ON THE SHAFAREVICH GROUP OF RESTRICTED RAMIFICATION EXTENSIONS OF NUMBER FIELDS IN THE TAME CASE

by

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Abstract. — Let K be a number field and S a finite set of places of K. We study the kernels $Ш^2_S$ of maps $H^2(\mathrm{G}_S, \mathbb{F}_p) \to \oplus_{v\in S} H^2(\mathrm{G}_v, \mathbb{F}_p)$. There is a natural injection $Ш^2_S \hookrightarrow Б_S$, into the dual $Б_S$ of a certain readily computable Kummer group V_S, which is always an isomorphism in the wild case. The tame case is much more mysterious. Our main result is that given a finite X coprime to p, there exists a finite set of places S coprime to p such that $Ш^2_{S\cup X} \overset{\simeq}{\hookrightarrow} Б_{S\cup X} \overset{\simeq}{\twoheadleftarrow} Б_X \hookleftarrow Ш^2_X$. In particular, we show that in the tame case $Ш^2_Y$ can *increase* with increasing Y. This is in contrast with the wild case where $Ш^2_Y$ is nonincreasing in size with increasing Y.

Let K be a number field, and let S be a finite set of places of K. Denote by K_S the maximal extension of K unramified outside S, and set $G_S = \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$. Given a prime number p, let $Ш^2_S$ be the 2-Shafarevich group associated to G_S and p: it is the kernel of the localization map of the cohomology group $H^2(\mathrm{G}_S, \mathbb{F}_p)$:

$$Ш^2_S := Ш^2(\mathrm{G}_S, \mathbb{F}_p) = \ker\big(H^2(\mathrm{G}_S, \mathbb{F}_p) \to \oplus_{v\in S} H^2(\mathrm{G}_v, \mathbb{F}_p)\big),$$

where G_S acts trivially on $\mathbb{F}_p$. We denote by G_v the absolute Galois group of the maximal extension of the completion K_v of K at v.

It is well-known that $Ш^2_S$ is closely related to $Б_S = (\mathrm{V}_S/\mathrm{K}^{\times p})^\vee$, where

$$\mathrm{V}_S = \{x \in \mathrm{K}^\times, v(x) \equiv 0 \pmod p\ \forall v; x \in \mathrm{K}_v^p\ \forall v \in S\}.$$

Clearly $\mathrm{K}^{\times p} \subset \mathrm{V}_S$ and $S \subset T \implies \mathrm{V}_T \subset \mathrm{V}_S$. Namely, in the wild case, when S contains all the places above p and all archimedean places, by Poitou-Tate duality Theorem one has $Ш^2_S \simeq Б_S$. See for example [**5**, Chapter X, §7]. It is important to note that algorithms exist to compute $Б_S$ via ray class group computations over $\mathrm{K}(\mu_p)$, so in the wild case one can, at least in theory, compute $d_p Ш^2_S$. For the more general tame situation, one only

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has the following injection (due to Shafarevich and Koch, see for example [**3**, Chapter 11, §2] or [**5**, Chapter 10, §7])

$$Ш^2_S \hookrightarrow Б_S. \tag{1}$$

At present there is no general algorithm to compute $d_p Ш^2_S$ *in the tame case, short of computing* G_S *itself.*

Let us write $\mathrm{K}_S(p)/\mathrm{K}$ as the maximal pro-p extension of K inside K_S, and put $\mathrm{G}_S(p) = \mathrm{Gal}(\mathrm{K}_S(p)/\mathrm{K})$. It is an exercise to see the quotient $\mathrm{G}_S \twoheadrightarrow \mathrm{G}_S(p)$ induces the injection $Ш^2_{S,p} \hookrightarrow Ш^2_S$, where $Ш^2_{S,p} := \ker\left(H^2(\mathrm{G}_S(p),\mathbb{F}_p) \to \oplus_{v\in S} H^2(\mathrm{G}_v,\mathbb{F}_p)\right)$. Observe that we can take $\mathrm{G}_v(p)$ instead of G_v, due to the fact that $H^2(\mathrm{G}_v(p),\mathbb{F}_p) \simeq H^2(\mathrm{G}_v,\mathbb{F}_p)$ (see for example [**5**, Chapter VII, §5]).
The Shafarevich group $Ш^2_S$ is central to the study of the maximal pro-p quotient $\mathrm{G}_S(p)$ of G_S, in particular when S is coprime to p: obviously, one gets

$$d_p H^2(\mathrm{G}_S(p),\mathbb{F}_p) \leqslant \sum_{v\in S} d_p H^2(\mathrm{G}_v,\mathbb{F}_p) + d_p Ш^2_S \leqslant \sum_{v\in S} \delta_{v,p} + d_p Ш^2_S \leqslant |S| + d_p \mathrm{V}_S/\mathrm{K}^{\times p},$$

which is sufficient to produce criteria involving the infinitess of $\mathrm{G}_S(p)$ (thanks to the Golod-Shafarevich Theorem). Here $\delta_{v,p} = 1$ or 0 as K_v contains the pth roots of unity or does not.

Observe that thanks to (1), one can force $Ш^2_S$ to be trivial (see the notion of saturated set S in §1.2), which can also yield situations where $\mathrm{G}_S(p)$ has cohomological dimension 2. See [**4**] for the first examples and [**6**] for general statements.

Before giving our main result, we make the following observation: given p a prime number, and two finite sets S and X of places of K, one has:

$$Ш^2_{S\cup X,p} \hookrightarrow Ш^2_{S\cup X} \hookrightarrow Б_{S\cup X} \twoheadleftarrow Б_X \hookleftarrow Ш^2_X \hookleftarrow Ш^2_{X,p} \tag{2}$$

where the middle surjection follows as $\mathrm{V}_{S\cup X} \subset \mathrm{V}_X$. To simplify, we consider only the case where the finite places X and S are coprime to p. Here we prove:

Theorem A. — *Let p be a prime number, and let* K *be a number field. Let X be a finite set of places of* K *coprime to p. There exist infinitely many finite sets S of finite places of* K*, all coprime to p, such that:*

$$Ш^2_{S\cup X,p} \simeq Ш^2_{S\cup X} \simeq Б_{S\cup X} \simeq Б_X.$$

Moreover such S can be chosen of size $|S| \leqslant d_p Б_\varnothing$.

Set $m := d_p Б_\varnothing$. Note $\mathrm{K}^{\times p} \subset \mathrm{V}_S$ for all S. In particular, we have the exact sequence

$$0 \to \mathscr{O}_\mathrm{K}^\times/\mathscr{O}_\mathrm{K}^{\times p} \to \mathrm{V}_\varnothing/\mathrm{K}^{\times p} \to Cl_\mathrm{K}[p] \to 0$$

so $m = d_p \mathrm{Cl}_\mathrm{K} + d_p \mathscr{O}_\mathrm{K}^\times$.

