Noise analysis of the measurement of group delay in Fourier white-light interferometric cross correlation

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The problem of noise analysis in measuring the group delay introduced by a dispersive optical element by use of white-light interferometric cross correlation is investigated. Two noise types, detection noise and position noise, are specifically analyzed. Detection noise is shown to be highly sensitive to the spectral content of the white-light source at the frequency considered and to the temporal acquisition window. Position noise, which arises from the finite accuracy of the measurement of the scanning mirror's position, can severely damage the estimation of the group delay. Such is shown to be the case for fast Fourier transform–based estimation algorithms. A new algorithm that is insensitive to scanning delay errors is proposed, and subfemtosecond accuracy is obtained without any postprocessing. © 2002 Optical Society of America

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1. INTRODUCTION

Measuring dispersion with subfemtosecond accuracy has been a longstanding problem. For the propagation and the generation of femtosecond light pulses, knowledge of the dispersion introduced by optical elements is compulsory. With the advent of chirped mirrors, which control the group delay on reflection on a femtosecond scale, this problem has become even more acute.[1,2] However, chirped mirrors are complex dielectric stacks, and deposition errors can cause the actual dispersion to move away from the design goal, hence causing the actual dispersion to differ notably from the predicted dispersion.[3] It is then essential to have an apparatus that can measure the group delay as a function of frequency with the best possible accuracy. The most widespread, and arguably the most precise, apparatus used for measuring the dispersion of mirrors or other samples is white-light interferometric cross correlation,[4] as depicted in Fig. 1. This instrument is basically a Michelson interferometer with one scanning mirror. The intensity is recorded as a function of the delay and is written as

$$I(\tau) = I_0 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega S(\omega)\exp\{-i[\omega\tau - \phi(\omega)]\}, \tag{1}$$

where I_0 is a constant, $S(\omega)$ is the spectral intensity or spectrum, and $\phi(\omega)$ is the spectral phase. We refer to $I(\tau)$ as the cross-correlation signal. The spectral phase is the difference of the phase on reflection over each mirror plus the dispersion of the beam splitter, and the spectrum is that of the white-light source filtered by the optical elements in the setup.

From the measurement described by Eq. (1), the spectrum and the phase can be obtained as the solution of an inverse problem. The chosen estimation algorithm will engender a trade off among accuracy, robustness to noise, and processing speed. In the setup reported in Ref. 4, a tunable narrow-band filter was used to alter the spectrum of the white-light source; then the group delay was computed as the center of each filtered light packet. This procedure is quite slow, because the filter has to be tuned for every frequency, but in turn it gives precise and stable results, although part of the apparent robustness to noise should be attributed to heavy spectral smoothing over the bandwidth of the filter. For an 8-nm-wide tunable filter near 800 nm, subfemtosecond precision was achieved.

In a proposed setup later,[5] Naganuma *et al.* avoided the use of a tunable filter by computing the fast Fourier transform (FFT) of the interferogram obtained with the whole white-light spectrum, i.e., of a sequence of sampling points in delay. This computation yielded at once the group-delay variation as a function of frequency, providing fast operation. We note, however, that there are several drawbacks in the FFT method that were not present in the first version of Ref. 4. First, the Fourier transform of Eq. (1) yields the spectral phase $\phi(\omega)$, and the group delay has to be obtained by differentiation, which is known to amplify estimation noise. Second, the FFT requires equally spaced sampling points in both the temporal and the spectral domains. In the research reported in Ref. 5, a He–Ne laser ranging technique is used to sample $I(\tau)$ at intervals of $\lambda/4$. Although the Nyquist criterion is satisfied, i.e., the signal is correctly sampled, use of a large spectral bandwidth results in a cross-correlation signal that is strongly concentrated about $\tau = 0$, so few sampling points actually convey useful information. As a result, sensitivity to noise is not optimized, and heavy data smoothing has to be performed to yield reasonable accuracy.[5,6]

Here we investigate the noise properties of the measurement of the group delay in Fourier white-light interferometric cross correlation. In Section 2 some prerequi-

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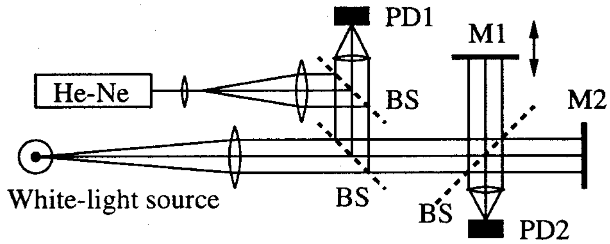


Fig. 1. White-light interferometric cross-correlation setup. Photodiode PD1 monitors the white-light dispersion signal and photodiode PD2 monitors the He–Ne. laser position signal (BSs, beam splitters; M1, scanning mirror; M2, mirror).

site definitions for the analysis are given. We consider two main sources of noise, in the detection process and in the position of the scanning mirror, which are discussed in Sections 3 and 4, respectively. Position noise arises, for instance, from optical detection of the scanning mirror's position. Section 5 gives some group-delay measurement examples and illustrates the main results of Sections 3 and 4.

