix an arbitrary prime ℓ. In the next section, we will work exclusively with $\ell = 2$. For a finitely generated pro-ℓ group G, we let $d(G) = \dim_{\mathbb{F}_\ell} H^1(G, \mathbb{F}_\ell)$, $r(G) = \dim_{\mathbb{F}_\ell} H^2(G, \mathbb{F}_\ell)$ be its generator and relation rank, respectively. For a number field K, and a finite set T of prime ideals of K, we say that T is "away from ℓ" if no prime in T is a divisor of ℓ. For such a T, let $\mathfrak{m}_T = \prod_{\mathcal{P} \in T} \mathcal{P}$ be the corresponding modulus and define $rd_{K,T} = rd_K (\mathbb{N}_{K/\mathbb{Q}} \mathfrak{m}_T)^{1/[K:\mathbb{Q}]}$. We let $\mathrm{Cl}(K), \mathrm{Cl}_T(K)$ be the ideal class group and ray class group modulo $\mathfrak{m}_T$ of K, respectively, and write $\rho_K, \rho_{K,T}$ for their respective ℓ-ranks. We write E_K for the unit group of K. Let K_T be the maximal ℓ-extension of K unramified outside T (in a fixed algebraic closure of K), and put $G_{K,T} = \mathrm{Gal}(K_T/K)$ for its Galois group. Note that, by the Burnside Basis Theorem, and class field theory, $d(G_{K,T}) = \rho_{K,T}$. When T is empty, K_T/K is the Hilbert class field tower of K. We say that an ideal of K is odd if its absolute norm is odd. We let

$$\theta_{K,T} = \begin{cases} 1 & \text{if } T \text{ is empty and } K \text{ contains a primitive } \ell\text{th root of unity} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *Suppose K is a number field with signature (r_1, r_2). Let T be a finite set of prime ideals of K away from ℓ.*

(1) *If*

$$\rho_{K,T} \ge 2 + 2\sqrt{r_1 + r_2 + \theta_{K,T}},$$

then $G_{K,T}$ is infinite.

(2) *If F is a finite extension of K contained in K_T, then $rd_F < rd_{K,T}$.*

Proof. (1) This is immediate by combining Theorems 1 and 5 in Shafarevich (1964) (or Satz 11.5 and 11.8 in Koch, 1970), which give an estimate on the minimal number of relations of $G_{K,T}$ with the Golod–Shafarevich theorem (Golod and Shafarevich, 1964) on the number of relations of a finite ℓ-group (as improved by Vinberg and Gaschütz, see Roquette, 1980, Theorem 10). (2) This is a simple calculation. For more details see Hajir and Maire (2001), Lemma 5. □

Theorem 2. (Genus Theory) *Suppose K/k is a Galois extension of degree ℓ. Suppose t places of k ramify in K. Then*

$$\rho_K \geq t - 1 - \dim_{\mathbb{F}_\ell}(E_k/E_k \cap \mathbb{N}_{K/k}U_K)$$

where U_K is the group of idèle units of K.

Proof. A proof can be found, for example, in Schoof (1986). □

We will single out a special case of the above Theorem which will be useful in our totally real example.

Corollary 3. *Let $\ell = 2$. Consider a number field k with signature (r_1, r_2) such that -1 is not a square in k, and a quadratic extension K/k unramified at the infinite places of k. Suppose further that all t primes of k ramified in K have absolute norm $\equiv 1 \bmod 4$. Then*

$$\rho_K \geq t - r_1 - r_2.$$

Proof. It suffices to show that $-1 \in \mathbb{N}_{K/k}U_K$. Since in an unramified extension of local fields, every unit is a norm, we need only consider the localizations at each of the ramified primes in K/k. Hensel's lemma reduces this to checking that -1 is a square in the corresponding residue field, which follows from the assumption that this residue field has cardinality $\equiv 1 \bmod 4$. □

3. Infinite Towers with Small Root Discriminant

Throughout this section, we let $\ell = 2$. We construct some infinite tamely ramified 2-towers with small root discriminant.