As mentioned above, the computation of $Ш^2_S$ is very difficult in the tame case. Indeed, the only examples we know of where the map $Ш^2_{\varnothing,p} \hookrightarrow Б_\varnothing$ is *not* an isomorphism are those in which we know the relation rank of $G_\varnothing(p)$ by knowing the full group itself. Using Theorem A, one may give situations where the value of $|Ш^2_S|$ is known without being trivial. As corollary, we get

Corollary A. — *There exist infinitely many finite sets $S_0 \subset S_1 \subset \cdots \subset S_m$ of finite places of* K *all coprime to p, such that for $i = 0, \cdots, m$, one has*

$$Ш^2_{S_i,p} \simeq Ш^2_{S_i} \simeq \mathbb{F}_p^{m-i}.$$

Remark. — *We will see that the sets S and S_i can be explicitly given by the Chebotarev density Theorem in some governing field extension over* K*.*

Remark. — *Let $\mathrm{K}_S^{ta}/\mathrm{K}$ be the maximal Galois extension of* K*, unramified outside S, and tamely ramified at S; put $\mathrm{G}_S^{ta} = \mathrm{Gal}(\mathrm{K}_S^{ta}/\mathrm{K})$. Then instead of considering G_S one may consider G_S^{ta} which also surjects onto $\mathrm{G}_S(p)$. Observe here that G_S^{ta} may be finite (typically when the discriminant of* K *and the norm of prime ideals of S are too small), even trivial (for example when $\mathrm{K} = \mathbb{Q}$ and $S = \emptyset$).*

Notations

– We fix a prime number p and a number field K.
– Put $\mathrm{K}' = \mathrm{K}(\zeta_p)$ and $\mathrm{K}'' = \mathrm{K}(\zeta_{p^2})$, where ζ_{p^2} is some primitive p^2th root of unity, and $\zeta_p = \zeta_{p^2}^p$.
– We denote by $\mathscr{O}_\mathrm{K}$ the ring of integers of K, by $\mathscr{O}_\mathrm{K}^\times$ the group of units of $\mathscr{O}_\mathrm{K}$, and by Cl_K the class group of K.
– We identify a prime ideal $\mathfrak{p} \subset \mathscr{O}_\mathrm{K}$ with the place v it defines. We write K_v for the completion of K at v and $\mathscr{U}_v$ for the units of the local field K_v; when v is archimedean, put $\mathscr{U}_v = \mathrm{K}_v^\times$.
– One says that a prime ideal $\mathfrak{p}$ is *tame* if $\#\mathscr{O}_\mathrm{K}/\mathfrak{p} \equiv 1 (\mathrm{mod}\ p)$, which is equivalent to $\mu_p \subset \mathrm{K}_v$, that is $\delta_{v,p} = 1$.
– If S is a finite set of places of K, we denote by $\mathrm{K}_S(p)/\mathrm{K}$ (resp. $\mathrm{K}_S^{ab}(p)/\mathrm{K}$) the maximal pro-$p$ extension (resp. abelian) of K unramified outside S, and we put $\mathrm{G}_S(p) = \mathrm{Gal}(\mathrm{K}_S(p)/\mathrm{K})$ (resp. $\mathrm{G}_S^{ab}(p) = \mathrm{Gal}(\mathrm{K}_S^{ab}(p)/\mathrm{K})$). For $S = \emptyset$, we denote by $\mathrm{H} := \mathrm{K}_\emptyset^{ab}(p)$ the Hilbert p-class field of K.
– By convention, the infinite places in S are only real. Let us write $S = S_0 \cup S_\infty$, where S_0 contains only the finite places and S_0 only the real ones. Put $\delta_{2,p} = \begin{cases} 1 & p = 2 \\ 0 & \text{otherwise} \end{cases}$
– The set S is said to be coprime to p, if all finite places v of S are coprime to p; it is said to be tame if S is coprime to p and $S_\infty = \emptyset$.
– Put $\mathrm{V}_S = \{x \in \mathrm{K}^\times, v(x) \equiv 0\ (\mathrm{mod}\ p)\ \forall v; x \in \mathrm{K}_v^p\ \forall v \in S\}$. Note $\mathrm{K}^{\times p} \subset \mathrm{V}_S$ for all S.

1. Preliminaries

1.1. Extensions with prescribed ramification. — Let p be a prime number.

1.1.1. Governing fields. — We recall a result of Gras-Munnier (see [**1**, Chapter V, §2, Corollary 2.4.2], as well as [**2**]) which gives a criterion for the existence of totally ramified p-extension at some set S (and unramified outside S). Put $\mathrm{K}' := \mathrm{K}(\zeta_p)$ and consider the governing field $\mathrm{L}' := \mathrm{K}'(\sqrt[p]{\mathrm{V}_\emptyset})$. The extension L'/K' has Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{r_1+r_2-1+\delta+d}$, where $d = d_p\mathrm{Cl}_\mathrm{K}$.

Given a place v of K, we choose some place $w|v$ of L' above v, and we consider $\sigma_v \in \mathrm{Gal}(\mathrm{L}'/\mathrm{K}')$ defined as follows:

– if v corresponds to a prime ideal $\mathfrak{p}$ coprime to p, and $\mathfrak{P}$ to w, then $\mathfrak{P}$ is unramified in L'/K', and then $\sigma_v = \sigma_{\mathfrak{p}} = \left(\dfrac{\mathrm{L}'/\mathrm{K}'}{\mathfrak{P}}\right)$ corresponds to the Frobenius elements at $\mathfrak{P}$ in $\mathrm{Gal}(\mathrm{L}'/\mathrm{K}')$;
– if v corresponds to a real place, then σ_v is the Artin symbol at w: $\sigma_v(\sqrt{\varepsilon}) = +1$ is ε is positive at w, and -1 otherwise.

While σ_v does in fact depend on the choice of $\mathfrak{P}$, it is easy to see a different choice of $\mathfrak{P}$ gives a nonzero multiple of the previous choice of σ_v in the $\mathbb{F}_p$-vector space $\mathrm{Gal}(\mathrm{L}'/\mathrm{K}')$. This is all we need when invoking Theorem 1.1 below. By abuse, we will also call the σ_v's Frobenius elements.

