Homogenization of the spectral equation in one-dimension

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Abstract

The asymptotic behavior of a one-dimensional spectral problem with periodic coefficient is addressed for high frequency modes by a method of Bloch wave homogenization. The analysis leads to a spectral problem including both microscopic and macroscopic eigenmodes. Numerical simulation results are provided to corroborate the theory.
Keywords. Homogenization, Bloch waves, spectral problem, two-scale transform.

Contents

1 Introduction **2**

2 Statement of the problem **3**

3 Notations and elementary properties **4**

4 Homogenization of the high-frequency eigenvalue problem **7**
4.1 Main result 8
4.2 Modal decomposition on the Bloch modes 11
4.3 Derivation of the macroscopic equation 12
4.3.1 Case $k \neq 0$ 12
4.3.2 Case $k = 0$ 14
4.4 Analytic solutions 15
4.4.1 The case $\rho = 1$ 15
4.4.2 The case $a = \rho = 1$ 16
4.5 Neumann boundary conditions 17

5 Homogenization based on a first order formulation **17**
5.1 Reformulation of the spectral problem and the main result 17
5.2 Model derivation 19
5.2.1 Modal decomposition on the Bloch modes 19
5.2.2 Derivation of the macroscopic equation 20

6 Numerical simulations **22**
6.1 Simulation methods and conditions 22
6.2 Approximation of physical modes by two-scale modes 23
6.3 The modeling problem 27
6.4 Order of convergence 28

1 Introduction

We consider the spectral problem

$$-\partial_x \left(a^\varepsilon \partial_x w^\varepsilon\right) = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \tag{1}$$

posed in an one-dimensional open bounded domain $\Omega \subset \mathbb{R}$ with Dirichlet boundary conditions. An asymptotic analysis of this problem is carried out where $\varepsilon > 0$ is a parameter tending to zero and the coefficients are ε-periodic, namely $a^\varepsilon = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon = \rho\left(\frac{x}{\varepsilon}\right)$ where $a(y)$ and $\rho(y)$ are 1-periodic in $\mathbb{R}$. The homogenization of such spectral problem has been studied in various works providing the asymptotic behaviour of eigenvalues and eigenvectors. The low frequency part of the spectrum has been investigated in [17], [18], [25]. Then, many configurations have been analyzed, as [16] and [13] for a fluid-structure interaction, [7], [3] for neutron transport, [22], [24] for ρ which changes sign or [4] for the first high frequency eigenvalue and eigenvector for a one-dimensional non-self-adjoint problem with Neumann boundary conditions. In [6], G. Allaire and C. Conca studied the asymptotic behaviour of both the low and high frequency spectrum. In order to analyze the asymptotic behaviour of the high frequency eigenvalues, they used the Bloch wave homogenization method. It is a blend of two-scale convergence, see e.g. [1], [2], [21], and Bloch wave decomposition, see e.g. [15], [12], [14], and was previously introduced in [5] to a fluid-solid interaction problem. They have shown that the limit of the set of renormalized eigenvalues $\varepsilon^2\lambda^\varepsilon$ is the union of the Bloch spectrum and the boundary layer spectrum, when ε goes to 0. However, the asymptotic behaviour of the corresponding eigenvectors was not addressed. This is the goal of the present work which focuses on the Bloch spectrum of the high frequency part. Precisely, we search eigenvalues λ^ε such that

$$\varepsilon^2\lambda^\varepsilon = \lambda_n^k + \varepsilon\lambda^1 + \varepsilon O(\varepsilon) \tag{2}$$

where λ_n^k is solution of the Bloch wave spectral problem, also called the microscopic equation in this work,

$$-\partial_y \left(a(y)\,\partial_y \phi_n^k(y)\right) = \lambda_n^k \rho(y)\,\phi_n^k(y) \quad \text{for} \quad n \in \mathbb{N}^* \tag{3}$$

with $k-$quasi-periodic boundary conditions for some $k \in \mathbb{R}$. From [6], it is known that each λ_n^k can be reached as a limit of a subsequence of $\varepsilon^2\lambda^\varepsilon$. For each $n \in \mathbb{N}^*$ and each k, λ_n^k is either a simple or a double eigenvalue and $\lambda_n^k = \lambda_n^{-k}$. We pose $I^k = \{-k, k\}$ if $k \neq 0$ and $I^k = \{0\}$ otherwise. To guarantee that Bloch waves are kept in the weak limit, we apply the modulated two-scale transform S_k^ε, defined in [8] from the usual two-scale transform in [20], [19], [10], [9] or [11]. Passing to the limit in the weak formulation, it is shown that $\sum_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ is weakly converging to two-scale modes

$$g_k(x,y) = \sum_{\sigma\in I^k}\sum_m u_m^\sigma(x)\,\phi_m^\sigma(y)$$

where the second sum runs over all modes ϕ_m^σ with the same eigenvalue λ_n^k. Here, the modes ϕ_m^σ are called microscopic modes. The factors $(u_m^\sigma)_m$ are solution of the macroscopic system of first order differential equation,

$$\sum_m c(\sigma, n, m) \partial_x u_m^\sigma + \lambda^1 b(\sigma, n, m) u_m^\sigma = 0 \text{ in } \Omega \text{ for each } \sigma \in I^k, \tag{4}$$

which boundary conditions and the constant $c(\sigma, n, m)$ are depending on the involved microscopic modes and eigenvalues. The physical solution w^ε is then approximated by two-scale modes

$$w^\varepsilon(x) \approx \sum_{\sigma \in I^k} \sum_m u_m^\sigma(x) \phi_m^\sigma\left(\frac{x}{\varepsilon}\right). \tag{5}$$

These results are established for Neumann boundary conditions.

In fact, this method is inspired from [8] dedicated to the wave equation, except that in the latter work the two-scale transforms $S_k^\varepsilon w^\varepsilon$ and $S_{-k}^\varepsilon w^\varepsilon$ were analyzed separately and the macroscopic boundary conditions were lacking. Moreover, the model derivation in [8] is starting from the wave equation written as a first order system. So, for the sake of comparison, we derive the homogenized spectral equation from a first order formulation.

In addition, we report exploration results regarding approximations of physical eigenmodes by two-scale modes. First, for a given ε and each high frequency physical eigenelement $(\lambda^\varepsilon, w^\varepsilon)$, we show how to find quadruplets $\left(\lambda_n^k, \lambda_1, \phi_n^k, u_n^k\right)_{n,k}$ satisfying the approximations (2) and (5). This shows that each high frequency eigenelement can be approximated by a two-scale mode. Conversely, the high-frequency physical eigenelements can be built from the two-scale eigenelements only. Namely, for a given Bloch mode $\left(\lambda_n^k, \phi_n^k\right)$, a macroscopic eigenelement $\left(\lambda^1, u_n^k\right)$ is minimizing the error on the physical equation (1) where w^ε and λ^ε are replaced by their approximations (2) and (5).

This paper is organized as follows. In Section 2 we state the physical spectral equation with Dirichlet boundary conditions. In Section 3 the notations and elementary properties, which are used throughout the paper, are introduced. In Section 4 and 5, the model homogenization is derived based on the second order and first order formulations respectively. Finally, the numerical results are reported in the last section.

2 Statement of the problem

We consider $\Omega = (0, \alpha) \subset \mathbb{R}^+$ an interval, which boundary is denoted by $\partial\Omega$, and two functions $(a^\varepsilon, \rho^\varepsilon)$ assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right), \tag{6}$$

where $\rho \in L^\infty(\mathbb{R})$, $a \in W^{1,\infty}(\mathbb{R})$ are both Y-periodic where Y is an open interval. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions:

$$\rho^0 \leq \rho \leq \rho^1 \text{ and } a^0 \leq a \leq a^1, \tag{7}$$

for some given strictly positive ρ^0, ρ^1, a^0 and a^1.