Since $\ell = 2$, every number field contains the ℓth roots of unity. Upon examination of the Golod–Shafarevich bound (Theorem 1), it is clear that the use of tame, as opposed to unramified, towers is most advantageous when

$$\lceil 2\sqrt{r_1 + r_2}\rceil < \lceil 2\sqrt{r_1 + r_2 + 1}\rceil,$$

where $\lceil x \rceil$ is the least integer greater than or equal to x. In this case, the bound for obtaining an infinite tame tower is less by one than the bound for obtaining an infinite unramified tower. It pays, therefore, to examine cases where the number of infinite places of K, namely $r_1 + r_2$, is of the form n^2 or $(n^2 - 1)/4$, for a natural number n. For example, in Hajir and Maire (2001), we made use of degree eight totally complex K, where $r_1 + r_2 = 4$ is a perfect square. For both of our key examples here, we take K of degree 12: in the totally real case, $r_1 + r_2 = 12 = (7^2 - 1)/4$ and in the totally complex case, $r_1 + r_2 = 6 = (5^2 - 1)/4$.

An outline of the construction is as follows. We used the PARI number field tables (Batut *et al.*, 2001) (which stop at degree seven) to find a suitable base field k, then used PARI to locate prime ideals of small norm whose product has a generator η which is a square mod $4\mathcal{O}_k$ and has the right signature at infinity; these conditions ensure that $K = k(\sqrt{\eta})$ is unramified at 2 and has the desired signature. In the totally real case, we take 15 ramifying primes all of norm congruent to 1 mod 4 and apply Corollary 3 to obtain $\rho_K \geq 9$. Theorem 1 then says that K_T/K is infinite for *any non-empty* set T of odd primes of K. In the totally complex case, the best results were found for k with signature $(4, 1)$. In that case, four infinite and eight finite places of k are ramified in K, so $\rho_K \geq 6$ by Theorem 2. We then take a set T of primes of K such that $\rho_{K,T} \geq 7$, ensuring that K has an infinite T-ramified 2-tower by Theorem 1.

The number field arithmetic which is at the heart of our construction takes place in degree six number fields; computer packages such as PARI (Batut *et al.*, 2001) and KANT (Pohst, 2001) make it easy to carry out these calculations. We also give approximate roots $\xi^{(j)}, j = 1, \ldots, 6$, accurate to 25 decimal places (which more than suffices) of a monic integral defining polynomial f for k, using which our claims can be verified by using an ordinary calculator.

3.1. TOTALLY REAL CASE

We now prove $\alpha_1 < 954.3$ by constructing a degree 12 totally real field with small discriminant and 2-class group of large rank.

Let $k = \mathbb{Q}(\xi)$ where ξ is a root of $f = x^6 - x^5 - 10x^4 + 4x^3 + 29x^2 + 3x - 13$. The roots of f are

$$\begin{aligned}
\xi^{(1)} &= -1.883\,173\,014\,899\,617\,292\,140\,105\,726\ldots \\
\xi^{(2)} &= -1.850\,277\,939\,491\,434\,625\,515\,659\,524\ldots \\
\xi^{(3)} &= -0.933\,888\,901\,249\,385\,718\,006\,746\,6209\ldots \\
\xi^{(4)} &= 0.636\,193\,411\,182\,150\,231\,090\,095\,9583\ldots \\
\xi^{(5)} &= 2.295\,319\,807\,404\,063\,434\,093\,464\,652\ldots \\
\xi^{(6)} &= 2.735\,826\,637\,054\,223\,970\,478\,951\,259\ldots.
\end{aligned}$$

Thus, k has signature $(6, 0)$. The prime factorization of the discriminant of f is $d_f = 7^4 \cdot 13 \cdot 113$. One can check that k is a quadratic extension of the maximal real subfield of the field of 7th roots of unity (for example, if $h(t) = -t^5 + 3t^4 + 4t^3 - 13t^2 - 3t + 7$, $h(\xi^{(j)}) = 2\cos(2\pi j/7)$ for $j = 1, 2, 3$). Since the latter field has discriminant 7^2, d_k is divisible by 7^4, hence $d_f = d_k$ and $\mathcal{O}_k = \mathbb{Z}[\xi]$. The (wide) class number of k is 1. The unit group of k is generated by $\{\xi^5 - 3\xi^4 - 4\xi^3 + 13\xi^2 + 3\xi - 8, \xi^5 - 3\xi^4 - 5\xi^3 + 14\xi^2 + 7\xi - 9, \xi^2 - \xi - 5, 3\xi^5 - 9\xi^4 - 13\xi^3 + 40\xi^2 + 12\xi - 22, \xi^5 - 3\xi^4 - 4\xi^3 + 14\xi^2 + 2\xi - 11, -1\}$.