Theorem 1.1 **(Gras-Munnier)**. — *Let $S = \{v_1, \cdots, v_t\}$ be a set of places of* K *coprime to p. There exists a cyclic degree p extension* L/K*, unramified outside S and totally ramified at each place of S, if and only if, for $i = 1, \cdots, t$, there exists $a_i \in \mathbb{F}_p^\times$, such that*

$$\prod_{i=1}^{t} \sigma_{v_i}^{a_i} = 1 \ \in \mathrm{Gal}(\mathrm{L}'/\mathrm{K}').$$

When S is as in the Gras-Munnier criterion, i.e. the necessary and sufficient condition of the theorem holds, one says that the elements σ_{v_i} satisfy a *strongly nontrivial relation*. When one only has $\prod_{i=1}^{t} \sigma_{v_i}^{a_i} = 1$, with the a_i not all zero, one says that the σ_{v_i}'s satisfy a *nontrivial relation*.

Remark 1.2. — In fact, we don't find Theorem 1.1 in [**1**] in this form, the difference coming from the real places (and then only for $p = 2$). Indeed, one starts with the following: for a real place v, in our context we speak of *ramification*, and in the context of [**1**] Gras speaks of *decomposition*. Hence the governing field in [**1**] is smaller than L' and the condition he obtains did not involve the σ_v's, $v \in S_\infty$ (in fact, in his case these σ_v are trivial). But the proof is the same, we can follow it without difficulty due to the fact that for $v \in S_\infty$, one has: $\mathscr{U}_v/\mathscr{U}_v^2 = \mathbb{R}^\times/\mathbb{R}^{\times 2} \simeq \mathbb{Z}/2\mathbb{Z}$; see Lemmas 2.3.1, 2.3.2, 2.3.4 and 2.3.5 of [**1**].

Remark 1.3. — The results of [**1**] allow us to also obtain the following: put $\mathrm{L}'_0 := \mathrm{K}'(\sqrt[p]{\mathscr{O}_\mathrm{K}^\times})$, then $\#\mathrm{G}_S^{ab}(p) > \#\mathrm{G}_\varnothing^{ab}(p)$ if and only if, there exists some nontrivial relation in $\mathrm{Gal}(\mathrm{L}'_0/\mathrm{K}')$ between the σ_v's, $v \in S$. See also Proposition 1.5.

As consequence of Theorem 1.1, one has:

Corollary 1.4. — *Given p and* K*, and two finite sets T and S of places of* K *coprime to p, there exists a cyclic degree p extension* L/K*, unramified outside $S \cup T$ and ramified at each place of S (no condition on the places of T), if and only if the σ_v's for $v \in S$ satisfy a strongly nontrivial relation in the quotient* $\mathrm{Gal}(\mathrm{L}'/\mathrm{K}')/\langle \sigma_v, v \in T\rangle$.

1.1.2. Extensions over the Hilbert p-class field of K *that are abelian over* K.— As noted in the beginning of Chapter V of [**1**], the result about the existence of a degree-p^e cyclic extension with prescribed ramification can be generalized in different forms. Let H be the Hilbert class field of K. In what follows, we only need the existence of a degree-p^2 cyclic extension of H, abelian over K, with prescribed ramification.

Now we follow the strategy of [**1**, Chapter V, §2, d)]. Put $B = \mathrm{Gal}(\mathrm{K}_S^{ab}(p)/\mathrm{H})$. Take Σ a finite set of places of K coprime to p (not necessarily satisfying the congruence $\mathrm{N}(\mathfrak{p}) \equiv 1(\bmod\ p^2)$ when $\mathfrak{p} \in \Sigma_0$). By class field theory, we get

$$1 \longrightarrow (B/B^{p^2})^* \xrightarrow{\rho} \bigoplus_{v\in\Sigma} (\mathscr{U}_v/(\mathscr{U}_v)^{p^2})^* \longrightarrow \left(\iota(\mathscr{O}_{\mathrm{K}}^\times)\right)^* \longrightarrow 1,$$

where $\iota : \mathscr{O}_{\mathrm{K}}^\times \longrightarrow \bigoplus_{v\in\Sigma} \mathscr{U}_v/(\mathscr{U}_v)^{p^2}$ is the diagonal embedding.

A cyclic degree-p^2 extension M of H, abelian over K and unramified outside Σ is given by a character ψ of B/B^{p^2} of order p^2 as follows:
Given $\psi_v \in (\mathscr{U}_v/(\mathscr{U}_v)^{p^2})^*$ for all $v \in \Sigma$, there exists a character ψ of B/B^{p^2} such that $\psi_{|\mathscr{U}_v} = \psi_v$ if and only if,

$$\forall \varepsilon \in \mathscr{O}_{\mathrm{K}}^\times,\ \prod_{v\in\Sigma} \psi_v(\varepsilon) = 1. \tag{3}$$

As M/H is totally ramified at at least one prime ideal, at least one ψ_v has order p^2.

Now we will focus on the case where Σ contains only finite places, and we use the notation $\mathfrak{p}$ instead of v.

Let S be a finite non-empty set of tame places of K where each prime $\mathfrak{p}$ (corresponding to $v \in S$) is such that $\mathrm{N}(\mathfrak{q}) \equiv 1(\bmod\ p^2)$. Let us write now $\Sigma_\mathfrak{q} = S \cup T_\mathfrak{q}$, where $T_\mathfrak{q} = \{\mathfrak{q}\}$ is also tame. We are interested in the existence of a degree-p^2 cyclic extension $\mathrm{K}_\mathfrak{q}/\mathrm{H}$, abelian over K and unramified outside $\Sigma_\mathfrak{q}$, such that $\mathrm{K}_\mathfrak{q}/\mathrm{H}$ has degree p^2 and for which the inertia degree at $\mathfrak{q}$ is exactly p.
For $\mathfrak{p} \in \Sigma_\mathfrak{q}$, let us fix $\chi_\mathfrak{p}$ a generator of $(\mathscr{U}_\mathfrak{p}/(\mathscr{U}_\mathfrak{p})^{p^2})^*$. By (3), $\mathrm{K}_\mathfrak{q}$ exists if and only if, there exist $a_\mathfrak{q} \in \mathbb{F}_p^\times$, and $b_\mathfrak{p} \in \mathbb{Z}/p^2$, $\mathfrak{p} \in S$, such that

$$\forall \varepsilon \in \mathscr{O}_{\mathrm{K}}^\times,\ \hat{\chi}_\mathfrak{q}^{a_\mathfrak{q}}(\varepsilon) \prod_{\mathfrak{p}\in S} \chi_\mathfrak{p}^{b_\mathfrak{p}}(\varepsilon) = 1,$$