Before turning to the main subject of this paper, let us note that the technique of white-light interferometric cross correlation is closely related to the technique of spectral interferometry, in which the signals from the two arms of the Michelson interferometer that have a fixed delay between them are made to interfere inside a spectrometer.[7,8] More precisely, the measurement provided by spectral interferometry is the Fourier transform of Eq. (1). An important difference, however, is that the dispersion information is sampled temporally for white-light interferometric cross correlation and spectrally for spectral interferometry. Although position noise is meaningless to spectral interferometry, other sources of error are specific to this technique.[9] A comparison of the noise sensitivities of the two techniques would be instructive but is however beyond the scope of this paper.

2. ESTIMATION OF THE GROUP DELAY

Figure 2(a) shows an example of a raw measurement of the cross-correlation signal as a function of the delay. This measurement was obtained with the setup described in Section 5 below that was devoted to experimental results but is shown here to facilitate the discussion. There are 40,000 samples taken over a 2.1-ps delay range at a rate of 20,000 samples/s; i.e., the total scanning time is ~2 s. The cross-correlation signal is a rapidly oscillating function of the delay. Figure 2(b) is a close-up of the central region of Fig. 2(a) from which the oscillatory behavior can better be seen. Figure 2(c) shows the recorded intensity for the He–Ne fringes. The position of the mirror is estimated from this signal. It can be seen that the period of the fringes fluctuates, indicating the influence of the low-frequency noise that can arise, for instance, from vibrations in the optical table, which is not stabilized in this case. As a consequence the samples are not taken to be uniform in delay, though they are in time, but the position of the mirror can still be monitored quite precisely.

It is well known that the spectrum and the spectral phase can be obtained by a Fourier transform of the measurement modeled by Eq. (1):

$$S(\omega)\exp[i\phi(\omega)] = F_T[I](\omega), \tag{2}$$

where the following notation has been used for the windowed Fourier transform:

$$F_T[f](\omega) = \frac{1}{T}\int_{-T/2}^{+T/2} f(\tau)\exp(i\omega\tau)d\tau. \tag{3}$$

In Eq. (2) the normalization by delay range T has been arbitrarily introduced, so both the spectrum and the cross-correlation signal are given in the same units, i.e., volts; this choice has no practical consequence for the forthcoming analysis and simplifies the expressions. Of course, Eq. (2) is an idealization when the delay window is sufficiently large to contain all useful data.

The group delay is defined as the derivative of the spectral phase, according to

$$t_g(\omega) = \frac{\mathrm{d}\phi(\omega)}{\mathrm{d}\omega} = \phi'(\omega). \tag{4}$$

Differentiating Eq. (2), we obtain

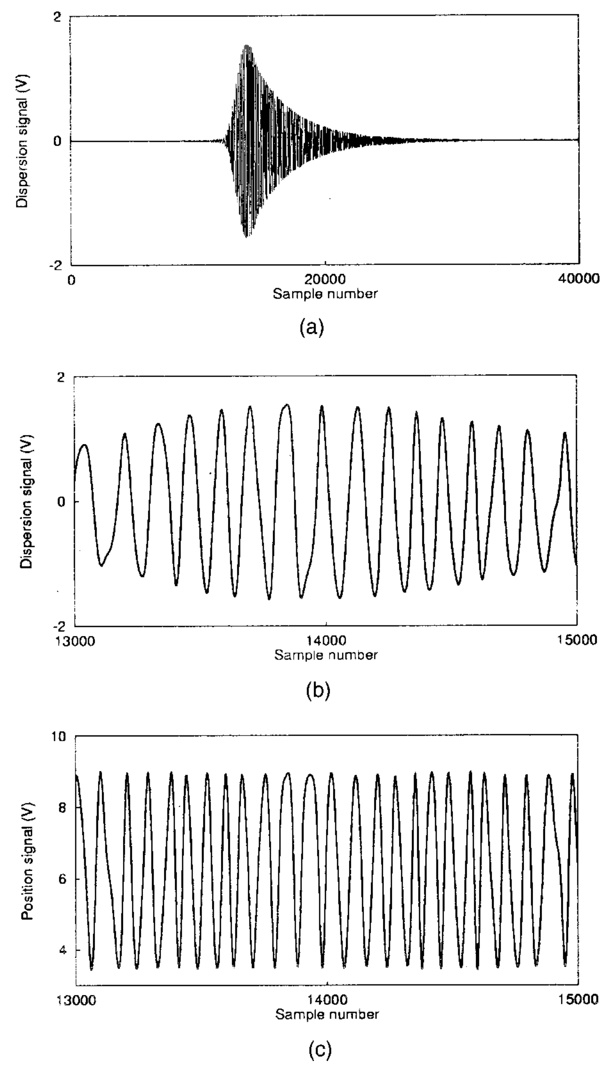


Fig. 2. Example of a cross-correlation measurement: (a) Dispersion signal on photodiode PD1 with white-light illumination, (b) closeup of (a), (c) position signal on photodiode PD2 with 632.8-nm He–Ne laser illumination.