In order to locate some prime ideals of small norm, we factor f modulo some small primes:

$$\begin{aligned}
f(x) &\equiv (x^2 + 2x + 2)^3 \bmod 7 \\
f(x) &\equiv x(x + 4)^2(x - 6)(x - 2)(x - 1) \bmod 13 \\
f(x) &\equiv (x + 7)(x + 11)(x + 12)(x - 9)(x^2 + 7x - 6) \bmod 29
\end{aligned}$$

$$f(x) \equiv (x+16)(x-20)(x-8)(x-3)(x^2+14x+1) \bmod 41$$
$$f(x) \equiv (x-35)(x-34)(x-32)(x-22)(x^2+25x-33) \bmod 97.$$

Generators for some of the prime ideals evident in the above factorizations are listed in the table below; here, $\pi_r = a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$ generates a prime ideal $\pi_r\mathcal{O}_k$ of absolute norm r. The second column matches the primes listed below with the ones above by giving $s(\xi)$, a degree one or two expression in ξ which is an $\mathcal{O}_k$-multiple of π_r. We also give h_{π_r}, the minimal polynomial of π_r so that the reader can verify that each of the algebraic integers listed has the claimed norm. (Instead of writing out the polynomial in full, we give a list of its coefficients in order of descending powers of x. For example the coefficient list $1, 2, 3, 4, 5, 6, 7$ represents the polynomial $x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7$.)

π	$s(\xi)$	$a_5, a_4, a_3, a_2, a_1, a_0$	h_π
π_{13}	$\xi+4$	$0, -1, 0, 5, 1, -2$	$1, 15, -50, 23, 51, -52, 13$
π'_{13}	$\xi-6$	$3, -8, -15, 35, 19, -18$	$1, -2, -12, 11, 44, 12, -13$
π''_{13}	$\xi-2$	$-1, 2, 5, -9, -5, 6$	$1, 16, 28, -23, -42, 8, 13$
π'''_{13}	$\xi-1$	$-2, 6, 9, -26, -11, 11$	$1, 8, 14, -30, -98, -47, 13$
π''''_{13}	ξ	$2, -6, -9, 26, 10, -13$	$1, 5, -5, -24, 16, 19, -13$
π_{29}	$\xi+7$	$-2, 5, 11, -22, -16, 10$	$1, 6, -8, -51, 44, 66, -29$
π'_{29}	$\xi+12$	$1, -2, -5, 8, 5, -3$	$1, -13, 35, -2, -91, 100, -29$
π''_{29}	$\xi-9$	$1, -3, -5, 14, 8, -10$	$1, 9, 21, -16, -105, -104, -29$
π'''_{29}	$\xi+11$	$-1, 3, 4, -13, -4, 7$	$1, 3, -16, -30, 57, 27, -29$
π_{41}	$\xi+16$	$0, 0, 1, 0, -5, -2$	$1, -2, -25, 54, 25, -93, 41$
π'_{41}	$\xi-8$	$-3, 8, 15, -35, -19, 19$	$1, -4, -7, 37, 0, -81, 41$
π''_{41}	$\xi-3$	$-1, 3, 5, -14, -6, 9$	$1, -5, -7, 40, -14, -55, 41$
π'''_{41}	$\xi-20$	$5, -14, -24, 62, 29, -35$	$1, 6, -1, -58, -89, 3, 41$
π_{49}	$\xi^2+2\xi+2$	$0, 0, -1, 1, 4, -1$	$0, 0, 0, 1, -7, 7$
π_{97}	$\xi-35$	$-5, 14, 23, -54, -31$	$1, 36, -1079, -8776, -14103, 6227, 97$
π'_{97}	$\xi-32$	$-3, 1, 33, 2, -90, -41$	$1, 71, -1667, -4884, -3708, -367, 97$

We let

$$\begin{aligned}\eta &= \pi_{13}\pi'_{13}\pi''_{13}\pi'''_{13}\pi''''_{13}\pi_{29}\pi'_{29}\pi''_{29}\pi'''_{29}\pi_{41}\pi'_{41}\pi''_{41}\pi'''_{41}\pi_{49}\pi_{97} \\ &= -2993\xi^5 + 7230\xi^4 + 18937\xi^3 - 38788\xi^2 - 32096\xi + 44590 \in \mathcal{O}_k.\end{aligned}$$