With the operators $P^\varepsilon = -\partial_x(a^\varepsilon \partial_x .)$, the spectral problem with Dirichlet boundary conditions is

$$P^\varepsilon w^\varepsilon = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \quad \text{in} \quad \Omega \ \text{ and } w^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{8}$$

where as usual $\varepsilon > 0$ denotes a small parameter intended to go to zero.

The eigenvectors $w^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ are normalized by

$$\|w^\varepsilon\|_{L^2(\Omega)} = \left(\int_\Omega |w^\varepsilon|^2 \, dx \right)^{\frac{1}{2}} = 1, \tag{9}$$

and we search the eigenvalues such that

$$\varepsilon^2 \lambda^\varepsilon = \lambda^0 + \varepsilon \lambda^1 + \varepsilon O(\varepsilon), \tag{10}$$

where λ^0 is a non negative real number and $O(\varepsilon)$ tends to zero with ε. The weak formulation of the spectral problem (8) is: find $w^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega a^\varepsilon \partial_x w^\varepsilon \partial_x v \; dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon v \; dx \;\; \text{for all } v \in H_0^1(\Omega). \tag{11}$$

Since $\varepsilon^2 \lambda^\varepsilon$ is bounded, it results the uniform bound

$$||\varepsilon \partial_x w^\varepsilon||_{L^2(\Omega)} \leq N_0. \tag{12}$$

3 Notations and elementary properties

The functional space $L^2(\Omega)$ of square integrable functions is over $\mathbb{C}$. Let $u = (u_i)_i$ and $v = (v_i)_i$ be m-dimensional complex vector valued functions in $L^2(\Omega)$, the dot product is denoted by $u.v := \sum_i u_i v_i$ and the hermitian inner product by

$$\int_\Omega u \cdot v \; dx = \int_\Omega u(x).\overline{v(x)} \; dx. \tag{13}$$

The notation $O(\varepsilon)$ refers to numbers or functions tending to zeros when $\varepsilon \to 0$ in a sense made precise in each case. The notations $\partial_x u = \frac{\partial u}{\partial x}, \partial_y u = \frac{\partial u}{\partial y}$ are for $x-$ and $y-$derivatives of a function u. The vectors n_Ω, n_Y are the outer unit normals of $\partial\Omega$ and ∂Y.

Bloch decomposition We follow the definition of Bloch decomposition in [8] with $N = 1$, $L = \mathbb{Z}$, and $Y = (0,1)$, so $\mathbb{R} = \overline{Y} + L$. The dual lattice is necessarily $L^* = \mathbb{Z}$, and the equivalence class $Y^* = \mathbb{R}/L^*$ is chosen as $Y^* = (-1/2, 1/2)$. For $K \in \mathbb{N}^*$, considering the dual lattices $KL = K\mathbb{Z}$ and $L^*/K = \mathbb{Z}/K$, we pose

$$L_K = \begin{cases} \{-\frac{K}{2}, .., \frac{K}{2} - 1\} \subset L \text{ if } K \text{ is even,} \\ \{-\frac{K-1}{2}, .., \frac{K-1}{2}\} \text{ if } K \text{ is odd,} \end{cases}$$

so that $L = L_K + KL$. Posing $L_K^* = L_K/K$ yields $L^*/K = L^* + L_K^*$.

Functional spaces of quasi-periodic functions For any $k \in Y^*$, we define the $k-$quasi-periodic L^2-vector space over $\mathbb{C}$ with the hermitian inner product (13) by

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}) \mid u(x + \ell) = u(x)e^{2i\pi k\ell} \text{ a.e. in } \mathbb{R} \text{ for all } \ell \in L\},$$

or equivalently

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}) \mid \exists v \in L_\sharp^2 \text{ such that } u(x) = v(x)e^{2i\pi kx} \text{ a.e.}\},$$

where $L_\sharp^2$ is the traditional notation for L_k^2 in the periodic case i.e. when $k = 0$. Likewise, for $s \geq 0$ we set

$$H_k^s := L_k^2 \cap H_{loc}^s(\mathbb{R})$$

bearing in mind that the subscript $\sharp$ would be more appropriate in the periodic case $k=0$.

The modulated two-scale transform Let us assume from now that the domain Ω is the union of a finite number of entire cells of size ε or equivalently that the sequence ε is exactly $\varepsilon_n = \frac{\alpha}{n}$ for $n \in \mathbb{N}^*$. Setting $C_\varepsilon := \{\omega_\varepsilon = \varepsilon l + \varepsilon Y \mid l \in L, \varepsilon l + \varepsilon Y \subset \Omega\}$ is the set of all cells of Ω.

Definition 1 *For any $k \in Y^*$, the modulated two-scale transform of the function $u \in L^2(\Omega)$, $S_k^\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y)$ is defined by*

$$S_k^\varepsilon u(x,y) = \sum_{\omega_\varepsilon \in C_\varepsilon} u(\varepsilon l_{\omega_\varepsilon} + \varepsilon y)\, \chi_{\omega_\varepsilon}(x)\, e^{-2i\pi k l_{\omega_\varepsilon}}, \tag{14}$$

where $\varepsilon l_{\omega_\varepsilon}$ stands for the unique node in εL of ω_ε and $\chi_{\omega_\varepsilon}$ is the characteristic function of ω_ε.

The three following properties can be checked by using (14) and are admitted. For $u, v \in L^2(\Omega)$

$$\|S_k^\varepsilon u\|^2_{L^2(\Omega\times Y)} = \int_{\Omega\times Y} |S_k^\varepsilon u|^2\, dxdy = \sum_\varepsilon \int_{\omega_\varepsilon} |u|^2\, dx = \|u\|^2_{L^2(\Omega)}, \tag{15}$$

$$S_k^\varepsilon(uv) = S_0^\varepsilon(u) S_k^\varepsilon(v),$$

$$\text{and } S_k^\varepsilon(\partial_x u)(x,y) = \frac{1}{\varepsilon}\partial_y S_k^\varepsilon u(x,y) \text{ for } u \in H^1(\Omega).$$

Remark 2 *Let $k \in Y^*$ and a sequence u^ε bounded in $L^2(\Omega)$ such that $S_k^\varepsilon u^\varepsilon$ converges to u^k in $L^2(\Omega\times Y)$ weakly when $\varepsilon \to 0$, then $S_{-k}^\varepsilon u^\varepsilon$ converges to some u^{-k} in $L^2(\Omega\times Y)$ weakly. Moreover, since $S_k^\varepsilon u^\varepsilon$ and $S_{-k}^\varepsilon u^\varepsilon$ are conjugate then u^k and u^{-k} are also conjugate.*

The adjoint $S_k^{\varepsilon *} : L^2(\Omega\times Y) \to L^2(\Omega)$ of S_k^ε, is defined by

$$\int_\Omega (S_k^{\varepsilon *} v)(x) \cdot w(x)\, dx = \int_{\Omega\times Y} v(x,y) \cdot (S_k^\varepsilon w)(x,y)\, dxdy, \tag{16}$$

for all $w \in L^2(\Omega)$ and $v \in L^2(\Omega\times Y)$, and we denote by $\mathfrak{R}$ the operator operating on functions $v(x,y)$ defined in $\Omega \times \mathbb{R}$,

$$(\mathfrak{R}v)(x) = v(x, \frac{x}{\varepsilon}). \tag{17}$$

The next Lemma shows that $\mathfrak{R}$ is an approximation of $S_k^{\varepsilon *}$ for $k-$quasi-periodic functions.

Lemma 3 *Let $v \in C^1(\Omega\times Y)$ a $k-$quasi-periodic function in y then*

$$S_k^{\varepsilon *} v = \mathfrak{R}v + O(\varepsilon) \quad \text{in the } L^2(\Omega) \text{ sense.} \tag{18}$$

Proof. The proof is carried out in two steps. First the explicit expression of $S_k^{\varepsilon *} v$ is derived, then the approximation is deduced.