$$[S'(\omega) + i\phi'(\omega)S(\omega)]\exp[i\phi(\omega)] = iF_T[\tau I](\omega). \quad (5)$$

We then note that the group delay can be estimated from the following integral formula:

$$t_g(\omega) = \Re\frac{\int_{-T/2}^{+T/2} \mathrm{d}\tau\tau I(\tau)\exp(i\omega\tau)}{\int_{-T/2}^{+T/2} \mathrm{d}\tau I(\tau)\exp(i\omega\tau)} = \Re\frac{F_T[\tau I](\omega)}{F_T[I](\omega)}, \quad (6)$$

where $\Re$ stands for the real part. The imaginary part is connected to the variations of the spectrum through

$$\frac{S'(\omega)}{S(\omega)} = -\Im\frac{F_T[\tau I](\omega)}{F_T[I](\omega)}, \quad (7)$$

where $\Im$ stands for the imaginary part. All expressions until now were written with continuous integrals. However, in the experiment we have access only to a certain number of sampling points, from which the integrals have to be estimated by use of some quadrature formula (see, e.g., Ref. 10 for practical quadrature formulas). The sampling points will not in general be equally spaced in delay if the experimental apparatus does not specifically impose the delay[5] or if resampling is performed. Assuming N equally spaced sampling points, we can approach the windowed Fourier transform of Eq. (3) by the simplest quadrature formula as

$$F_T[I](\omega) \simeq \frac{1}{N}\sum_p I(\tau_p)\exp(i\omega\tau_p), \quad (8)$$

in which $d\tau$ has been approximated by T/N. If this integral has to be evaluated only at equally spaced angular frequencies separated by $\delta\omega = 2\pi/T$, the summation in expression (8) will be the discrete Fourier transform, which can be computed efficiently by use of a FFT algorithm. For a typical temporal window of 2 ps, as used in this study, the spectral spacing is 500 GHz, or approximately 1 nm for a wavelength of 800 nm.

Approaching this continuous integral with a finite summation over sampling points requires care that is not taken in the FFT algorithm. It is well known that the FFT is relatively immune to additive white detection noise. However, another source of error stems from the positioning of the scanning mirror. Even a $\lambda/100$ positioning error will result in degraded estimation performance, because τ is the integrant of the two integrals in Eq. (6), as is shown in Section 4 below.

3. DETECTION NOISE

In this section we discuss the sensitivity of the estimation of the group delay with respect to detection noise. This analysis is rather independent of the choice made regarding the evaluation of integrals of the type of Eq. (3); for convenience we choose the approximation of expression (8), as it simplifies the mathematics. We assume that detection noise is a white Gaussian stochastic process, i.e., that many photons are available for each measurement. The measured intensity is then written for each sample as

$$J(\tau_p) = I(\tau_p) + n(\tau_p), \quad (9)$$

where $I(\tau_p)$ is the true cross-correlation signal given by Eq. (1). We have

$$\langle n(\tau_p)\rangle = 0 \quad (10)$$

$$\langle n(\tau_p)n(\tau_q)\rangle = \sigma^2\delta_{p-q}, \quad (11)$$

where $\langle\rangle$ denotes an ensemble average, σ^2 is the noise variance, and δ_{p-q} is the Kronecker delta function. A noise analysis of the estimation procedure requires the evaluation of stochastic integrals of the form

$$F_T[nf](\omega) \simeq \frac{1}{N}\sum_p n(\tau_p)f(\tau_p)\exp(i\omega\tau_p). \quad (12)$$

It is straightforward to show that

$$\langle F_T[nf](\omega)\rangle = 0 \quad (13)$$

and that

$$\langle F_T[nf](\omega)F_T[ng](\omega)^*\rangle = \frac{\sigma^2}{N^2}\left[\sum_p f(\tau_p)g^*(\tau_p)\right], \quad (14)$$

with the special cases that $\langle F_T[nf](\omega)^2\rangle$ equals σ^2/N and $\sigma^2T^2/(12N)$ if $f(\tau) = 1$ and $f(\tau) = \tau$, respectively.