One checks easily that η is totally positive. Its minimal polynomial is $g(y) = y^6 - 56966y^5 + 959048181y^4 - 5946482981439y^3 + 14419821937918124y^2 - 12705425979835529941y + 3527053069602078368989$. We let $K = k(\sqrt{\eta})$, a totally real field of degree 12. A defining polynomial for K is $g(y^2)$. We note that η is congruent to a square modulo $4\mathcal{O}_K$; explicitly, $\eta = \beta^2 + 4\gamma$ with $\beta = \xi^5 + \xi^3 + \xi^2 + \xi + 1$ and $\gamma = -811\xi^5 + 1617\xi^4 + 5013\xi^3 - 8847\xi^2 -$

$8002\xi + 10754$. Thus, the relative discriminant $\mathfrak{D}_{K/k}$ is simply (η), and K/k is ramified at the 15 primes dividing η and nowhere else. The root discriminant of K is

$$rd_K = rd_k(\mathbb{N}_{K/k}\mathfrak{D}_{K/k})^{1/12} = (7^4 \cdot 13 \cdot 113)^{1/6}(7^2 \cdot 13^5 \cdot 29^4 \cdot 41^4 \cdot 97)^{1/12} = 770.643\ldots.$$

Now let us estimate the 2-class rank of K. By Corollary 3, since all 15 places ramifiying in K/k have norm congruent to 1 mod 4, the 2-rank of the ideal class group of K is at least $15 - 6 = 9 > 2 + 2\sqrt{12}$. By Theorem 1, K admits an infinite T-tamely ramified 2-tower, where T consists of any odd prime of K at all; for instance we can take T to consist of any one of the five primes of absolute norm 13. By Theorem 1, the root discriminant of the fields in this tower are bounded by $rd_{K,T} = 13^{1/12} rd_K$, so

$$\alpha_1 \le rd_{K,T} = 7^{5/6}13^{2/3}(29 \cdot 41)^{1/3}97^{1/12}113^{1/6} = 954.293\ldots.$$

Remarks.

1. A standard calculation of the class group, using PARI, revealed that $\mathrm{Cl}_K = (\mathbb{Z}/2\mathbb{Z})^9$, confirming our lower bound of nine for the 2-rank of the class group. Not surprisingly, the ray class group modulo any one of the primes of norm 13 was found to be isomorphic to the ideal class group. We thank Bill Allombert for performing this PARI calculation on PARI 2.1.0; this new release of PARI contains an improved algorithm for calculating class groups of number fields with many subfields which allowed the calculation to be made in less than 1 h; previous attempts with earlier versions of PARI had not succeeded.
2. One can take, instead of η, the number $\eta' = \eta \cdot \pi'_{97}/\pi_{97}$. Again, η' is totally positive, is congruent to a square modulo $4\mathcal{O}_k$ and $K' = k(\sqrt{\eta'})$ has the required properties.
3. We expect that K in fact has an infinite *unramified* 2-tower, which would give $\alpha_1 < 770.7$, but are unable to prove this. See Section 5 of Hajir and Maire (2001).

3.2. TOTALLY COMPLEX CASE

We now produce an infinite tower of number fields with root discriminant bounded by $82.100\ldots$.

Let $k = \mathbb{Q}(\xi)$ where ξ is a root of $f = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$. The prime factorization of the discriminant of f is $d_f = -23 \cdot 35509$; thus, $d_f = d_k$ is also the discriminant of k, and $\mathcal{O}_k = \mathbb{Z}[\xi]$. The roots of f are

$$\xi^{(1)} = -0.76166245384468100791784609 70\ldots$$
$$\xi^{(2)} = -0.69953796284372129907057255 39\ldots$$
$$\xi^{(3)} = 0.29522571317729963668939709 80\ldots$$
$$\xi^{(4)} = 1.83015782341636731046020011 5\ldots$$
$$\xi^{(5)} = -0.33209155995263232008058928 12\ldots + 1.83394227605082629317069415 2\ldots i$$
$$\xi^{(6)} = -0.33209155995263232008058928 12\ldots - 1.83394227605082629317069415 2\ldots i.$$

Thus, k has signature $(4, 1)$. The narrow class number of k is 1. The unit group of k is generated by $\{\xi, 4\xi^5 - 3\xi^4 + 6\xi^3 - 20\xi^2 - 13\xi + 6, 6\xi^5 - 4\xi^4 + 9\xi^3 - 30\xi^2 - 21\xi + 8, \xi^5 - \xi^4 + 2\xi^3 - 6\xi^2 - \xi + 1, -1\}$.