(i) Let us prove that

$$(S_k^{\varepsilon *} v)(x) = \sum_{\omega_\varepsilon \in C_\varepsilon} \varepsilon^{-1} \int_{\omega_\varepsilon} v\left(z, \frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dz\ \chi_{\omega_\varepsilon}(x) e^{2i\pi k l_{\omega_\varepsilon}}.$$

From the definition of the two-scale transform with $r = \varepsilon l_{\omega_\varepsilon} + \varepsilon y \in \omega_\varepsilon$,

$$\int_{\Omega\times Y} v(x,y) \cdot (S_k^\varepsilon w)(x,y)\, dxdy = \sum_{\omega_\varepsilon \in C_\varepsilon} \int_{\Omega\times\omega_\varepsilon} \varepsilon^{-1} v\left(x, \frac{r - \varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) \cdot w(r)\, \chi_{\omega_\varepsilon}(x) e^{-2i\pi k l_{\omega_\varepsilon}}\ dxdr$$

or equivalently,

$$= \int_\Omega \sum_{\omega_\varepsilon \in C_\varepsilon} \varepsilon^{-1} \int_{\omega_\varepsilon} v\left(x, \frac{r-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dx \cdot w\,(r)\chi_{\omega_\varepsilon}(r) e^{-2i\pi k l_{\omega_\varepsilon}}\ dr.$$

Changing the variable names and using the definition of $S_k^{\varepsilon *}$,

$$\int_\Omega \left(S_k^{\varepsilon *} v\right)(x) \cdot w\,(x)\ dx = \int_\Omega \sum_{\omega_\varepsilon \in C_\varepsilon} \varepsilon^{-1} \int_{\omega_\varepsilon} v\left(z, \frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dz e^{2i\pi k l_{\omega_\varepsilon}} \cdot w\,(x)\chi_{\omega_\varepsilon}(x)\ dx.$$

This establishes the explicit expression of $S_k^{\varepsilon *}$.

(ii) Let us derive the expected approximation for $v \in C^1(\Omega \times Y)$ and $k-$quasi-periodic in y. Since $\varepsilon\,|Y| = |\omega_\varepsilon|$ and

$$v\,(z,y) = v\,(x,y) + \partial_x v\,(x,y)\,(z-x) + \varepsilon O\,(\varepsilon) \text{ in } L^2(\Omega) \text{ for a.e. } y \in Y$$

then

$$\left(S_k^{\varepsilon *} v\right)(\varepsilon l_{\omega_\varepsilon} + \varepsilon y) = \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} v\,(x,y) + \partial_x v\,(x,y)\,(z-x)\ \ dz\ e^{2i\pi k l_{\omega_\varepsilon}} + O\,(\varepsilon)$$

for a.e. $y \in Y$ and all $\omega_\varepsilon \in C_\varepsilon$. Remarking that $z - x = (z - \varepsilon l_{\omega_\varepsilon}) + (\varepsilon l_{\omega_\varepsilon} - x)$ and

$$\int_{\omega_\varepsilon} (z - \varepsilon l_{\omega_\varepsilon})\,dz = -\frac{1}{2}\varepsilon O\,(\varepsilon)\,.$$

So for all ω_ε and $y \in Y$,

$$e^{-2i\pi k l_{\omega_\varepsilon}}\,|\omega_\varepsilon|\left(S_k^{\varepsilon *} v\right)(\varepsilon l_{\omega_\varepsilon} + \varepsilon y) = |\omega_\varepsilon|\,v\,(x,y) + (-\frac{1}{2}\varepsilon O\,(\varepsilon) + \left(\varepsilon^2 y\right))\partial_x v\,(x,y) + \varepsilon O\,(\varepsilon)\,.$$

Therefore,

$$\left(S_k^{\varepsilon *} v\right)(x) = \sum_{\omega_\varepsilon} v\left(x, \frac{x}{\varepsilon} - l_{\omega_\varepsilon}\right)\chi_{\omega_\varepsilon}(x)\,e^{2i\pi k l_{\omega_\varepsilon}} + O\,(\varepsilon)\,.$$

Using the $k-$quasi-periodic of v in y,

$$\left(S_k^{\varepsilon *} v\right)(x) = \sum_{\omega_\varepsilon} v\left(x, \frac{x}{\varepsilon}\right)\chi_{\omega_\varepsilon}(x) + O\,(\varepsilon)$$

in $L^2(\Omega)$, hence the formula (18) follows. ■

In the proof, we constantly use the following consequence.

Corollary 4 *Let $v \in C^1(\Omega \times Y)$ and $k-$quasi-periodic in y, for any sequence u^ε bounded in $L^2(\Omega)$ such that $S_k^\varepsilon u^\varepsilon$ converges to u in $L^2(\Omega \times Y)$ weakly when $\varepsilon \to 0$ then*

$$\int_\Omega u^\varepsilon \cdot \mathfrak{R} v\ dx \to \int_{\Omega \times Y} u \cdot v\ dxdy \quad \textit{when } \varepsilon \to 0.$$

Note that for $k = 0$, this corresponds to the definition of two-scale convergence in [1] and [23].

Two-scale operators For a function $v(x,y)$ defined in $\Omega \times \mathbb{R}$, we pose

$$P^0 v = -\partial_x\left(a\partial_x v\right),\ P^1 v = -\partial_x\left(a\partial_y v\right) - \partial_y\left(a\partial_x v\right) \text{ and } P^2 v = -\partial_y\left(a\partial_y v\right),$$

so that

$$P^{\varepsilon}\Re v=\sum_{n=0}^{2}\varepsilon^{-n}\Re P^{n}v. \tag{19}$$

Bloch waves For a given $k\in Y^*$, we denote by (λ_n^k,ϕ_n^k) the Bloch wave eigenelements indexed by $n\in\mathbb{N}^*$ that are solution to

$$\mathcal{P}(k): P^2\phi_n^k=\lambda_n^k\rho\phi_n^k \text{ in } Y \text{ with } \phi_n^k\in H_k^2(Y) \text{ and } \left\|\phi_n^k\right\|_{L^2(Y)}=1. \tag{20}$$

The corresponding weak formulation is: find $\phi_n^k\in H_k^1(Y)$ solution to

$$\int_Y a\partial_y\phi_n^k\cdot\partial_y v-\lambda_n^k\rho\phi_n^k\cdot v\; dy=0 \text{ for all } v\in H_k^1(Y). \tag{21}$$

Since the operator $P^2: H_k^2(Y)\subset L_k^2(Y)\to L_k^2(Y)$ is self-adjoint, its spectra is real. Furthermore, for $n,m\in\mathbb{N}^*$, we introduce the coefficients

$$c(k,n,m)=\int_Y a\partial_y\phi_m^k\cdot\phi_n^k-\phi_m^k\cdot a\partial_y\phi_n^k\; dy \text{ and } b(k,n,m)=\int_Y \rho\phi_m^k\cdot\phi_n^k\; dy \tag{22}$$

and observe that the following properties hold,

$$c(k,n,m)=\overline{c(-k,n,m)},\; c(k,m,n)=-\overline{c(k,n,m)},\; c(k,n,m)=-c(-k,m,n)$$

and

$$b(k,n,m)=\overline{b(k,m,n)},\; b(k,n,m)=\overline{b(-k,m,n)},\; b\left(k,n,n\right)>0.$$

In particular for $k=0$, if the eigenvectors are chosen as real functions thus $c\left(0,n,n\right)=0$. In the special case $\rho=1$, $b(k,n,m)=1$ for $n=m$ and $b(k,n,m)=0$ otherwise.