Let us first consider the estimation of the spectrum and of the phase, which we obtain by computing the Fourier integral $F_T[J](\omega)$. This estimation is unbiased because

$$\langle F_T[J](\omega)\rangle = F_T[I](\omega) = S(\omega)\exp[i\phi(\omega)], \quad (15)$$

and the variance of the estimation is

$$\begin{aligned}\mathrm{VAR}_1 &= \langle|F_T[J](\omega) - F_T[I](\omega)|^2\rangle \\ &= \langle|F_T[n](\omega)|^2\rangle = \sigma^2/N. \quad (16)\end{aligned}$$

This estimation is seen to be independent of the spectral content of the white-light source, and moreover the signal-to-noise ratio (SNR) improves proportionally to the number of samples, as is usual for sums of independent measurements:

$$\mathrm{SNR}_1 = N\frac{S^2(\omega)}{\sigma^2}. \quad (17)$$

The situation is much different for the estimation of the group-delay, as we shall now see. To obtain tractable expressions we expand the estimator to first order in the detection noise; by this procedure we seek approximate expressions for the mean and the variance of the estimator. We write the group-delay estimator as

$$\hat{t}_g(\omega) = \Re\frac{F_T[\tau J](\omega)}{F_T[J](\omega)}. \quad (18)$$

Its first-order expansion reads as

$$\hat{t}_g(\omega) \simeq \Re\frac{F_T[\tau I](\omega)}{F_T[I](\omega)}\left[1 + \frac{F_T[\tau n](\omega)}{F_T[\tau I](\omega)} - \frac{F_T[n](\omega)}{F_T[I](\omega)}\right]. \quad (19)$$

To first order the estimator is unbiased because

$$\langle\hat{t}_g(\omega)\rangle \simeq \Re\frac{F_T[\tau I](\omega)}{F_T[I](\omega)} = t_g(\omega). \quad (20)$$

It is shown in Appendix A that the estimator variance is

$$\mathrm{VAR}_2 = \frac{1}{2N}\frac{\sigma^2}{S^2(\omega)}\left\{\frac{T^2}{12} + t_g{}^2(\omega) + \left[\frac{S'(\omega)}{S(\omega)}\right]^2\right\}. \quad (21)$$

It can be seen that this estimation variance is inversely proportional to the SNR of the estimation of the spectrum; i.e., the more energy for a particular frequency in the spectrum, the better the estimation of the group delay at that frequency. Moreover, the estimation degrades with the size of the delay window T, as is probably less natural. This is so because the useful signal occupies a given delay window, although whether this is so cannot be known before the measurement, and because adding more samples outside this window only introduce more noise into the estimation. It can be noted that this effect is absent from the estimation of the spectrum. The presence of the last two terms in braces in Eq. (21) indicates that the estimation degrades with the dispersion of the signal and with the variations of the spectrum, respectively. These last-named effects will in general be smaller than the effect of the delay window, especially if care is taken that the maximum group delay that has to be measured be smaller than the delay window used for measurement. However, Eq. (21) also shows that the spectrum of the light source should be chosen to be very smooth. In any case, the standard deviation for the estimation of the group delay is seen to have a lower bound as

$$\sqrt{\mathrm{VAR}_2} \geqslant \frac{T}{2\sqrt{6N}}\frac{\sigma}{S(\omega)}. \quad (22)$$

A numerical example of the influence of detection noise is given in Section 5 below.

4. POSITION NOISE

Let us write the general sampled version of Eqs. (2) and (6), using an arbitrary quadrature formula, as

$$S(\omega)\exp[i\phi(\omega)] = \sum_p W_p g(\tau_p), \quad (23)$$

$$t_g(\omega) = \Re\frac{\sum_p W_p f(\tau_p)}{\sum_p W_p g(\tau_p)}, \quad (24)$$

where W_p are weighting factors, τ_p are sampling delays, $f(\tau) = \tau I(\tau)\exp(i\omega\tau)$, and $g(\tau) = I(\tau)\exp(i\omega\tau)$. For the FFT case, delay samples τ_p are equally sampled, and the weights $W_p = 1/N$ are uniform. It is the simplest quadrature formula that one can think of. For accuracy, it is generally preferable to use a more-refined quadrature formula, e.g., a Newton–Cotes or a Gauss formula. In these cases the weights depend explicitly on the delay samples τ_p.

As was discussed in Section 1, the actual position of the scanning mirror cannot be known with arbitrary precision; i.e., there is noise in the delay samples. In Eqs. (23) and (24), errors in τ_p will to the first order be proportional to the derivatives of functions f and g and possibly of the weights. Indeed, if we write

$$\tau_p = \bar{\tau}_p + \vartheta_p, \quad (25)$$

where $\bar{\tau}_p$ is the actual mirror delay and ϑ_p is the error, then

$$\sum_p W_p f(\tau_p) \simeq \sum_p W_p f(\bar{\tau}_p) + \sum_p W_p \vartheta_p f'(\bar{\tau}_p) + \sum_p \sum_q \vartheta_q \frac{\partial W_p}{\partial \tau_q} f(\bar{\tau}_p), \quad (26)$$