Generators for some $\mathcal{O}_k$-ideals of small norm are listed in the table below where, as before, $\pi_r = a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$ generates a prime ideal $\pi_r\mathcal{O}_k$ of norm

r and the coefficients of h_{π_r}, the minimal polynomial of π_r, are listed in descending powers.

π_r	$a_5, a_4, a_3, a_2, a_1, a_0$	h_{π_r}
π_3	$-6, 4, -9, 30, 21, -7$	$1, 0, -5, 2, 5, -5, 3$
π_7	$-9, 6, -13, 44, 31, -12$	$1, 1, -29, 98, 624, -449, -7$
π_{13}	$-7, 5, -11, 36, 23, -9$	$1, 3, -4, -24, -23, 7, 13$
π_{19}	$5, -4, 8, -26, -15, 6$	$1, 11, 50, 120, 151, 89, 19$
π'_{19}	$5, -3, 7, -24, -20, 6$	$1, -3, -10, 13, 29, -8, -19$
π_{23}	$-5, 4, -8, 26, 15, -9$	$1, 7, 20, 30, 16, -20, -23$
π'_{23}	$6, -4, 9, -30, -22, 6$	$1, 6, 11, 0, -30, -46, -23$
π_{29}	$11, -8, 17, -56, -35, 16$	$1, -7, 3, 52, -82, 55, -29$
π_{31}	$7, -5, 11, -36, -22, 7$	$1, 9, 22, 13, -15, -38, -31$

The fact that $19\mathcal{O}_K$ has two prime factors of residue degree 1 can be seen, for instance, from the factorization of f over $\mathbb{F}_{19}$: $f(x) \equiv (x+7)(x-2)(x^4+14x^3+2x^2+11x+4) \bmod 19$. Similarly, f factors over $\mathbb{F}_{23}$ as $f(x) \equiv (x+10)^2(x-5)(x^3+8x^2+19x+4) \bmod 23$. To see that the pairs π_{19}, π'_{19} and π_{23}, π'_{23} generate different prime ideals, one can check that the minimal polynomials of π_{19}/π'_{19} and π_{23}/π'_{23} are not integral.

The element $\eta = -671\xi^5 + 467\xi^4 - 994\xi^3 + 3360\xi^2 + 2314\xi - 961 \in \mathcal{O}_k$ is totally negative. Its minimal polynomial is $g(y) = y^6 + 339y^5 - 19752y^4 - 2188735y^3 + 284236829y^2 + 4401349506y + 15622982921$. The ideal (η) factors into eight prime ideals of $\mathcal{O}_k$; in fact, one can check that $\eta = \pi_7\pi_{13}\pi_{19}\pi'_{19}\pi_{23}\pi'_{23}\pi_{29}\pi_{31}$. We let $K = k(\sqrt{\eta})$, a totally complex field of degree 12. A defining polynomial for K is $g(y^2)$. We note that η is congruent to a square modulo $4\mathcal{O}_K$; explicitly, $\eta = \beta^2 + 4\gamma$ with $\beta = \xi^5 + \xi^4 + \xi^3 + 1$ and $\gamma = -173\xi^5 + 112\xi^4 - 270\xi^3 + 815\xi^2 + 576\xi - 237$. Thus, the relative discriminant $\mathfrak{D}_{K/k}$ is simply (η), and K/k is ramified at the infinite places, at the eight primes dividing η, and nowhere else. The root discriminant of K is

$$rd_K = rd_k(\mathbb{N}_{K/k}\mathfrak{D}_K/k)^{1/12} = (23 \cdot 35509)^{1/6}(7 \cdot 13 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31)^{1/12} = 68.363\ldots.$$