Notation 5 *For $k\neq 0$, $\overline{\phi_n^k}\in H_{-k}^2(Y)$, the conjugate of ϕ_n^k, is solution of $\mathcal{P}(-k)$. We choose the numbering of eigenvectors ϕ_n^{-k} so that $\phi_n^{-k}=\overline{\phi_n^k}$ and remark that $\lambda_n^{-k}=\lambda_n^k$.*

Remark 6 *In one dimension, for $k\in Y^*$, it is well-known that all eigenvalue λ_n^k are simple, except for $k=0$ where they are double.*

Finally, we denote

$$I^k=\{k,-k\} \text{ if } k\in Y^*\diagdown\{0\} \text{ and } I^0=\{0\} \text{ otherwise.}$$

4 Homogenization of the high-frequency eigenvalue problem

For $k\in Y^*$, we decompose

$$\frac{\alpha k}{\varepsilon}=h_\varepsilon^k+l_\varepsilon^k \text{ with } h_\varepsilon^k=\left[\frac{\alpha k}{\varepsilon}\right] \text{ and } l_\varepsilon^k\in[0,1)\,, \tag{23}$$

and assume that the sequence of the ε is varying in a set $E_k\subset\mathbb{R}^{+*}$ depending on k so that

$$l_\varepsilon^k\to l^k \text{ when } \varepsilon\to 0 \text{ and } \varepsilon\in E_k \text{ with } l^k\in[0,1)\,. \tag{24}$$

We note that for $k=0$, $h_\varepsilon^k=0$, $l_\varepsilon^k=0$, so $l^k=0$ and $E_0=\mathbb{R}^{+*}$.

4.1 Main result

The macroscopic equation is stated for each $k \in Y^*$ and each Bloch wave eigenvalue λ_n^k. For $k \neq 0$, we assume that $c(\sigma, n, n) \neq 0$ for each $\sigma \in I^k$, so it is stated as an eigenvalue problem

$$c(\sigma, n, n)\, \partial_x u_n^\sigma + \lambda^1 b(\sigma, n, n)\, u_n^\sigma = 0 \quad \text{in} \quad \Omega \tag{25}$$

for each σ, with the boundary conditions

$$\sum_{\sigma \in I^k} u_n^\sigma(x)\, \phi_n^\sigma(0)\, e^{sign(\sigma) 2 i \pi \frac{l^k x}{\alpha}} = 0 \text{ on } x \in \partial\Omega, \tag{26}$$

where l^k is defined in (24). We observe that the first order operator $c(k, n, n) \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}$ of this system is self-adjoint on the domain

$$D^k = \left\{ (u_n, v_n) \in H^1(\Omega)^2 \text{ satisfying (26)} \right\}$$

so λ^1 is real.

For $k = 0$, assuming that λ_n^0 is a double eigenvalue corresponding to two eigenvectors ϕ_n^0 and ϕ_m^0, and that $c(0, n, m) \neq 0$, the macroscopic system states

$$\sum_{q \in \{n,m\}} c(0, p, q)\, \partial_x u_q^0 + \lambda^1 b(0, p, q)\, u_q^0 = 0 \text{ in } \Omega \text{ for } p \in \{n, m\}, \tag{27}$$

with the boundary conditions

$$\sum_{q \in \{n,m\}} u_q^0(x)\, \phi_q^0(0) = 0 \text{ on } x \in \partial\Omega. \tag{28}$$

Again $\lambda^1 \in \mathbb{R}$ since $c(0, n, m) \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}$ is self-adjoint on

$$D^0 = \left\{ (u_n, u_m) \in H^1(\Omega)^2 \text{ satisfying (28)} \right\}.$$

Remark 7 *(i) If $c(k, n, n) = 0$ for $k \neq 0$ or $c(0, p, q) = 0$ for all p, q varying in $\{n, m\}$, the macroscopic equations (25) or (27) are $\lambda^1 = 0$ or $u = (u_n^\sigma)_{n,\sigma} = 0$. But $u = 0$ is impossible since $\|w^\varepsilon\|_{L^2(\Omega)} = 1$ for all eigenmodes w^ε. So $\lambda^1 = 0$ and this model does not provide any equation for u_n^σ.*

(ii) For $k \neq 0$, if $\phi_m^k(0) = 0$ then $\phi_m^k(1) = 0$ and ϕ_m^k is a periodic solution that is a solution of $k = 0$. So, we consider always that $\phi_m^k(0) \neq 0$ for $k \neq 0$.

(iii) For $k = 0$, in case where $\phi_n(0) = \phi_m(0) = 0$ the boundary conditions of the macroscopic equation vanishes.

Remark 8 *This work focuses on the Bloch spectrum. To avoid eigenmodes related to the boundary spectrum, according to Proposition 7.7 in [6] we shall assume that the weak limit of $S_k^\varepsilon w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ is not vanishing.*

The main Theorem states as follows.

Theorem 9 *For $k \in Y^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be solution of (8) then $\sum_{\sigma\in I^k} S^\varepsilon_\sigma w^\varepsilon$ is bounded in $L^2(\Omega; H^1(Y))$. For $\varepsilon \in E_k$, as in (23, 24), assuming that the weak limit of $S^\varepsilon_k w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ is non-vanishing and the renormalized sequence $\varepsilon^2\lambda^\varepsilon$ satisfies the decomposition (10), there exists $n \in \mathbb{N}^*$ such that $\lambda^0 = \lambda^k_n$ with λ^k_n an eigenvalue of the Bloch wave spectrum and the limit g_k of any weakly converging extracted subsequence of $\sum_{\sigma\in I^k} S^\varepsilon_\sigma w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ can be decomposed on the Bloch modes*

$$g_k(x,y) = \sum_{\sigma\in I^k} u^\sigma_n(x)\,\phi^\sigma_n(y) \text{ for } k \neq 0 \text{ and } g_0(x,y) = \sum_{q\in\{n,m\}} u^0_q(x)\,\phi^0_q(y) \text{ otherwise} \tag{29}$$

Moreover, $u^\sigma_m \in H^1(\Omega)$ and $(u^\sigma_m)_{m,\sigma}$ are solutions of the macroscopic equations (25, 26) and (27, 28). Finally, u^k_m and u^{-k}_m are conjugate.

Thus, it follows from (29) that the physical solution w^ε is approximated by two-scale modes

$$w^\varepsilon(x) \approx \sum_{\sigma\in I^k} u^\sigma_n(x)\,\phi^\sigma_n\left(\frac{x}{\varepsilon}\right) \text{ for } k \neq 0 \text{ and } w^\varepsilon(x) \approx \sum_{q\in\{n,m\}} u^0_q(x)\,\phi^0_q\left(\frac{x}{\varepsilon}\right) \text{ otherwise.} \tag{30}$$

The boundary conditions (26) and (28) can be directly derived by replacing w^ε in the physical boundary condition by its approximations,

$$\sum_{\sigma\in I^k} u^\sigma_n(x)\,\phi^\sigma_n\left(\frac{x}{\varepsilon}\right) = 0 \text{ for } k \neq 0 \text{ and } \sum_{q\in\{n,m\}} u^0_q(x)\,\phi^0_q\left(\frac{x}{\varepsilon}\right) = 0 \text{ otherwise at } x \in \partial\Omega. \tag{31}$$

For $k \neq 0$, they result from

$$\phi^\sigma_n\left(\frac{x}{\varepsilon}\right) = \phi^\sigma_n(0)\,e^{2i\pi\sigma\frac{x}{\varepsilon}} = \phi^\sigma_n(0)\,e^{sign(\sigma)2i\pi x\frac{h^k_\varepsilon + l^k_\varepsilon}{\alpha}} = \phi^\sigma_n(0)\,e^{sign(\sigma)2i\pi x\frac{l^k_\varepsilon}{\alpha}} \text{ for } x \in \partial\Omega$$

and the assumption $l^k_\varepsilon \to l^k$. For $k = 0$, the conditions follow from the periodicity of ϕ^0_n. Furthermore, we observe that $g_k(x,0)$ and $g_k(x,1)$ are generally not vanishing except for $k = 0$.

