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Equality of the Energy and Group Velocities of Bulk Acoustic Waves in Piezoelectric Media

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***Abstract*—The equality of the energy and group velocities of bulk acoustic waves in a lossless piezoelectric medium is demonstrated, with the energy velocity defined from the generalized energy density and the generalized Poynting vector.**

I. Introduction

It is well-known that the energy and the group velocities of bulk acoustic waves (BAW) in elastic media are equal [1], [2]. The energy velocity, v_e, is defined as the ratio of the Poynting vector to the energy density, with the help of the Poynting theorem. The group velocity, v_g, is defined as the derivative of the phase velocity with respect to the direction of propagation. The group velocity thus is defined on a purely geometrical ground and is by construction normal to the slowness surface. Further properties are the equality of the BAW kinetic and the potential (or strain) energies, and the orthogonality of the slowness vector to the wave-front surface.

In the case of piezoelectric media, we are not aware of a general demonstration of equivalent properties, though generalized expressions have long been known for the energy density and the Poynting vector [1]. Nevertheless, the collinearity of the generalized Poynting vector and the group velocity is routinely used to predict the beam-steering angle, or direction of energy transport. However, Zaitsev and Kuznetsova [3] recently identified a possible discrepancy that can occur if the generalized Poynting vector is used to predict the direction of energy transport in strong piezoelectrics such as lithium niobate. Their argument relies on a dissymmetry of the mechanoelectrical and electromechanical contributions to the generalized Poynting vector. Their work has motivated us to establish the equality of the energy and group velocities of BAW in an arbitrary lossless piezoelectric medium, with the energy velocity obtained from the generalized forms of the Poynting vector and the energy density. This property settles the apparent discrepancy.

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II. Basic Relations

We summarize in this section some well-known energetic relations for BAW propagating in linear piezoelectric media. Let us first consider a perturbance of an arbitrary linear piezoelectric medium, characterized by the strain S_{ij}, the stress T_{ij}, the electric field strength E_k, and the electric displacement D_k. These fields are related by the constitutive relations [1], [4]:

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \tag{1}$$
$$D_j = e_{jkl} S_{kl} + \epsilon_{jk} E_k, \tag{2}$$

where c_{ijkl}, e_{kij}, and ϵ_{jk} are the elastic, piezoelectric, and dielectric tensors, respectively. The repeated summation index convention is used throughout this paper. The time-averaged generalized energy density is given by [1]:

$$W = \frac{1}{2} \mathrm{Re} \, (T_{ij} S_{ij}^* + D_k E_k^*), \tag{3}$$

and the time-averaged generalized Poynting vector is:

$$P_j = \frac{1}{2} \mathrm{Re} \, (-T_{ij} v_i^* + (\mathbf{E} \times \mathbf{H}^*)_j), \tag{4}$$

where v_i is the particle velocity, H_i is the magnetic field strength, and * denotes complex conjugation.

We then consider specifically a time-harmonic plane wave with angular frequency ω and slowness vector s_j. The displacements u_i then are of the form:

$$u_i(\mathbf{x}, t) = u_i \exp(\jmath\omega(t - s_j x_j)), \tag{5}$$

and similar expressions hold for all fields. We specifically demand that the slowness vector be real valued, i.e., the medium is lossless and evanescent or inhomogeneous BAW are not considered.

Considering further the quasistatic approximation [1], Maxwell's equations give:

$$E_i = \jmath\omega s_i \Phi, \tag{6}$$
$$(\mathbf{s} \times \mathbf{H})_j = -D_j, \tag{7}$$
$$s_j D_j = 0, \tag{8}$$

where Φ is the electric potential. The constitutive relations become:

$$T_{ij} = -\jmath\omega (c_{ijkl} s_k u_l + e_{kij} s_k \Phi), \tag{9}$$
$$D_j = -\jmath\omega (e_{jkl} s_k u_l - \epsilon_{jk} s_k \Phi). \tag{10}$$

The generalized energy density simplifies to:

$$W = \frac{1}{2} \mathrm{Re} \, (\jmath\omega T_{ij} s_j u_i^*). \tag{11}$$

It can be seen that the electromagnetic part of the energy density vanishes. The generalized Poynting vector becomes:

$$P_j = \frac{1}{2}\mathrm{Re}\,(\jmath\omega(T_{ij}u_i^* + D_j\Phi^*)), \tag{12}$$

where $v_i = \jmath\omega u_i$ has been used in (4).