and a similar equation holds for function $g(\tau)$. In expression (26) the weights should be chosen to minimize sensitivity to position noise. In practice, the influence of position noise on the group-delay estimation given by Eq. (24) will be much more noticeable on the integral of function f than it will be for the integral of function g because the first involves a further multiplication of its integrant by the delay. We can then approximate the effect of position noise by transforming Eq. (24) into

$$t_g(\omega) = \Re\frac{\sum_p W_p f(\bar{\tau}_p)}{\sum_p W_p g(\bar{\tau}_p)} + \Re\frac{\sum_p W_p \vartheta_p f'(\bar{\tau}_p) + \sum_p \sum_q \vartheta_q \frac{\partial W_p}{\partial \tau_q} f(\bar{\tau}_p)}{\sum_p W_p g(\bar{\tau}_p)}. \quad (27)$$

A. Case of the Fast Fourier Transform

In the case of the FFT, the weights are constant, and expression (26) becomes

$$\frac{1}{N}\sum_p f(\tau_p) \simeq \frac{1}{N}\sum_p f(\bar{\tau}_p) + \frac{1}{N}\sum_p \vartheta_p f'(\bar{\tau}_p). \quad (28)$$

We assume that the position noise has white-noise statistics, at least to second order; i.e., that

$$\langle \vartheta_p \rangle = 0, \quad (29)$$

$$\langle \vartheta_p \vartheta_q \rangle = \sigma_\vartheta{}^2 \delta_{p-q}. \quad (30)$$

From these definitions it is easy to obtain that the estimation is unbiased, because its mean is

$$\left\langle \frac{1}{N}\sum_p f(\tau_p) \right\rangle \simeq \frac{1}{N}\sum_p f(\bar{\tau}_p), \quad (31)$$

and that its variance is

$$\mathrm{VAR}_3 \simeq \frac{\sigma_\vartheta{}^2}{N^2}\sum_p |f'(\bar{\tau}_p)|^2 = \sigma_\vartheta{}^2 h^2 T^{-2}\sum_p |f'(\bar{\tau}_p)|^2, \quad (32)$$

where we have used the notation $h = T/N$. The variance of the estimation of the group delay can then be approximated as

$$\mathrm{VAR}_4 \simeq \frac{\sigma_\vartheta{}^2 h^2 \sum_p |f'(\bar{\tau}_p)|^2}{T^2 S^2(\omega)}. \tag{33}$$

As can be seen from Fig. 2, $f(\tau)$ is a fast varying function of τ, which implies that, even if position noise variance $\sigma_\vartheta{}^2$ is small, the estimation variance will be increased notably, as is shown by a numerical example in Section 5 below.

B. Case of the Trapezoidal Rule

Given a sequence of sampling points in an interval on which a function is to be integrated, we can set up Newton–Cotes formulas by specifying that all polynomials to a given degree be integrated exactly.[10] For instance, for the trapezoidal rule this degree is 2 and for Simpson's rule it is 3. When the sampling points are not known in advance, the trapezoidal rule is the only formula that will retain an unchanging expression, which is given by

$$\sum_p W_p f(\tau_p) = \frac{1}{2T} \sum_{p=2}^{N} (\tau_p - \tau_{p-1})[f(\tau_p) + f(\tau_{p-1})]. \tag{34}$$

It is shown in Appendix B that to first order in position noise the estimation is again unbiased and that its variance is approximately

$$\mathrm{VAR}_5 \simeq \frac{\sigma_\vartheta{}^2}{16T^2} \sum_{p=2}^{N-1} \left| f''(\bar{\tau}_p)(\delta\tau_p{}^2 - \delta\tau_{p+1}{}^2) - \frac{1}{3} f'''(\bar{\tau}_p)(\delta\tau_p{}^3 + \delta\tau_{p+1}{}^3) \right|^2, \tag{35}$$

where $\delta\tau_p = \bar{\tau}_p - \bar{\tau}_{p-1}$. Expression (35) shows that the estimation error for the trapezoidal rule is at least of second order with regard to the sampling interval, whereas it is of first order for the FFT, as is apparent from expression (32). The variance of the estimation of the group delay can then be approximated as

$$\mathrm{VAR}_6 \simeq \frac{\sigma_\vartheta{}^2 \sum_{p=2}^{N-1} \left| f''(\bar{\tau}_p)(\delta\tau_p{}^2 - \delta\tau_{p+1}{}^2) - \frac{1}{3} f'''(\bar{\tau}_p)(\delta\tau_p{}^3 + \delta\tau_{p+1}{}^3) \right|^2}{16T^2 S^2(\omega)}. \tag{36}$$

Furthermore, if the signal is equally sampled, then $\delta\tau_p = h$, and the estimation error is of third order:

$$\mathrm{VAR}_5 \simeq \frac{\sigma_\vartheta{}^2}{36T^2} h^6 \sum_{p=2}^{N-1} |f'''(\bar{\tau}_p)|^2, \tag{37}$$

and the variance of the estimation of the group delay can then be approximated as

$$\mathrm{VAR}_6 \simeq \frac{\sigma_\vartheta{}^2 h^6 \sum_{p=2}^{N-1} |f'''(\bar{\tau}_p)|^2}{36T^2 S^2(\omega)}, \tag{38}$$

as is shown by a numerical example in Section 5.