where

$$\hat{\chi}_\mathfrak{q} = \begin{cases} \chi_\mathfrak{q} & \text{if } \mathrm{N}(\mathfrak{q}) \not\equiv 1(\bmod\ p^2) \\ \chi_\mathfrak{q}^p & \text{if } \mathrm{N}(\mathfrak{q}) \equiv 1(\bmod\ p^2) \end{cases},$$

and such that at least one $b_\mathfrak{p} \in (\mathbb{Z}/p^2\mathbb{Z})^\times$.
This last condition can be rephrased thanks to Kummer theory with the following governing field (see [**1**, Chapter V, §2, d)]):

$$\mathrm{L} = \mathrm{K}''(\sqrt[p^2]{\mathscr{O}_{\mathrm{K}}^\times}),$$

where $\mathrm{K}'' = \mathrm{K}(\zeta_{p^2})$. For each prime $\mathfrak{p} \in \Sigma_\mathfrak{q}$ let us choose a prime $\mathfrak{P}|\mathfrak{p}$ of K'', and denote by $\sigma_\mathfrak{p}$ the Frobenius of $\mathfrak{P}$ in $\mathrm{Gal}(\mathrm{L}/\mathrm{K}'')$. As before, $\sigma_\mathfrak{p}$ depends on $\mathfrak{P}|\mathfrak{p}$ only up to a power coprime to p.
The above discussion allows us to obtain the following:

Proposition 1.5. — *There exists a degree-p^2 cyclic extension* $\mathrm{K}_\mathfrak{q}/\mathrm{H}$*, abelian over* K*, unramified outside* $\Sigma_\mathfrak{q}$*, for which the inertia degree at* $\mathfrak{q}$ *is exactly* p*, if and only if, there exists* $a_\mathfrak{q} \in \mathbb{F}_p^\times$*, and* $b_\mathfrak{p} \in \mathbb{Z}/p^2\mathbb{Z}$*,* $\mathfrak{p} \in S$*, such that*

$$\hat{\sigma}_\mathfrak{q}^{a_\mathfrak{q}} \prod_{\mathfrak{p}\in S} \sigma_\mathfrak{p}^{b_\mathfrak{p}} = 1 \in \mathrm{Gal}(\mathrm{L}/\mathrm{K}''), \tag{4}$$

where

$$\hat{\sigma}_{\mathfrak{q}} = \begin{cases} \sigma_{\mathfrak{q}} & \text{if } \mathrm{N}(\mathfrak{q}) \not\equiv 1 (\mathrm{mod}\ p^2) \\ \sigma_{\mathfrak{q}}^p & \text{if } \mathrm{N}(\mathfrak{q}) \equiv 1 (\mathrm{mod}\ p^2) \end{cases},$$

with at least one $b_{\mathfrak{p}} \in (\mathbb{Z}/p^2\mathbb{Z})^\times$.

Remark. — *Infinitely many such sets exist by the Chebotarev Density Theorem.*

1.2. Saturated sets. — Take p, K as before, and let S be a finite set of places of K, coprime to p.

Definition 1.6. — The S set of places K is called saturated if $\mathrm{V}_S/(\mathrm{K}^\times)^p = \{1\}$.

Recall the following equality due to Shafarevich (see for example [**5**, Chapter X, §7, Corollary 10.7.7]):

$$d_p\mathrm{G}_S = |S_0| + |S_\infty|\delta_{2,p} - (r_1 + r_2) + 1 - \delta + d_p\mathrm{V}_S/(\mathrm{K}^\times)^p, \tag{5}$$

showing that $d_p\mathrm{G}_S$ is easy to compute when S is saturated.

Proposition 1.7. — *Let S and T be two finite sets of places of* K *coprime to p. Suppose S is saturated. Then*

- *if $S \subset T$, then T is saturated;*
- *for every tame place $v \notin S$, one has $d_p\mathrm{G}_{S\cup\{\mathfrak{p}\}} = d_p\mathrm{G}_S + 1$.*

Proof. — The first point is due to the fact that $\mathrm{V}_T \subset \mathrm{V}_S$, and the second point is a consequence of (5) along with the first point. □

Theorem 1.8. — *A finite set S coprime to p is saturated if and only if, the Frobenii σ_v, $v \in S$, generate the whole group* $\mathrm{Gal}(\mathrm{K}'(\sqrt[p]{\mathrm{V}_\varnothing})/\mathrm{K}')$.

Proof. — • Suppose the Frobenii generate the full Galois group. By hypothesis, for each degree-p extension L/K' in $\mathrm{K}'(\sqrt[p]{\mathrm{V}_\varnothing})/\mathrm{K}'$, there exists a place $v \in S$ such that v is inert in L/K' (when $v \in S_\infty$, v is ramified in L/K'). Let us take now $x \in \mathrm{V}_S$: then every $v \in S$ splits totally in $\mathrm{K}'(\sqrt[p]{x})/\mathrm{K}'$. As $\mathrm{K}'(\sqrt[p]{x}) \subset \mathrm{K}'(\sqrt[p]{\mathrm{V}_\varnothing})$, one deduces that $\mathrm{K}'(\sqrt[p]{x}) = \mathrm{K}'$, and then $x \in (\mathrm{K}')^p$. As $[\mathrm{K}' : \mathrm{K}]$ is coprime to p, one finally obtains that $x \in \mathrm{K}^{\times p}$, so $Б_S = \{0\}$.
• If S is saturated, then for every finite set T of tame places of K with $T \cap S = \varnothing$, one has $d_p\mathrm{G}_{S\cup T} = d_p\mathrm{G}_S + |T|$ by Proposition 1.7. Then by the Gras-Munnier criterion, one has $\langle \sigma_v, v \in S\rangle = \mathrm{Gal}(\mathrm{L}'/\mathrm{K}')$. □

Corollary 1.9. — *The finite set S coprime to p is saturated if and only if, for every finite set T of tame places of* K*, there exists a cyclic degree p-extension of* K *unramified outside $S \cup T$ but ramified at each place of T.*

Proof. — • If S is saturated, then by Theorem 1.8 the Frobenii σ_v, $v \in S$, generate $\mathrm{Gal}(\mathrm{L}'/\mathrm{K}')$, and the result follows by using Corollary 1.4.
• Suppose that S is such that for every finite set T of tame places of K, there exists a cyclic degree p-extension unramified outside $S \cup T$ and ramified at each place of T. Then by Corollary 1.4 and the Chebotarev density theorem, $\mathrm{Gal}(\mathrm{L}'/\mathrm{K}') = \langle \sigma_v,\ v \in S\rangle$. By Theorem 1.8, S is saturated. □