Proposition 10 *For $k \in Y^*$, $n \in \mathbb{N}^*$, if the macroscopic solution u^k_n is a non-vanishing constant, then any two-scale mode (30) is a physical eigenmode i.e. a solution to (8).*

Proof. For $k \in Y^*$, $n \in \mathbb{N}^*$, if the macroscopic solution u^k_n is constant then $\lambda^1 = 0$ and $(u^\sigma_m)_{m,\sigma}$ are constant for all $\sigma \in I^k$ and $m \in \mathbb{N}^*$ such that $\lambda^\sigma_m = \lambda^\sigma_n$. Now, we consider $\rho = 1$ and the proof is similar for $\rho \neq 1$. Based on Remark 14 about the macroscopic solutions in Section 4.4, $\lambda^1 = 0$ is equivalent to $\ell = \frac{2k\alpha}{\varepsilon}$. From the $\sigma-$quasi-periodicity of ϕ^σ_n,

$$\phi^\sigma_n\left(\frac{\alpha}{\varepsilon}\right) = \phi^\sigma_n(0)\,e^{sign(\sigma)2i\pi k\frac{\alpha}{\varepsilon}} = \phi^\sigma_n(0)\,e^{sign(\sigma)i\pi\ell} = \pm\phi^\sigma_n(0),$$

then ϕ^σ_n is $\alpha-$periodic or $\alpha-$anti-periodic for $\sigma \in I^k$. Hence $\phi^\sigma_n\left(\frac{x}{\varepsilon}\right)$ is a solution of the equation

$$\partial_x\left(a\left(\frac{x}{\varepsilon}\right)\partial_x\phi^\sigma_n\left(\frac{x}{\varepsilon}\right)\right) = -\frac{\lambda^\sigma_n}{\varepsilon^2}\phi^\sigma_n\left(\frac{x}{\varepsilon}\right) \text{ in } \Omega \tag{32}$$
$$\text{and } \phi^\sigma_n\left(\frac{x}{\varepsilon}\right) \text{ is } \alpha-\text{periodic or } \alpha-\text{anti-periodic,}$$

and $u_m^\sigma \phi_m^\sigma \left(\frac{x}{\varepsilon}\right)$ is also a solution of (32). Denote by $w^\varepsilon := \sum\limits_{\sigma\in I^k} \sum\limits_m u_m^\sigma \phi_m^\sigma \left(\frac{x}{\varepsilon}\right)$ and observe that w^ε is a solution of the equation

$$\partial_x \left(a^\varepsilon \partial_x w^\varepsilon\right) = -\lambda^\varepsilon w^\varepsilon \text{ in } \Omega$$

with the boundary conditions

$$w^\varepsilon(0) = \sum_{\sigma\in I^k} \sum_m u_m^\sigma \phi_m^\sigma(0) = 0 \text{ and } w^\varepsilon(\alpha) = \sum_{\sigma\in I^k} \sum_m u_m^\sigma \phi_m^\sigma \left(\frac{x}{\varepsilon}\right) = \pm w^\varepsilon(0) = 0.$$

Finally, Proposition 10 is concluded. ■

Remark 11 *The converse is probably true, and is numerically studied in Section 6.2, i.e. for any* $(\lambda^\varepsilon, w^\varepsilon)$ *solution to (8), there exist* $k \in Y^*$, $n \in \mathbb{N}^*$ *and two complex numbers* ξ_1 *and* ξ_2 *such that* $\lambda^\varepsilon = \lambda_n^k / \varepsilon^2$ *and*

$$w^\varepsilon(x) = \xi_1 \phi_n^k \left(\frac{x}{\varepsilon}\right) + \xi_2 \phi_n^{-k} \left(\frac{x}{\varepsilon}\right) \textit{ if } k \neq 0 \textit{ and } w^\varepsilon(x) = \xi_1 \phi_n^0 \left(\frac{x}{\varepsilon}\right) + \xi_2 \phi_m^0 \left(\frac{x}{\varepsilon}\right) \textit{ otherwise} \tag{33}$$

for ξ_1, ξ_2 *two numbers such that the boundary conditions (28), respectively (26), are satisfied for* $k = 0$, *respectively for* $k \neq 0$. *In the later case* ξ_1 *and* ξ_2 *are conjugate.*

Remark 12 *(i) The case of non-constant coefficients* u_n^k *is used for approximations of the solution to the homogenized wave equation that may be derived from [8]. In such case* k *belongs to a finite subset* L_K^* *of* Y^* *made with values distant from* $1/K$ *and including* 0. *We cannot expect that there always exists a pair* (k, n) *such that* u_n^k *is a constant.*

(ii) The case of non-constant coefficients u_n^k *is also seen as a preparation to derive homogenized spectral problems in higher dimension where the boundary conditions constitute a more difficult problem and may require a more general solution than constant* u_n^k.

Proof of Theorem 9

Proof. The proof is based on Lemma 13 in Section 4.2 and on the macroscopic model derivation in Section 4.3. For a given $k \in Y^*$, let w^ε be solution of (8) which is bounded in $L^2(\Omega)$, the property (15) yields the uniform boundness of $\|S_\sigma^\varepsilon w^\varepsilon\|_{L^2(\Omega\times Y)}$ for any $\sigma \in I^k$. So there exist $w^\sigma \in L^2(\Omega \times Y)$ such that up the extraction of a subsequence $S_\sigma^\varepsilon w^\varepsilon \to w^\sigma$ in $L^2(\Omega \times Y)$ weakly. Since $\|S_\sigma^\varepsilon(\varepsilon\partial_x w^\varepsilon)\|_{L^2(\Omega\times Y)} = \|\partial_y S_\sigma^\varepsilon w^\varepsilon\|_{L^2(\Omega\times Y)}$ is uniformly bounded as $\|\varepsilon\partial_x w^\varepsilon\|_{L^2(\Omega)}$. Hence

$$\lim_{\varepsilon\to 0} \int_{\Omega\times Y} \partial_y S_\sigma^\varepsilon w^\varepsilon \cdot v dx dy = \lim_{\varepsilon\to 0} \int_{\Omega\times Y} -S_\sigma^\varepsilon w^\varepsilon \cdot \partial_y v dx dy = -\int_{\Omega\times Y} w^\sigma \cdot \partial_y v dx dy$$

for all $v \in L^2(\Omega; H_0^1(Y))$, and $w^\sigma \in L^2(\Omega; H^1(Y))$ then

$$\lim_{\varepsilon\to 0} \int_{\Omega\times Y} \partial_y S_\sigma^\varepsilon w^\varepsilon \cdot v dx dy = \int_{\Omega\times Y} \partial_y w^\sigma \cdot v dx dy.$$

Therefore $S_\sigma^\varepsilon w^\varepsilon$ tends weakly to w^σ also in $L^2(\Omega; H^1(Y))$. Hence, $\sum\limits_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ converges to

$$g_k(x,y) = \sum_{\sigma\in I^k} w^\sigma(x,y).$$

Using the decomposition (34) of w^σ in Lemma 13, for $\left(\phi_p^\sigma\right)_{\sigma,p}$ the Bloch wave eigenmodes corresponding to λ^0,

$$\begin{cases} g_k\left(x,y\right) = \sum_{\sigma\in I^k} u_n^\sigma\left(x\right)\phi_n^\sigma\left(y\right) \text{ for } k\neq 0, \\ g_0\left(x,y\right) = \sum_{p\in\{n,m\}} u_p^0\left(x\right)\phi_p^0\left(y\right) \text{ for } k=0. \end{cases}$$

Finally, $\left(u_p^\sigma\right)_{\sigma,p}$ is solution of the macroscopic problem as proved in Section 4.3. ■

4.2 Modal decomposition on the Bloch modes

Lemma 13 *For $(\lambda^\varepsilon, w^\varepsilon)$ solution of (8) and satisfying (10), for a fixed $k\in Y^*$ there exists at least a subsequence of $S_k^\varepsilon w^\varepsilon$ converging weakly towards non-vanishing function w^k in $L^2\left(\Omega\times Y\right)$ when ε tends to zero. If $w^k\in L^2(\Omega;H^2(Y))$ then $\left(\lambda^0, w^k\right)$ is solution of the Bloch wave equation (20) and w^k admits the modal decomposition,*

$$w^k\left(x,y\right) = \sum_m u_m^k\left(x\right)\phi_m^k\left(y\right) \;\; \textit{for } u_m^k\in L^2\left(\Omega\right) \tag{34}$$

where the sum is over all Bloch modes ϕ_m^k associated to λ^0. Moreover for $k\neq 0$ the two factors u_m^k and u_m^{-k} are conjugate.