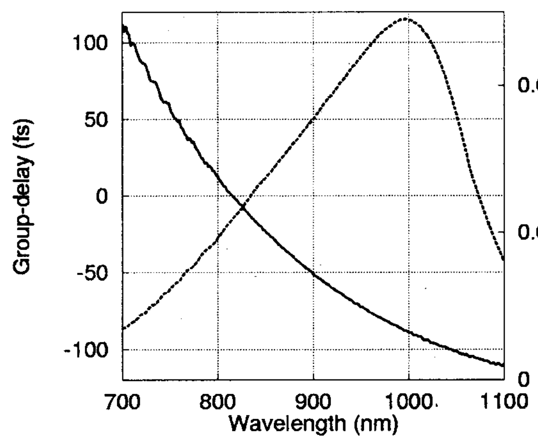
C. Comparison

A direct comparison of expressions (33) and (36) or (38) for the FFT and the trapezoidal rule, respectively, proves that the latter has better robustness to position noise because, when one is using the trapezoidal rule, the factor h is raised to the sixth power, whereas with the FFT it is raised only to the second power. The sums involved in these expressions are of the same order, because the integrated functions are fast oscillating and can be approximated by the amplitude-modulated sinusoid. This fact will be confirmed by the numerical examples given in Section 5. In addition, in both cases using more sampling points helps to improve accuracy, because the estimation SNR depends dramatically on the sampling interval $h = T/N$. Using more sampling points would be seen as oversampling the signal in the FFT sense, because according to the Nyquist criterion the number of samples is limited by the bandwidth of the signal, but an increase in the number of samples greatly reduces the noise sensitivity. Such is also true for the case of detection noise, because the estimation variance of Eq. (21) is inversely proportional to the number of samples N, but in this case the improvement is the same for both the FFT and the trapezoidal rule.

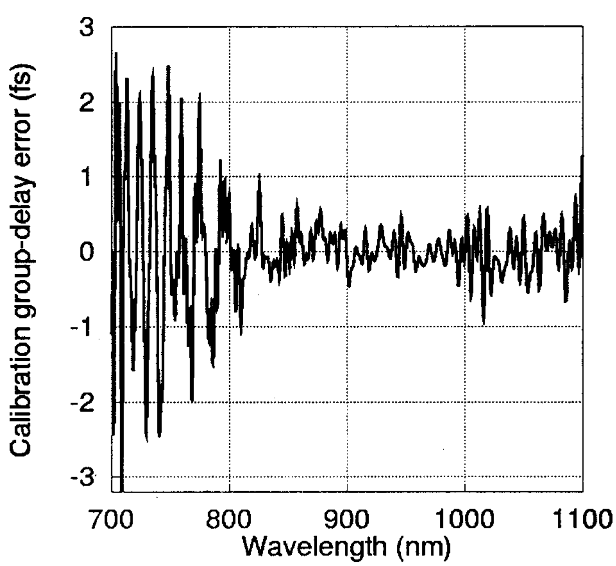
5. EXPERIMENTS

We set up a white-light interferometric cross correlator that is basically similar to that described in Ref. 5. One difference is that we take sampling points at approximately $\lambda/40$ intervals instead of at $\lambda/4$ intervals, where $\lambda = 632.8$ nm is the He–Ne laser wavelength. Furthermore, we do not impose the condition that these sampling points be equally spaced, unlike for the FFT, because the trapezoidal rule of Eq. (34) is used. With ~40,000 sampling points, a temporal window of 2.1 ps is scanned. The total scanning time is 2 s, with a sampling rate of 20,000/s. Figure 3(a) shows a typical calibration result obtained with two bare gold mirrors. The observed dispersion is that of the beam splitter. No data smoothing was performed. It can be observed that the accuracy is maximum where the spectral content is highest, i.e., near 1 μm. Figure 3(b) shows the difference between the measured group delay and a polynomial fit to illustrate the variations in degree of accuracy.