1.3. Spectral sequence. — Let S and T be two finite sets of places of K coprime to p. Consider the following exact sequence of pro-p groups

$$1 \longrightarrow \mathrm{H}_{S,T} \longrightarrow \mathrm{G}_{S\cup T}(p) \longrightarrow \mathrm{G}_S(p) \longrightarrow 1. \tag{6}$$

Definition 1.10. — Put

$$\mathscr{X}_{S,T} := \mathrm{H}_{S,T}/[\mathrm{H}_{S,T},\mathrm{H}_{S,T}]\mathrm{H}_{S,T}^p,$$

and

$$\mathrm{X}_{S,T} := (\mathscr{X}_{S,T})_{\mathrm{G}_S(p)} = \mathrm{H}_{S,T}/[\mathrm{H}_{S,T},\mathrm{G}_S(p)]\mathrm{H}_{S,T}^p.$$

Recall that as $\mathrm{G}_S(p)$ is a pro-p group, then $\mathbb{F}_p[\![\mathrm{G}_S(p)]\!]$ is a local ring.

Lemma 1.11. — *The abelian group $\mathscr{X}_{S,T}$ is a $\mathbb{F}_p[\![\mathrm{G}_S(p)]\!]$-module (with continuous action) that can be generated by $d_p\mathrm{X}_{S,T}$ generators. Moreover, $d_p\mathrm{X}_{S,T} \leqslant |T|$.*

Proof. — The first part follows from Nakayama's lemma. For the second, the fact that $\mathrm{G}_S(p)$ acts transitively on the inertia groups I_w of $w|v \in T$ in $\mathscr{X}(S,T)$ implies

$$\bigoplus_{i=1}^{t} \mathbb{F}_p[\![\mathrm{G}_S(p)]\!] \twoheadrightarrow \langle I_w, w|v \in T\rangle = \mathscr{X}_{S,T},$$

where $t = |T|$. Taking the $\mathrm{G}_S(p)$-coinvariants, we obtain $\mathbb{F}_p^t \twoheadrightarrow \mathrm{X}_{S,T}$. □

Applying the Hochschild-Serre spectral sequence to (6), one gets:

Lemma 1.12. — *Let S, T be two finite sets of places of K coprime to p. Then one has :*

$$1 \longrightarrow H^1(\mathrm{G}_S(p),\mathbb{F}_p) \longrightarrow H^1(\mathrm{G}_{S\cup T}(p),\mathbb{F}_p) \longrightarrow \mathrm{X}_{S,T}^\vee \longrightarrow Ш^2_{S,p} \longrightarrow Ш^2_{S\cup T,p}.$$

Furthermore, the cokernel of the natural injection $Ш_{X,p} \hookrightarrow Б_X$ decreases in dimension as X increases.

Proof. — The Hochschild-Serre spectral sequence gives the exact commutative diagram:

$$\begin{array}{ccccccccc}
H^1(\mathrm{G}_S(p),\mathbb{F}_p) & \hookrightarrow & H^1(\mathrm{G}_{S\cup T}(p),\mathbb{F}_p) & \longrightarrow & \mathrm{X}_{S,T}^\vee & \longrightarrow & H^2(\mathrm{G}_S(p),\mathbb{F}_p) & \longrightarrow & H^2(\mathrm{G}_{S\cup T}(p),\mathbb{F}_p) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \oplus_{v\in S} H^2(\mathrm{G}_v,\mathbb{F}_p) & \hookrightarrow & \oplus_{v\in S\cup T} H^2(\mathrm{G}_v,\mathbb{F}_p)
\end{array}$$

Chasing the trangression map $\mathrm{X}_{S,T}^\vee \xrightarrow{tg} H^2(G_S(p))$ to the right gives that its image lies in $Ш^2_{S,p}$ whose image to the right lies in $Ш^2_{S\cup T,p}$. We now have the diagram

$$\begin{array}{ccccccccc}
1 \longrightarrow H^1(\mathrm{G}_S(p),\mathbb{F}_p) & \longrightarrow & H^1(\mathrm{G}_{S\cup T}(p),\mathbb{F}_p) & \longrightarrow & \mathrm{X}_{S,T}^\vee & \longrightarrow & Ш^2_{S,p} & \longrightarrow & Ш^2_{S\cup T,p} \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & Б_S & \twoheadrightarrow & Б_{S\cup T}
\end{array}$$

where the bottom horizontal map is surjective as the inclusion $\mathrm{V}_{S\cup T}/\mathrm{K}^{\times p} \hookrightarrow \mathrm{V}_S/\mathrm{K}^{\times p}$ is immediate from the definition of V_X. The second result follows. □

Corollary 1.13. — *If the natural injection $Ш_{X,p} \hookrightarrow Б_X$ is an ismorphism, then for any set Y we have $Ш_{X\cup Y,p} \overset{\simeq}{\hookrightarrow} Б_{X\cup Y}$*

Let us give an obvious consequence of Lemma 1.12.

Lemma 1.14. — *Suppose that $H^1(\mathrm{G}_S(p),\mathbb{F}_p) \simeq H^1(\mathrm{G}_{S\cup T}(p),\mathbb{F}_p)$, then $\mathrm{X}_{S,T}^\vee \hookrightarrow Ш^2_{S,p}$. If moreover $S \cup T$ is saturated then $\mathrm{X}_{S,T}^\vee \simeq Ш^2_{S,p}$.*

Proof. — If $S \cup T$ is saturated then $\mathrm{V}_{S\cup T}/\mathrm{K}^{\times p} = \{1\}$, which implies that $Б_{S\cup T} = \{1\}$. Hence, by (1) $Ш^2_{S\cup T} = \{0\}$, and the same holds for $Ш^2_{S\cup T,p}$. We conclude with Lemma 1.12. □

An important consequence of Lemmas 1.12 and 1.14 is that elements of $\mathrm{X}_{S,T}^\vee$ can give rise to elements of $Ш^2_{S,p}$. The former can be found via ray class group computations. We thus have a method of producing independent elements of $Ш^2_{S,p}$. If we find $d_p Б_S$ such elements, we have established $Ш^2_{S,p} \overset{\simeq}{\hookrightarrow} Ш^2_S \overset{\simeq}{\hookrightarrow} Б_S$, and thus computed $d_p Ш^2_S$.

2. Proof of the results

2.1. A key Proposition. — Let p be a prime number. Let K be a number field and let X be a finite set of places of K coprime to p. The proof of Theorem 1.1 is a consequence of the following proposition.