Proof. The test functions of the weak formulation (11) are chosen as

$$v^\varepsilon := \mathfrak{R}v \in H_0^1(\Omega)\cap H^2(\Omega), \tag{35}$$

with

$$v\in H_0^1(\Omega;L_k^2(Y))\cap L^2(\Omega;H_k^2(Y))\cap H^2\left(\Omega;L_k^2\left(Y\right)\right). \tag{36}$$

Applying two integrations by parts and the boundary conditions satisfied by w^ε and by $\mathfrak{R}v$, it remains

$$\int_\Omega w^\varepsilon\cdot(P^\varepsilon-\lambda^\varepsilon\rho^\varepsilon)v^\varepsilon\;dx = 0. \tag{37}$$

From (19) multiplied by ε^2 and (10),

$$\int_\Omega w^\varepsilon\cdot\mathfrak{R}((P^2-\lambda^0\rho)v)\;dx = O(\varepsilon).$$

Since $(P^2-\lambda^0\rho)v$ is $k-$quasi-periodic and $S_k^\varepsilon w^\varepsilon\to w^k$ in $L^2(\Omega\times Y)$ weakly, Corollary 4 allows to pass to the limit

$$\int_{\Omega\times Y} w^k\cdot(P^2-\lambda^0\rho)v\;dxdy = 0,$$

or equivalently

$$\int_{\Omega\times Y} w^k\cdot\partial_y\left(a\partial_y v\right)+w^k\cdot\lambda^0\rho v\;dxdy = 0. \tag{38}$$

Using the assumption $w^k\in L^2(\Omega;H^2(Y))$ and applying integrations by parts,

$$\int_{\Omega\times Y}\partial_y\left(a\partial_y w^k\right)\cdot v + w^k\cdot\lambda^0\rho v\;dxdy + \int_{\Omega\times\partial Y} aw^k\cdot\partial_y v - a\partial_y w^k\cdot v\,dxdy = 0.$$

Then, choosing test functions $v \in L^2(\Omega; H_0^2(Y))$ comes the strong form

$$-\partial_y \left(a\partial_y w^k\right) = \lambda^0 \rho w^k \text{ in } \Omega \times Y. \tag{39}$$

So, it remains

$$\int_\Omega \left[a w^k \cdot \partial_y v - a\partial_y w^k \cdot v\right]_0^1 \, dx = 0$$

for general test functions (36), which implies that w^k and $\partial_y w^k$ are $k-$quasi-periodic in the variable y.

As we know that λ^0 is an eigenvalue λ_n^k of the Bloch wave spectrum, then w^k is a Bloch eigenvector and is decomposed as

$$w^k(x,y) = \sum_m u_m^k(x)\,\phi_m^k(y) \text{ with } u_m^k \in L^2(\Omega)$$

the sum being over all Bloch modes ϕ_m^k associated to λ^0 where $u_m^k(x) = \int_Y w^k(x,y) \cdot \phi_m^k(y)\,dy$. For $k \neq 0$, $\phi_m^k = \overline{\phi_m^{-k}}$ and from Definition 1 of modulated two-scale transform, $S_k^\varepsilon w^\varepsilon = \overline{S_{-k}^\varepsilon w^\varepsilon}$ thus u_m^k and u_m^{-k} are conjugate i.e. $u_m^k = \overline{u_m^{-k}}$. ■

4.3 Derivation of the macroscopic equation

In the macroscopic model derivation, we distinguish between the two cases $k \neq 0$ and $k = 0$.

4.3.1 Case $k \neq 0$

We consider $\lambda^0 = \lambda_n^k$ and the two conjugate eigenvectors ϕ_n^k and ϕ_n^{-k} discussed in Notation 5. We restart from the very weak formulation (37) with the test function

$$v^\varepsilon(x) := \Re(v^k + v^{-k}) \in H_0^1(\Omega) \cap H^2(\Omega). \tag{40}$$

Furthermore, we pose $v^\sigma(x,y) = \psi^\sigma(x)\phi_n^\sigma(y)$ with $\psi^\sigma \in H^2(\Omega)$ for $\sigma \in I^k$ and use the $\sigma-$quasi-periodicity of ϕ_n^σ, i.e. $\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) = \phi_n^\sigma(0)\,e^{2i\pi k\frac{x}{\varepsilon}}$ at any $x \in \partial\Omega$. So the boundary condition in (40) is equivalent to

$$\psi^k(x)\,\phi_n^k(0)e^{2i\pi k\frac{x}{\varepsilon}} + \psi^{-k}(x)\,\phi_n^{-k}(0)e^{-2i\pi k\frac{x}{\varepsilon}} = 0 \text{ at any } x \in \partial\Omega.$$

Applying the relation (23),

$$\psi^k(x)\phi_n^k(0)e^{2i\pi x\frac{h_\varepsilon^k + l_\varepsilon^k}{\alpha}} + \psi^{-k}(x)\phi_n^{-k}(0)e^{-2i\pi x\frac{h_\varepsilon^k + l_\varepsilon^k}{\alpha}} = 0.$$

Since $x\frac{h_\varepsilon^k}{\alpha} = 0$ at $x = 0$ and $x\frac{h_\varepsilon^k}{\alpha} = h_\varepsilon^k$ at $x = \alpha$ with $h_\varepsilon^k \in \mathbb{Z}$ then $e^{\pm 2i\pi x\frac{h_\varepsilon^k}{\alpha}} = 1$. From (24), $e^{\pm 2i\pi\frac{l_\varepsilon^k x}{\alpha}} \to e^{\pm 2i\pi\frac{l^k x}{\alpha}}$ when $\varepsilon \to 0$. Passing to the limit, the boundary conditions of the test function are

$$\psi^k(x)\phi_n^k(0)e^{2i\pi\frac{l^k x}{\alpha}} + \psi^{-k}(x)\phi_n^{-k}(0)e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega. \tag{41}$$

From (19) multiplied by ε, (10) and $P^2 v^\sigma - \lambda^0 \rho v^\sigma = 0$,

$$\sum_{\sigma \in I^k} \int_\Omega w^\varepsilon \cdot \Re(-P^1 v^\sigma + \lambda^1 \rho v^\sigma)\,dx = O(\varepsilon). \tag{42}$$

Extracting a subsequence of w^ε so that $S_k^\varepsilon w^\varepsilon$ and $S_{-k}^\varepsilon w^\varepsilon$ are converging to w^k and w^{-k} in $L^2(\Omega\times Y)$ weak, since $-P^1 v^\sigma + \lambda^1 \rho v^\sigma$ is $\sigma-$quasi-periodic then Corollary 4 infers that

$$\sum_{\sigma\in I^k}\int_{\Omega\times Y} w^\sigma\cdot\left(-P^1 v^\sigma+\lambda^1\rho v^\sigma\right)\,dxdy=0,$$

i.e.

$$\sum_{\sigma\in I^k}\int_{\Omega\times Y} w^\sigma\cdot\left(\partial_x\left(a\partial_y v^\sigma\right)+\partial_y\left(a\partial_x v^\sigma\right)+\lambda^1\rho v^\sigma\ \right)dxdy=0.$$