The group delay's standard deviation owing to detection noise in the experiment is estimated from inequality (22) as follows: The detection noise is taken to be mostly the quantization noise of the 12-bit analog-to-digital converter with a signal range of 4 V, resulting in a standard deviation of approximately 2.8×10^{-4} V. For comparison, a hypothetical 16-bit analog-to-digital converter is also considered, with the assumption that the detection noise is still given by the quantization noise. Figure 4 shows the standard deviation as a function of frequency, in femtoseconds. From expressions (33) and (36), and using the experimental data of Fig. 2 that were used for the estimation results shown in Fig. 3, we can compute the group-delay variances caused by position noise that would be obtained with a FFT algorithm and with the trapezoidal rule of Eq. (34), respectively. Figure 4 shows the standard deviation as a function of frequency as computed from expressions (33) and (36), assuming a conservative value of $\lambda/100$ for position accuracy. It can be seen that using the trapezoidal rule, greatly reduces the effect of the position noise compared with the noise for the FFT algorithm. In the reported experiments the standard deviation that is due to position noise when the trapezoidal rule is used is then negligible compared with that which is due to detection noise with a 12-bit analog-to-digital converter, which would not have been the case for a FFT-based estimation algorithm. Assuming a 16-bit analog-to-digital converter, we predict that the position and detection noise will be comparable.



(a)



(b)

Fig. 3. Calibration example: (a) calibration intensity and group delay, (b) difference between the calibration group delay and a polynomial fit to third order.

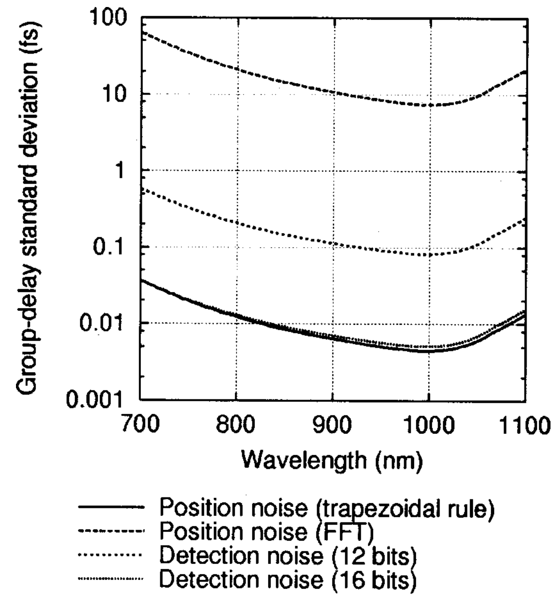


Fig. 4. Group-delay standard deviation caused by detection and position noise.

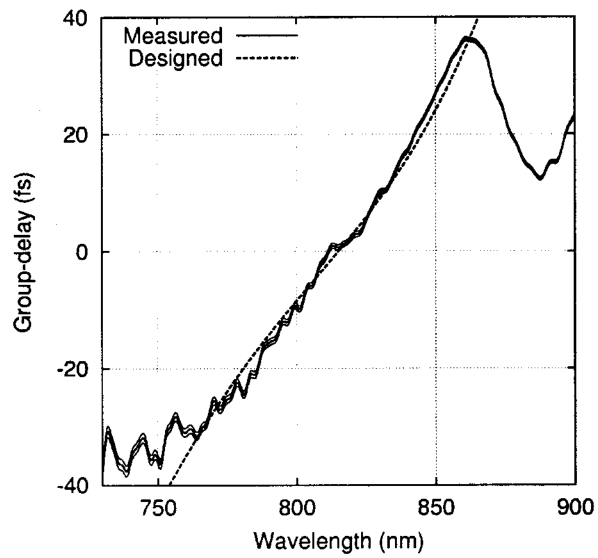


Fig. 5. Chirped-mirror measurement. The two thin solid curves are one standard deviation away from the measurement (thicker solid curve).

Figure 5 shows the measured group delay on reflection from a chirped mirror designed for the 765–865-nm bandwidth as related to compression of ultrashort pulses from a Ti:sapphire laser in air.[11] Note that this bandwidth is not the best part of the spectrum for estimation accuracy, as one can see from Fig. 4. The design was specified to fifth order with a group-delay dispersion of $-200\ \mathrm{fs}^2$, a third-order dispersion of $500\ \mathrm{fs}^3$, a fourth-order dispersion of $-16{,}250\ \mathrm{fs}^4$, and a fifth-order dispersion of $125{,}000\ \mathrm{fs}^5$.

Two thin solid curves were added at a distance of one standard deviation away from the measurement (thick solid curve) to show the accuracy of the measurement. The oscillations in the measured group delay can be attributed to the mirror itself and not to the measurement.

6. CONCLUSION

We have investigated the problem of noise analysis during the measurement of the group delay introduced by a dispersive optical element, using white-light interferometric cross correlation. Two noise types, i.e., detection noise and position noise, were specifically analyzed. Detection noise is the additive noise that arises from measurement of the intensity of the white-light fringes with a photodetector. Detection noise was shown to be highly sensitive to the spectral content of the white-light source at the frequency considered and to the temporal acquisition window. Position noise, which arises from the finite accuracy of the measurement of the scanning mirror's position, was found to be a severe source of error in the estimation of the group delay with discrete Fourier-transform-based estimation algorithms. A new estimation algorithm that is insensitive to position noise was proposed, and subfemtosecond accuracy was obtained without any postprocessing.