Proposition 2.1. — *There exist (infinitely many) pairs of finite sets of tame places S and T of* K *such that:*

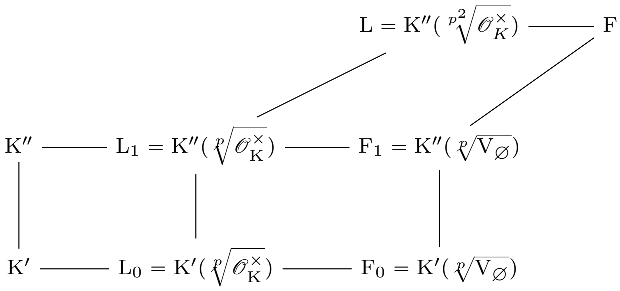
- *(i)* $T \cup X$ *is saturated and* $d_p\mathrm{G}_{T\cup X} = d_p\mathrm{G}_X$*;*
- *(ii)* $d_p\mathrm{G}_{S\cup T\cup X} = d_p\mathrm{G}_{S\cup X}$*;*
- *(iii)* $|T| \leqslant d_p\mathrm{Cl}_\mathrm{K} + r_1 + r_2 - 1 + \delta$ *and* $|S| \leqslant r_1 + r_2 - 1 + \delta$*;*
- *(iv) for each prime $\mathfrak{q} \in T$, there exists a degree-p^2 cyclic extension $\mathrm{K}_\mathfrak{q}$ of K^H, abelian over* K*, unramified outside $S \cup X \cup \{\mathfrak{q}\}$, where the inertia group at $\mathfrak{q}$ is of order p.*

Put $\mathrm{F}_0 = \mathrm{K}'(\sqrt[p]{\mathrm{V}_\varnothing})$, $\mathrm{L}_0 = \mathrm{K}'(\sqrt[p]{\mathscr{O}_\mathrm{K}^\times})$, $\mathrm{K}'' = \mathrm{K}(\zeta_{p^2})$, $\mathrm{L}_1 = \mathrm{K}''(\sqrt[p^2]{\mathscr{O}_\mathrm{K}^\times})$, $\mathrm{F}_1 = \mathrm{K}''(\sqrt[p]{\mathrm{V}_\varnothing})$, and $\mathrm{F} = \mathrm{L}\mathrm{F}_0 = \mathrm{K}''(\sqrt[p^2]{\mathscr{O}_\mathrm{K}^\times}, \sqrt[p]{\mathrm{V}_\varnothing})$. Put $\mathrm{G} = \mathrm{Gal}(\mathrm{F}/\mathrm{K}')$.

Proof. — (of Proposition 2.1.)
Given a prime $\mathfrak{p}$ of $\mathscr{O}_\mathrm{K}$, coprime to p, we choose a prime $\mathfrak{P}|\mathfrak{p}$ of F, and we consider its Frobenius $\sigma_\mathfrak{p} := \sigma_\mathfrak{P}$ in the Galois group $\mathrm{Gal}(\mathrm{F}/\mathrm{K}')$ and its quotients. As mentioned earlier, this is well-defined up to a nonzero scalar multiple in $\mathrm{Gal}(\mathrm{F}/\mathrm{K}')$ and that is all we need.

Put $E_X = \langle \sigma_{\mathfrak{p}|\mathrm{F}_0}, \mathfrak{p} \in X\rangle \subset \mathrm{Gal}(\mathrm{F}_0/\mathrm{K}')$ the subgroup of $\mathrm{Gal}(\mathrm{F}_0/\mathrm{K}')$ generated by the Frobenii of the primes $\mathfrak{p} \in X$. Put $m_\mathrm{X} = d_p\mathrm{V}_\varnothing - d_pE_X$.

a) Assume first that $\mathrm{F}_0 \cap \mathrm{K}'' = \mathrm{K}'$.



We choose S and T as follows:

– let T be *any* set of primes $\mathfrak{q}$ whose Frobenii $\sigma_{\mathfrak{q}}$ in G are such that the restriction in $\mathrm{Gal}(\mathrm{F}_0/\mathrm{K}')$ forms an $\mathbb{F}_p$-basis of a subspace in direct sum with E_X: in other words,

$$\mathrm{Gal}(\mathrm{F}_0/\mathrm{K}') = \langle \sigma_{\mathfrak{q}|\mathrm{F}_0}, \mathfrak{q} \in T\rangle \bigoplus E_X,$$

and $\langle \sigma_{\mathfrak{q}|\mathrm{F}_0}, \mathfrak{q} \in T\rangle = \bigoplus_{\mathfrak{q}\in T} \langle \sigma_{\mathfrak{q}|\mathrm{F}_0}\rangle$.

– let $\tilde{X}$ be those places of X whose Frobenii lie in $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_1)$ and let S be *any* set of primes $\mathfrak{p}$ whose Frobenii $\sigma_{\mathfrak{p}}$ in G form in direct sum with the Frobenii in $\tilde{X}$ a basis of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_1)$.

As $\mathrm{Gal}(\mathrm{F}_1/\mathrm{K}')$ has exponent p, we see for each $\mathfrak{q} \in T$, $\sigma_{\mathfrak{q}}^p \in \mathrm{Gal}(\mathrm{F}/\mathrm{F}_1)$. Observe also that if $\sigma_{\mathfrak{q}|\mathrm{K}''}$ is not trivial (which is equivalent to $\mathrm{N}(\mathfrak{q}) \neq 1 \pmod{p^2}$), then $\sigma_{\mathfrak{q}}^p$ is the Frobenius at $\mathfrak{P}$ in $\mathrm{Gal}(\mathrm{F}/\mathrm{F}'')$; otherwise $\sigma_{\mathfrak{q}}^p$ is the p-power of the Frobenius at $\mathfrak{Q} \mid \mathfrak{q}$ in $\mathrm{Gal}(\mathrm{F}/\mathrm{F}'')$.

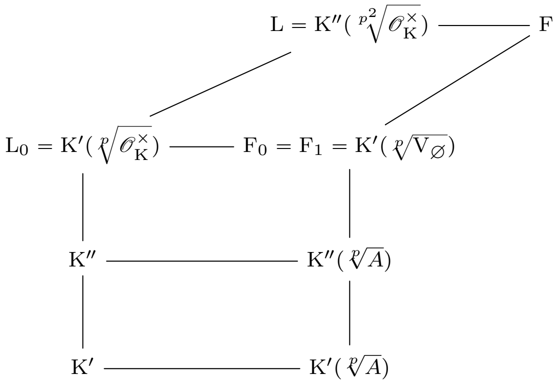
By Theorem 1.8 the set $T \cup X$ is saturated. Moreover thanks to the condition on the direct sum for the Frobenius at $\mathfrak{p} \in T$, by Theorem 1.1, there is no cyclic degree-p extension of K, unramified outside $T \cup X$ and totally ramified at any nonempty subset of places of T: thus $d_p\mathrm{G}_{T\cup X} = d_p\mathrm{G}_X$, and (i) holds.