This is the very weak form of the macroscopic equation for all test functions $v^\sigma\in H^1\left(\Omega;H_k^1\left(Y\right)\right)$, reached by density, satisfying (41). Now, we derive the strong formulation. We assume that $w^\sigma\in H^1(\Omega;L^2(Y))$, since $w^\sigma\in L^2(\Omega;H^1(Y))$ after two integrations by parts,

$$\begin{aligned}\sum_{\sigma\in I^k}\Bigg[\int_{\Omega\times Y}&\partial_y\left(a\partial_x w^\sigma\right)\cdot v^\sigma+\partial_x\left(a\partial_y w^\sigma\right)\cdot v^\sigma+\lambda^1\rho w^\sigma\cdot v^\sigma\ dxdy\\&+\int_{\partial\Omega\times Y} w^\sigma\cdot a\partial_y v^\sigma-a\partial_y w^\sigma\cdot v^\sigma\ dxdy\\&+\int_{\Omega\times\partial Y} w^\sigma\cdot a\partial_x v^\sigma-a\partial_x w^\sigma\cdot v^\sigma\ dxdy\Bigg]=0.\end{aligned}$$

From Lemma 13, w^σ is solution to the Bloch mode equation and is decomposed as

$$w^\sigma(x,y)=u^\sigma(x)\phi_n^\sigma(y). \tag{43}$$

After replacement,

$$\begin{aligned}\sum_\sigma\Bigg[\int_Y\partial_y(a\phi_n^\sigma)\cdot\phi_n^\sigma+a\partial_y\phi_n^\sigma\cdot\phi_n^\sigma\ dy\int_\Omega\partial_x u^\sigma\cdot\psi^\sigma dx+\lambda^1\int_Y\rho\phi_n^\sigma\cdot\phi_n^\sigma\ dy\int_\Omega u^\sigma\cdot\psi^\sigma dx\\+\int_Y\phi_n^\sigma\cdot a\partial_y\phi_n^\sigma-a\partial_y\phi_n^\sigma\cdot\phi_n^\sigma\ dy\int_{\partial\Omega}u^\sigma\cdot\psi^\sigma\ dx\\+\int_{\partial Y}\phi_n^\sigma\cdot a\phi_n^\sigma\ dy\int_\Omega u^\sigma\cdot\partial_x\psi^\sigma-\partial_x u^\sigma\cdot\psi^\sigma\ dx\Bigg]=0.\end{aligned} \tag{44}$$

Let us recall that $b(.,.,.)$ and $c(.,.,.)$ have been defined in (22). For the sake of simplicity, we use $c(\sigma,n):=c(\sigma,n,n)$ and $b(\sigma,n):=b(\sigma,n,n)$ and observe that

$$\int_Y\partial_y(a\phi_n^\sigma)\cdot\phi_n^\sigma+a\partial_y\phi_n^\sigma\cdot\phi_n^\sigma\ dy=c(\sigma,n),$$

which results from integrations by parts and from the $\sigma-$quasi-periodicity of ϕ_n^σ. So, using the σ-quasi-periodicity of ϕ_n^σ, (44) can be rewritten as

$$\sum_\sigma\left[\int_\Omega(c(\sigma,n)\partial_x u^\sigma+\lambda^1 b\left(\sigma,n\right)u^\sigma)\cdot\psi^\sigma\ dx-c(\sigma,n)\int_{\partial\Omega}u^\sigma\cdot\psi^\sigma\ dx\right]=0.$$

Choosing the test function $\psi^\sigma=0$ on $\partial\Omega$, the boundary condition (41) is satisfied and by density of $H_0^1\left(\Omega\right)$ in $L^2\left(\Omega\right)$, the internal equation satisfied by u^σ follows,

$$c(\sigma,n)\partial_x u^\sigma+\lambda^1 b\left(\sigma,n\right)u^\sigma=0\text{ in }\Omega\text{ for each }\sigma. \tag{45}$$

Choosing general $\psi^{\sigma} \in H^1(\Omega)$ satisfying (41) yields the boundary conditions

$$\sum_{\sigma} c(\sigma, n) u^{\sigma} \overline{\psi^{\sigma}} = 0 \text{ on } \partial\Omega. \tag{46}$$

We introduce the matrices $C_1 = diag((c(\sigma, n))_{\sigma})$, $C_2 = diag((b(\sigma, n))_{\sigma})$ and the vectors $u = (u^{\sigma})_{\sigma}$, $\psi = (\psi^{\sigma})_{\sigma}$, $\varphi = \left(\phi_n^{\sigma}(0) e^{sign(\sigma) 2i\pi \frac{l^k x}{\alpha}}\right)_{\sigma}$ with $\sigma \in I^k$, so that (41, 45, 46) can be written on the matrix form

$$C_1 \partial_x u + \lambda^1 C_2 u = 0 \text{ in } \Omega \ ,$$
$$\text{and } C_1 u(x).\overline{\psi}(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \overline{\varphi}(x,0).\overline{\psi}(x) = 0 \text{ on } \partial\Omega.$$

The boundary condition is equivalent to $C_1 u(x)$ is collinear with $\overline{\varphi}(x,0)$ i.e. $\det(C_1 u(x), \overline{\varphi}(x,0)) = 0$. Equivalently

$$\begin{cases} c(k,n) u^k(0) \overline{\phi^{-k}(0)} - c(-k,n) u^{-k}(0) \overline{\phi^k(0)} = 0, \\ c(k,n) u^k(\alpha) \overline{\phi^{-k}(0) e^{-2i\pi l^k}} - c(-k,n) u^{-k}(\alpha) \overline{\phi^k(0) e^{2i\pi l^k}} = 0. \end{cases}$$

Finally, since $c(k,n) = -c(-k,n)$ and $c(k,n)$ is assumed to do not vanish, the boundary conditions of macroscopic equation (45) are

$$u^k(x) \phi_n^k(0) e^{2i\pi \frac{l^k x}{\alpha}} + u^{-k}(x) \phi_n^{-k}(0) e^{-2i\pi \frac{l^k x}{\alpha}} = 0 \text{ at } x \in \partial\Omega.$$

4.3.2 Case $k = 0$

In case $k = 0$, to avoid any confusion with λ^0, the upper indices $k = 0$ are removed. We denote by ϕ_n, ϕ_m the eigenvectors associated to $\lambda^0 = \lambda_n = \lambda_m$, solutions to $\mathcal{P}(0)$ in (20), and by $\sum_p$, $\sum_q$ the sums over p or q varying in $\{n, m\}$. We restart with a test function

$$v^{\varepsilon}(x) := \Re(\sum_p v_p) \in H_0^1(\Omega) \cap H^2(\Omega) \tag{47}$$

for the very weak formulation (42). We pose $v_p(x,y) = \psi_p(x)\phi_p(y)$ with $\psi_p(x) \in H^1(\Omega)$ for $p \in \{n, m\}$. Since ϕ_p is periodic thus $\phi_p(\frac{x}{\varepsilon}) = \phi_p(0)$ at $x \in \partial\Omega$ and the boundary condition in (47) is equivalent to

$$\sum_p \psi_p(x)\phi_p(0) = 0 \text{ at } x \in \partial\Omega.$$

By setting $c(p,q) := c(0,p,q)$ for $p, q \in \{n, m\}$, using the expression in Lemma 13 of the weak limit w^0 of $S_0^{\varepsilon} w^{\varepsilon}$,

$$w^0(x,y) = \sum_p u_p(x) \phi_p(y), \tag{48}$$

using the periodicity of $\left(\phi_p\right)_p$ and conducting the same calculations as for $k \neq 0$, we obtain

$$\sum_{p,q} \left[\int_{\Omega} (c(p,q)\partial_x u_q + \lambda^1 b(p,q) u_q) \cdot \psi_p \, dx - \int_{\partial\Omega} c(p,q) u_q \cdot \psi_p \, dx \right] = 0.$$