III. Energy Velocity

We now introduce the generalized notation:

$$\bar{c}_{ijkl} = \begin{cases} c_{ijkl} & i,l = 1,2,3 \\ e_{kij} & l = 4,\ i = 1,2,3 \\ e_{jkl} & i = 4,\ l = 1,2,3 \\ -\epsilon_{jk} & i,l = 4 \end{cases}. \tag{13}$$

This notation should not be confused with the piezoelectrically stiffened elastic constants [4]. With the generalized displacements, $\bar{u}_i$, and stresses, $\bar{T}_{ij}$, defined as:

$$\bar{u}_i = u_i,\ i = 1,2,3, \tag{14}$$

$$\bar{u}_4 = \Phi, \tag{15}$$

$$\bar{T}_{ij} = T_{ij},\ i = 1,2,3, \tag{16}$$

$$\bar{T}_{4j} = D_j, \tag{17}$$

the constitutive relations can be written in a compact way as:

$$\bar{T}_{ij} = -\jmath\omega\bar{c}_{ijkl}s_k\bar{u}_l, \tag{18}$$

and the Christoffel equation [1] governing the wave dynamics assumes the form:

$$\bar{c}_{ijkl}s_js_k\bar{u}_l = \rho_{il}\bar{u}_l, \tag{19}$$

with:

$$\rho_{il} = \rho\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Contracting the Christoffel (19) with $\bar{u}_i^*$ yields the relation:

$$\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^* = \rho_{il}\bar{u}_l\bar{u}_i^* = \rho u_i^*u_i. \tag{21}$$

Inserting (9) into (11) we obtain:

$$W = \frac{\omega^2}{2}\mathrm{Re}\,(c_{ijkl}s_js_ku_lu_i^* + e_{kij}s_js_k\Phi u_i^*), \tag{22}$$

or by inspection:

$$W = \frac{\omega^2}{2}\mathrm{Re}\,(\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^*). \tag{23}$$

Using (21), we observe that this quantity is real and thus:

$$W = \frac{\omega^2}{2}\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^* = \frac{\omega^2}{2}\rho u_i^*u_i. \tag{24}$$

This relation proves the equality of potential and kinetic energies for BAW as in the case of elastic media [2].

The expression for the Poynting vector similarly can be transformed by inserting (9) and (10) into (12), yielding:

$$\begin{aligned} P_j = \frac{\omega^2}{2}\mathrm{Re}\,(&c_{ijkl}s_ku_lu_i^* + e_{kij}s_k\Phi u_i^* \\ &+ e_{jkl}s_ku_l\Phi^* - \epsilon_{jk}s_k\Phi\Phi^*), \end{aligned} \tag{25}$$

or simply:

$$P_j = \frac{\omega^2}{2}\mathrm{Re}\,(\bar{c}_{ijkl}s_k\bar{u}_l\bar{u}_i^*). \tag{26}$$

Defining the energy velocity as for BAW in elastic media by the ratio of the Poynting vector to the energy density, we obtain at once:

$$(V_e)_j = \frac{P_j}{W} = \frac{\mathrm{Re}\,(\bar{c}_{ijkl}s_k\bar{u}_l\bar{u}_i^*)}{\rho u_i^*u_i}. \tag{27}$$

Furthermore, contracting this expression with s_j, the following useful relation is obtained:

$$s_j(V_e)_j = 1, \tag{28}$$

as in the case of elastic media [1], [2].

IV. Group Velocity

The components of the slowness vector can be written $s_i = sn_i$ for which the n_i are the components of a unit vector, i.e., $n_in_i = 1$. With this notation, the Christoffel (19) becomes a generalized eigenvalue equation for the square of the phase velocity, $V = s^{-1}$,

$$\bar{c}_{ijkl}n_jn_k\bar{u}_l = V^2\rho_{il}\bar{u}_l. \tag{29}$$

The group velocity is defined as [2]:

$$(V_g)_j = \frac{\partial V}{\partial n_j}, \tag{30}$$

which implies that the group velocity vector is normal to the slowness surface by construction. The group velocity vector can be obtained by differentiating with respect to n_j the identity formed by contracting (29) with $\bar{u}_i^*$, or:

$$\bar{c}_{ijkl}n_jn_k\bar{u}_l\bar{u}_i^* = V^2\rho_{il}\bar{u}_l\bar{u}_i^*, \tag{31}$$

which is merely a restatement of (21). In this equation, $\bar{u}_l$ is a function of the unit vector $\mathbf{n}$ because it is the eigenvector associated with the eigenvalue V^2 of the Christoffel (29) for a fixed propagation direction. The differentiation of (31) with respect to n_j results in:

$$\begin{aligned} &2\bar{c}_{ijkl}n_k\bar{u}_l\bar{u}_i^* + \bar{c}_{i\beta kl}n_\beta n_k\frac{\partial\bar{u}_l}{\partial n_j}\bar{u}_i^* + \bar{c}_{i\beta kl}n_\beta n_k\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j} \\ &= 2V(V_g)_j\rho_{il}\bar{u}_l\bar{u}_i^* + V^2\rho_{il}\frac{\partial\bar{u}_l}{\partial n_j}\bar{u}_i^* + V^2\rho_{il}\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j}. \end{aligned} \tag{32}$$

This equation can be simplified by considering two different contractions of the Christoffel equation (29). First, the contraction by $\frac{\partial \bar{u}_i^*}{\partial n_j}$ yields:

$$\bar{c}_{i\beta kl} n_\beta n_k \bar{u}_l \frac{\partial \bar{u}_i^*}{\partial n_j} = V^2 \rho_{il} \bar{u}_l \frac{\partial \bar{u}_i^*}{\partial n_j}. \tag{33}$$

Second, complex conjugation of (29) and subsequent contraction by $\frac{\partial \bar{u}_i}{\partial n_j}$ result in:

$$\bar{c}_{i\beta kl} n_\beta n_k \bar{u}_l^* \frac{\partial \bar{u}_i}{\partial n_j} = V^2 \rho_{il} \bar{u}_l^* \frac{\partial \bar{u}_i}{\partial n_j}. \tag{34}$$

Permutation of indices i and l, and β and k, respectively, and consideration of the symmetries $\bar{c}_{i\beta kl} = \bar{c}_{lk\beta i}$ and $\rho_{il} = \rho_{li}$ lead to:

$$\bar{c}_{i\beta kl} n_\beta n_k \bar{u}_i^* \frac{\partial \bar{u}_l}{\partial n_j} = V^2 \rho_{il} \bar{u}_i^* \frac{\partial \bar{u}_l}{\partial n_j}. \tag{35}$$

Eventually, (32) simplifies to:

$$(V_g)_j = \frac{\bar{c}_{ijkl} n_k \bar{u}_l \bar{u}_i^*}{V \rho u_i u_i^*} = \frac{\bar{c}_{ijkl} s_k \bar{u}_l \bar{u}_i^*}{\rho u_i u_i^*}. \tag{36}$$

This is identical with the expression (27) for the energy velocity and incidentally shows that the $Re(.)$ operators in (26) and (27) can be dropped. Thus we have shown the equality of energy and group velocities for BAW in a lossless piezoelectric medium.

V. Conclusions

We have demonstrated that the energy and group velocities for BAW are equal in a lossless piezoelectric medium. Our derivation closely parallels and generalizes for piezoelectric media the derivation of [2], that was limited to purely elastic media. The possible discrepancy pointed out in [3], that the mechanoelectrical and electromechanical power flows calculated from the generalized form of the Poynting vector for piezoelectric media do not compensate each other, then has no significance for the estimation of the power flow direction. Moreover, this observation is supported by the experimental results of Havlice *et al.* [5], who demonstrated that the generalized Poynting vector indeed predicts the correct beam-steering angle of longitudinal waves along the y axis of lithium niobate. We observed that attempting to separate the generalized energy density or the generalized Poynting vector into their purely electrical, purely mechanical, mechanoelectrical and electromechanical parts can be confusing, because the wave displacements are obtained as the solution of an eigenvalue problem involving the mixed elastic, electrical, and piezoelectric properties of the medium.

References

[1] B. A. Auld, *Acoustic Fields and Waves in Solids.* vol. 1, New-York: Wiley, 1973.

[2] D. Royer and E. Dieulesaint, *Elastic Waves in Solids.* vol. 1, New York: Wiley, 1999.

[3] B. D. Zaitsev and I. E. Kuznetsova, "The energy density and power flow of acoustic waves propagating in piezoelectric materials," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 50, no. 12, pp. 1762–1765, 2003.

[4] "IEEE Standard on Piezoelectricity 176-1987," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 43, no. 5, p. 717, 1996.

[5] J. F. Havlice, W. L. Bond, and L. B. Wigton, "Elastic Poynting vector in a piezoelectric medium," *IEEE Trans. Sonics Ultrason.*, vol. SU-17, no. 4, pp. 246–249, 1970.