Moreover as each place of S splits totally in the governing extension F_0/K', then again by Theorem 1.1, $d_p\mathrm{G}_{S\cup T\cup X} = d_p\mathrm{G}_{S\cup X}$, and (ii) holds.

The condition on S gives a relation of type (4) in $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_1) \subset \mathrm{Gal}(\mathrm{F}/\mathrm{L}_1)$ for the set $S \cup \tilde{X} \cup \{\mathfrak{q}\}$, $\mathfrak{q} \in T$. After taking the quotient of this relation by $\mathrm{Gal}(\mathrm{F}/\mathrm{L})$, we obtain by Proposition 1.5 that for each prime $\mathfrak{q} \in T$, the existence of a degree-p^2 cyclic extension $\mathrm{K}_{\mathfrak{q}}/\mathrm{H}$, abelian over K and unramified outside $S \cup X \cup \{\mathfrak{q}\}$ for which the inertia at $\mathfrak{q}$ is of order p, proving (iv).

(iii) is obvious.

b) Assume now that that $\mathrm{K}'' \subset \mathrm{F}_0$.
Let $\mathfrak{A}_i, i = 1, \cdots, d$ be ideals of $\mathscr{O}_\mathrm{K}$, whose classes are a system of minimal generators of $\mathrm{Cl}_\mathrm{K}[p]$, and let $a_i \in \mathscr{O}_\mathrm{K}^\times$ such that $(a_i) = \mathfrak{A}_i^p$. Put $A = \langle a_1, \cdots, a_d\rangle \mathrm{K}^{\times p}/\mathrm{K}^{\times p} \subset \mathrm{V}_\varnothing/\mathrm{K}^{\times p}$.
Note $\mathrm{K}'(\sqrt[p]{\mathrm{V}_\varnothing}) = \mathrm{K}'(\sqrt[p]{A}, \sqrt[p]{\mathscr{O}_\mathrm{K}^\times})$.
As F_0/K' and K''/K' are abelian p-extensions, the containment $\mathrm{K}'' \subset \mathrm{F}_0$ implies $\mathrm{K}' = \mathrm{K}$.
Moreover $\mathrm{L}_0 \cap \mathrm{K}''(\sqrt[p]{A}) = \mathrm{K}''$.



Now take T and S as in case a). □

Remark 2.2. — Observe that one can take T such that $|T| \leqslant m_\mathrm{X} = d_p\mathrm{V}_\varnothing - d_pE_X$.

2.2. Proof of Theorem A. — Let S and T as in Proposition 2.1. As $X \cup T$ is saturated, by (i) of Proposition 2.1 and (5), one obtains $|T| = d_p Б_X$. Moreover, $S \cup X \cup T$ is also saturated and in particular, $Б_{S\cup X\cup T} \simeq Ш^2_{S\cup X\cup T,p} = \{0\}$. With (ii), we see that $\mathrm{d}_p Б_{S\cup X} = |T|$ so (i) and (ii) imply: $Б_{S\cup X} \simeq Б_X$.

Now let us take the spectral sequence of the short exact sequence

$$1 \longrightarrow \mathrm{H}_{S\cup X,T} \longrightarrow \mathrm{G}_{S\cup X\cup T}(p) \longrightarrow \mathrm{G}_{S\cup X}(p) \longrightarrow 1$$

to obtain by Lemma 1.12:

$$1 \to H^1(\mathrm{G}_{S\cup X}(p), \mathbb{F}_p) \to H^1(\mathrm{G}_{S\cup X\cup T}(p), \mathbb{F}_p) \to \mathrm{X}^\vee_{S\cup X,T} \to Ш^2_{S\cup X,p} \to Ш^2_{S\cup X\cup T,p} = \{0\}.$$

Hence, $\mathrm{X}^\vee_{S\cup X,T} \simeq Ш^2_{S\cup X,p}$. Now (iv) of Proposition 2.1 implies that $d_p \mathrm{X}_{S\cup X,T} \geqslant |T|$, and as obviously $d_p \mathrm{X}_{S\cup X,T} \leqslant |T|$, we finally get $d_p Ш^2_{S\cup X,p} = |T|$.

Hence $d_p Ш^2_{S\cup X,p} = |T| = d_p Б_{S\cup X} = d_p Б_X$. Thanks to (2), one has

$$Ш^2_{S\cup X,p} \simeq Ш^2_{S\cup X} \simeq Б_{S\cup X} \simeq Б_X.$$

2.3. Proof of Corollary A. — Let us choose S and T as in proof of Proposition 2.1. Let us write $T = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_{m_X}\}$, where $m_X = d_p Б_\varnothing - d_p E_X$. Put $S_0 = S \cup X$ and, for $i \geqslant 0$, $S_{i+1} = S \cup X \cup \{\mathfrak{p}_i\}$. Here, as $d_p \mathrm{G}_{S_i} = d_p \mathrm{G}_{S_{m_X}}$, the spectral sequence shows that

$$\mathbb{F}_p \hookrightarrow Ш^2_{S_i,p} \longrightarrow Ш^2_{S_{i+1},p}, \tag{7}$$

in particular $d_p Ш^2_{S_i,p} \leqslant d_p Ш^2_{S_{i+1},p} + 1$. After noting that $d_p Ш^2_{S_{m_X},p} = 0$ (the set $X \cup T$ is saturated) and that $d_p Ш^2_{S_0,p} = |T| = m_X$, then we conclude that $d_p Ш^2_{S_i,p} = m_{\mathrm{X}} - i$. Observe also that (7) induces:

$$\mathbb{F}_p \hookrightarrow Ш^2_{S_i} \longrightarrow Ш^2_{S_{i+1}},$$

and as before $d_p Ш^2_{S_i} = m - i$. The isomorphisms $Ш^2_{S_i,p} \simeq Ш^2_{S_i}$'s become obvious.

We have proved:

Corollary 2.3. — *One has* $Ш^2_{S_i} \simeq \mathbb{F}_p^{m_X - i}$.

Take $X = \varnothing$ to have Corollary A.

3. Examples

In this section we give a few examples of fields K and sets S such that in the diagram

$$Ш^2_\varnothing \hookrightarrow Б_\varnothing \twoheadrightarrow Б_S \hookleftarrow Ш^2_S,$$

the two maps on the right are isomorphisms. In our first two examples we show the left map is *not* an isomorphism. Thus we give explicit examples where $Ш^2_X$ increases as X does, in contrast to the wild case.