Now k admits a quadratic extension in which the only ramified finite primes are (π_3) and (π_{19}). To see this, we note that $\pi_3\pi_{19} = 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 14 = \rho^2 + 4\sigma$, where $\rho = \xi^5 + \xi^3 + \xi^2 + 1$, and $\sigma = 2\xi^5 - 8\xi^4 - 14\xi^3 - 28\xi^2 - 9\xi + 5$. Thus, $k(\sqrt{\pi_3\pi_{19}})/k$ is ramified at $\pi_3\mathcal{O}_k$ and at $\pi_{19}\mathcal{O}_k$ but at no other finite prime. Now since $\pi_{19}\mathcal{O}_k$ already ramifies in K, $K(\sqrt{\pi_3\pi_{19}})/K$ is a quadratic extension unramified outside T where T is the set of primes of K dividing $\pi_3\mathcal{O}_K$ (it is not necessary, but one can check that T has one element, $\mathfrak{P}_9$, a prime of absolute norm 9). Note that since $\pi_3\mathcal{O}_k$ does not ramify in K, $K(\sqrt{\pi_3\pi_{19}})/K$ is actually ramified at $\mathfrak{P}_9$.

Now let us estimate class ranks. By Theorem 2, since eight finite and four infinite places ramifiy in K/k, the 2-rank of the ideal class group of K is at least 6. Moreover, by the previous paragraph, the 2-rank of $\mathrm{Cl}_T(K)$ is at least one more than that of $\mathrm{Cl}(K)$, so 2-rank of $\mathrm{Cl}_T(K) \geq 6 + 1 \geq 2 + 2\sqrt{6}$.

By Theorem 1, K admits an infinite T-tamely ramified 2-tower, and the root discriminant of the fields in this tower are bounded by $rd_{K,T} = 9^{1/12} rd_K$, so

$$\alpha_0 \leq rd_{K,T} = 23^{1/3}(3 \cdot 19 \cdot 35509)^{1/6}(7 \cdot 13 \cdot 29 \cdot 31)^{1/12} = 82.100\ldots.$$

Remarks. We found one other example almost as good as the one above; we sketch it briefly, omitting the details which are similar. We take $k = \mathbb{Q}(\xi)$ where ξ is a root of $f = x^6 - 2x^5 - 2x^4 + 7x^3 - 2x^2 - 4x + 1$, a field with signature $(4, 1)$ and prime discriminant $d_k = -658403$. The element $\eta = -33\xi^5 - 9\xi^4 + 166\xi^3 - 70\xi^2 - 100\xi + 17$, is totally negative, is congruent to a square modulo $4\mathcal{O}_k$, generates an ideal divisible by eight primes and has norm $13^3 \cdot 17 \cdot 23^2 \cdot 37^2$. Its minimal polynomial is $g(y) = y^6 + 302y^5 - 20535y^4 + 11631690y^3 + 511386746y^2 + 6643287248y + 27048183149$. The field $K = \mathbb{Q}(\sqrt{\eta})$ has 2-class rank at least 6 and root discriminant $rd_K = 658403^{1/6} \cdot (13^3 \cdot 17 \cdot 23^2 \cdot 37^2)^{1/12} = 69.032\ldots$. The number $\pi_3 = -\xi^4 + \xi^3 + \xi^2 - \xi$ generates a prime of norm 3 in $\mathcal{O}_k$, and is congruent to a square $\bmod 4\mathcal{O}_k$. Thus $K(\sqrt{\pi_3})/K$ is ramified only at T where T is the set of primes of K above $\pi_3\mathcal{O}_k$ (it has cardinality 2). Thus, $\mathrm{Cl}_T(K)$ has 2-rank at least 7 and K_T/K is infinite. We have $rd_{K,T} = 9^{1/12} rd_K = 82.903\ldots$.

In closing, we briefly mention that our totally complex example achieves records for two other invariants. The Ihara–Tsfasman–Vladut deficiency δ for this example satisfies $\delta \leq 1 - \frac{\gamma + \log(8\pi)}{\log(rd_{K,T})} = 0.137\ldots$. The integer lattice of the fields in our tower (under the usual trace norm) have arbitrarily large dimension and asymptotic packing density $\frac{1}{n_i}\log_2(\Delta_i) \geq -2.132\ldots$. For details, see Hajir and Maire (2001); Tsfasman and Vladut (1998) and Conway and Sloane (1988).

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