APPENDIX A

In this appendix we demonstrate Eq. (21). Starting from expressions (19) and (20), the variance of the estimation is

$$\langle [t_g(\omega) - \hat{t}_g(\omega)]^2 \rangle \simeq \left\langle \left[\Re \frac{F_T[\tau n](\omega)}{F_T[I](\omega)} - \Re \frac{F_T[\tau I](\omega) F_T[n](\omega)}{F_T[I](\omega)^2} \right]^2 \right\rangle, \quad \text{(A1)}$$

which we write as

$$\langle [t_g(\omega) - \hat{t}_g(\omega)]^2 \rangle \simeq \frac{1}{4} \left\langle \left[\frac{F_T[\tau n](\omega)}{F_T[I](\omega)} - \frac{F_T[\tau I](\omega) F_T[n](\omega)}{F_T[I](\omega)^2} + \text{c.c.} \right]^2 \right\rangle, \quad \text{(A2)}$$

where c.c. stands for the complex conjugate. The right-hand side of expression (A2) has 10 terms. However, six of them are proportional to expectations of the form

$$\langle F_T[nf](\omega) F_T[ng](\omega) \rangle = \frac{\sigma^2}{N^2} \left[\sum_p f(\tau_p) g(\tau_p) \exp(2i\omega\tau_p) \right] \quad \text{(A3)}$$

or its complex conjugate, where functions f and g are either 1 or τ, or approximately

$$\langle F_T[nf](\omega) F_T[ng](\omega) \rangle = \frac{\sigma^2}{NT} \left[\int_{-T/2}^{T/2} f(\tau) g(\tau) \exp(2i\omega\tau) d\tau \right]. \quad \text{(A4)}$$

The integrants in Eq. (A4) are rapidly oscillating functions of the delay, resulting in destructive interference. They can then be neglected. Furthermore, we have

$$\langle F_T[\tau n](\omega) F_T[n](\omega)^* \rangle = 0. \quad \text{(A5)}$$

Then the variance of expression (A2) has only two dominant terms and can be written as

$$\text{VAR}_2 \simeq \frac{1}{2|F_T[T](\omega)|^2} \left[\langle |F_T[\tau n](\omega)|^2 \rangle + \frac{|F_T[\tau I](\omega)|^2}{|F_T[I](\omega)|^2} \langle |F_T[n](\omega)|^2 \rangle \right]. \quad \text{(A6)}$$

Using Eqs. (2), (5) and (14), we conclude that

$$\text{VAR}_2 = \frac{1}{2N} \frac{\sigma^2}{S^2(\omega)} \left\{ \frac{T^2}{12} + t_g^2(\omega) + \left[\frac{S'(\omega)}{S(\omega)} \right]^2 \right\}, \quad \text{(A7)}$$

which is Eq. (21).

APPENDIX B

In this appendix we demonstrate expression (35). Inserting Eq. (25) into Eq. (34), we have for the integral of any function f evaluated from the trapezoidal rule

$$\sum_p W_p f(\tau_p) = \bar{S} + \frac{1}{2T} \sum_{p=2}^{N} [(\vartheta_p - \vartheta_{p-1})(f_p + f_{p-1}) + (\bar{\tau}_p - \bar{\tau}_{p-1})(\vartheta_p f'_p + \vartheta_{p-1} f'_{p-1})], \quad \text{(B1)}$$

with the notation $f_p = f(\bar{\tau}_p)$ and

$$\bar{S} = \sum_p W_p f_p . \quad \text{(B2)}$$

Neglecting side effects for the first and the last samples and performing index shifts, we find for Eq. (B1) that

$$\sum_p W_p f(\tau_p) = \bar{S} + \frac{1}{2T} \sum_{p=2}^{N-1} \vartheta_p \times [f_{p-1} - f_{p+1} + (\bar{\tau}_{p+1} - \bar{\tau}_{p-1}) f'_p]. \quad \text{(B3)}$$

Then, using the Taylor expansion of function f to third order,

$$f(\bar{\tau}_p + \epsilon) \simeq f_p + \epsilon f'_p + \frac{\epsilon^2}{2} f''_p + \frac{\epsilon^3}{6} f'''_p , \quad \text{(B4)}$$

yields for Eq. (B3)

$$\sum_p W_p f(\tau_p) = \bar{S} + \frac{1}{4T} \sum_{p=2}^{N-1} \vartheta_p \times \left[f''_p (\delta\tau_p^2 - \delta\tau_{p+1}^2) - \frac{1}{3} f'''_p (\delta\tau_p^3 + \delta\tau_{p+1}^3) \right] \quad \text{(B5)}$$

with the notation $\delta\tau_p = \bar{\tau}_p - \bar{\tau}_{p-1}$. Taking the expectation of Eq. (B5), we obtain that the estimation is unbiased, with a variance given by expression (35).

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