In the third example we establish

$$Б_\varnothing \overset{\simeq}{\twoheadrightarrow} Б_S \overset{\simeq}{\hookleftarrow} Ш^2_S,$$

but do not know whether $d_p Ш^2_\varnothing < d_p Ш^2_S$. Indeed, we suspect equality in that case.

In the examples below, p_i refers to the ith prime of K above the rational prime p as MAGMA presents the factorization. All code was run unconditionally, that is we did *not* use GRH bounds for computing ray class groups.

Example 1. — *Let* K *be the unique degree* 3 *subfield of* $\mathbb{Q}(\zeta_7)$ *and let* $p = 2$. *Then one can easily compute that* K *has trivial class group and, since* K *is totally real,* $d_p\text{Б}_\varnothing = d_p\mathscr{O}_\text{K}^\times/\mathscr{O}_\text{K}^{\times 2} + d_p Cl_\text{K}[2] = 3$. *Clearly* $G_\varnothing = \{e\}$ *and* $d_p\text{Ш}^2_\varnothing = 0$ *so* $\text{Ш}^2_\varnothing \hookrightarrow \text{Б}_\varnothing$ *has* 3*-dimensional cokernel. Set* $S = \{37_1, 181_1, 293_1\}$ *and* $T = \{307_1, 311_1, 349_1\}$. *One computes* $d_pH^1(G_T, \mathbb{F}_2) = 0$ *so* T *and* $S \cup T$ *are saturated. The* 2*-parts of the ray class groups for conductors* $S \cup T$ *and* S *are* $(\mathbb{Z}/4)^3$ *and* $(\mathbb{Z}/2)^3$ *respectively, so the the map* $H^1(G_S, \mathbb{F}_2) \to H^1(G_{S\cup T}, \mathbb{F}_2)$ *is an isomorphism and* $d_p\text{X}^\vee_{S\cup X,T} \geqslant 3$. *As* $d_p\text{Ш}^2_S \leqslant d_p\text{Б}_S \leqslant d_p\text{Б}_\varnothing = 3$, *we see* $d_p\text{Ш}^2_S = 3$.

Example 2. — *Let* K *be the unique degree* 3 *subfield of* $\mathbb{Q}(\zeta_{349})$ *and let* $p = 2$. *Here* K *has class group* $(\mathbb{Z}/2)^2$ *and is again totally real, so* $d_p\text{Б}_\varnothing = d_p\mathscr{O}_\text{K}^\times/\mathscr{O}_\text{K}^{\times 2} + d_p Cl_\text{K}[2] = 5$. *One computes the class group of the Hilbert class field of* K *is trivial so* $\text{G}_\varnothing = \mathbb{Z}/2 \times \mathbb{Z}/2$ *and has three relations. Thus* $d_p\text{Ш}^2_\varnothing = d_pH^2(G_\varnothing, \mathbb{F}_2) = 3$ *so the map* $\text{Ш}^2_\varnothing \hookrightarrow \text{Б}_\varnothing$ *has* 2*-dimensional cokernel. Set* $S = \{701_1, 2857_1, 3169_1\}$ *and* $T = \{367_1, 397_1, 401_1, 409_1, 449_1\}$. *One computes* $d_pH^1(G_T, \mathbb{F}_2) = 2$ *so* T *and* $S\cup T$ *are saturated. The* 2*-parts of the ray class groups for conductors* $S\cup T$ *and* S *are* $\mathbb{Z}/4\times(\mathbb{Z}/8)^2\times\mathbb{Z}/16\times\mathbb{Z}/32$ *and* $(\mathbb{Z}/2)^5$ *respectively, so the the map* $H^1(G_S, \mathbb{F}_2) \to H^1(G_{S\cup T}, \mathbb{F}_2)$ *is an isomorphism and* $d_p\text{X}^\vee_{S\cup X,T} \geqslant 5$. *As* $d_p\text{Ш}^2_S \leqslant d_p\text{Б}_S \leqslant d_p\text{Б}_\varnothing = 5$, *we see* $d_p\text{Ш}^2_S = 5$.

Example 3. — *Let* $\text{K} = \mathbb{Q}[x]/(f(x))$ *where* $f(x) = x^{12} + 339x^{10} - 19752x^8 - 2188735x^6 + 284236829x^4 + 4401349506x^2 + 15622982921$. *This polynomial is irreducible and* K *is totally complex with small root discriminant and has class group* $(\mathbb{Z}/2)^6$. *The field* K *has been used as a starting point in finding infinite towers of totally complex number fields whose root discriminants are the smallest currently known. Set*

$$S = \{7_2, 11_1, 43_1, 47_3, 67_3, 97_1\},\ T = \{5_1, 13_1, 19_1, 19_2, 23_1, 23_2, 23_3, 29_1, 31_1, 61_1, 149_1, 149_4\}.$$

As K *is totally complex,*

$$d_p\text{Б}_\varnothing = d_p\mathscr{O}_\text{K}^\times/\mathscr{O}_\text{K}^{\times 2} + d_p Cl_\text{K}[2] = 6 + 6 = 12 = \#T.$$

One computes $d_pH^1(G_T, \mathbb{F}_2) = 6$ *so* T *and* $S \cup T$ *are saturated. The* 2*-parts of the ray class groups for conductors* $S\cup T$ *and* S *are* $(\mathbb{Z}/4)^5 \times (\mathbb{Z}/8)^4 \times (\mathbb{Z}/16)^3$ *and* $(\mathbb{Z}/2)^{11} \times \mathbb{Z}/8$. *respectively, so the the map* $H^1(G_S, \mathbb{F}_2) \to H^1(G_{S\cup T}, \mathbb{F}_2)$ *is an isomorphism. From this data one can only conclude* $d_p\text{X}^\vee_{S\cup X,T} \geqslant 11$. *On the other hand, for every* $v \in T$ *one computes the* 2*-part of the ray class group for conductor* $S \cup \{v\}$ *has order at least* $2^{15} > 2^{14}$. *As the latter quantity is the order of the* 2*-part of the ray class group with conductor* S, *we get* $\#T = 12$ *independent elements of* $\text{X}^\vee_{S\cup X,T}$ *so* $d_p\text{Ш}^2_S \geqslant 12$. *As* $d_p\text{Б}_S \leqslant d_p\text{Б}_\varnothing = 12$, *we have* $d_p\text{Ш}^2_S = 12$. *We suspect that in this case* $d_p\text{Ш}^2_\varnothing = 12$.

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