With $u = (u_p)_p$, $\psi = (\psi_p)_p$, $\phi = \left(\phi_p\right)_p$ and $C_1 = (c(p,q))_{p,q}$, $C_2 = (b(p,q))_{p,q}$, the macroscopic problem turns to be

$$C_1 \partial_x u + \lambda^1 C_2 u = 0 \text{ in } \Omega, \tag{49}$$

with the boundary conditions

$$C_1 u(x).\psi(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \psi(x).\phi(0) = 0 \text{ on } \partial\Omega.$$

Equivalently, $C_1 u(x)$ is collinear to $\phi(0)$ on $\partial\Omega$ or

$$\det\left(C_1 u(x), \phi(0)\right) = 0 \text{ on } \partial\Omega. \tag{50}$$

But $c(p,p) = 0$, so (50) simplifies to

$$\begin{cases} c\,(n,m)\,u_m\,(0)\,\phi_m\,(0) - c\,(m,n)\,u_n\,(0)\,\phi_n\,(0) = 0, \\ c\,(n,m)\,u_m\,(\alpha)\,\phi_m\,(0) - c\,(m,n)\,u_n\,(\alpha)\,\phi_n\,(0) = 0. \end{cases}$$

Finally, since $c\,(n,m) = -c\,(m,n)$ and $c\,(n,m) \neq 0$, the boundary conditions are

$$u_n\,(x)\,\phi_n\,(0) + u_m\,(x)\,\phi_m\,(0) = 0 \text{ on } \partial\Omega.$$

4.4 Analytic solutions

For $k \in Y^*$ and $\rho = 1$, we solve the macroscopic equations In Section 4.4.1. These solutions are used to validate the numerical results in the final Section. Moreover, in Section 4.4.2, the exact formulations of the two-scale eigenmodes are found for $\rho = 1$ and $a = 1$.

4.4.1 The case $\rho = 1$

For $k \neq 0$ and $b\,(n,n) = 1$, the exact solutions of the macroscopic equation (25) are

$$u_n^\sigma\,(x) = d^\sigma e^{-\lambda^1 c(\sigma,n)^{-1} x} \text{ for each } \sigma \in I^k$$

where d^σ is any complex number. Applying the boundary condition (26) and assuming that $\phi_n^k\,(0) \neq 0$, the eigenvalue is

$$\lambda^1 = \frac{c(k,n)}{\alpha}\left(2i\pi l^k - i\ell\pi\right) \text{ for } \ell \in \mathbb{Z}. \tag{51}$$

Furthermore, $u_n^k = \overline{u_n^{-k}}$ and $\phi_n^k\,(0) = \overline{\phi_n^{-k}}\,(0)$ then $\operatorname{Re}\left(d^k\phi_n^k\,(0)\right) = 0$, or $d^k\phi_n^k\,(0) = i\delta$ for any $\delta \in \mathbb{R}$. Thus,

$$d^k = \frac{i\delta}{\phi_n^k\,(0)} \text{ and } d^{-k} = -\frac{i\delta}{\phi_n^{-k}\,(0)} \text{ for any } \delta \in \mathbb{R}.$$

For $k = 0$, using the equalities $c\,(n,n) = c\,(m,m) = 0$, $b\,(n,m) = b\,(m,n) = 0$ and $b\,(n,n) = b\,(m,m) = 1$, the macroscopic equation (27) is rewritten

$$\begin{cases} c\,(n,m)\,\partial_x u_m^0 + \lambda^1 u_n^0 = 0 & \text{in} \quad \Omega, \\ c\,(m,n)\,\partial_x u_n^0 + \lambda^1 u_m^0 = 0 & \text{in} \quad \Omega. \end{cases} \tag{52}$$

If $\lambda^1 = 0$, $\partial_x u_m^0 = 0$ and $\partial_x u_n^0 = 0$ in Ω, then u_m^0 and u_n^0 are independent on x, equivalently, u_m^0 and u_n^0 are complex numbers.

If $\lambda^1 \neq 0$, the first equation gives $u_n^0 = -\frac{c(n,m)\partial_x u_m^0}{\lambda^1}$ in Ω and since $c(n,m) = -c(m,n)$ then

$$\partial_{xx} u_m^0 = -\left(\frac{\lambda^1}{c\,(n,m)}\right)^2 u_m^0 \tag{53}$$

and

$$u_m^0(x) = d_1 \cos\left(\frac{\lambda^1}{c(n,m)}x\right) + d_2 \sin\left(\frac{\lambda^1}{c(n,m)}x\right)$$

for two constants for $d_1, d_2 \in \mathbb{C}$ and u_n^0 follows by its above expression. Applying the boundary condition (28), if $\phi_m^0(0) \neq 0$,

$$\lambda^1 = \frac{\ell\pi c(n,m)}{\alpha} \text{ for } \ell \in \mathbb{Z} \text{ and } d_1 = -d_2\frac{\phi_n^0(0)}{\phi_m^0(0)} \tag{54}$$

for any $\ell \in \mathbb{Z}$ and $d_2 \in \mathbb{C}$. If $\phi_m^0(0) = 0$ then $\phi_n^0(0) = 0$ or $u_n^0(x) = 0$ on $\partial\Omega$. In the case $\phi_n^0(0) = 0$, the macroscopic equation is lacking of boundary conditions and their solutions are not unique, they depend on arbitrary coefficients d_1, d_2 and λ^1. When $u_n^0(x) = 0$ at $\partial\Omega$, there is an alternative, or u_n^0 is the trivial solution or

$$\det\begin{pmatrix} 0 & 1 \\ -\sin\left(\frac{\lambda_1}{c(n,m)}\alpha\right) & \cos\left(\frac{\lambda_1}{c(n,m)}\alpha\right)\end{pmatrix} = 0$$

and then $d_2 = 0$, $\lambda^1 = \frac{\ell\pi c(n,m)}{\alpha}$ for any $\ell \in \mathbb{Z}$ and $d_1 \in \mathbb{C}$.

Remark 14 *According to (51) and (54), $\lambda^1 = 0$ iff $\ell = 2l^k$ for $k \neq 0$ and iff $\ell = 0$ otherwise. So, in any case small values of $\lambda^{1,\ell}$ correspond to indices ℓ in a vicinity of $2l^k$ or to $\frac{2k\alpha}{\varepsilon}$ when $\varepsilon > 0$.*

4.4.2 The case $a = \rho = 1$

We consider the spectral problem

$$-\partial_{yy}^2\phi^k = \lambda^k\phi^k \text{ in } Y$$

with the $k-$quasi-periodicity conditions.

For $k \neq 0$, for a mapping $m \mapsto n(m)$ from $\mathbb{Z}$ to $\mathbb{N}^*$ not detailed here, $\lambda_{n(m)}^k = 4\pi^2(m+k)^2$ and there are exactly two conjugated solutions $\phi_{n(m)}^\sigma(y) = e^{sign(\sigma)2i\pi(m+k)y}$ for any $m \in \mathbb{Z}$ and $\sigma \in I^k$. It follows that $c(\sigma, n(m)) = sign(\sigma)\, 4i\pi(m+k)$, $b(\sigma, n(m)) = 1$ and $\lambda^1 = -\frac{4\pi^2}{\alpha}(2l^k - \ell)(m+k)$ for any $\ell \in \mathbb{Z}$, so

$$u_{n(m)}^\sigma(x) = d^\sigma e^{\frac{sign(\sigma)i\pi}{\alpha}(2l^k-\ell)x}$$

and the resulting two-scale eigenmode is

$$w^\sigma(x,y) = d^\sigma e^{\frac{sign(\sigma)i\pi}{\alpha}(2l^k-\ell)x} e^{sign(\sigma)2i\pi(n+k)y}.$$

For $k = 0$, for each $\lambda_{n(m)}^0 = (2\pi m)^2$ there are two eigenvectors $\phi_{n(m)}(y) = \cos(2\pi my)$ and $\phi_{n(m)+1}(y) = \sin(2\pi my)$ so

$$C_1 = 2m\pi\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},\ C_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\ \begin{pmatrix}\phi_{n(m)}(0) \\ \phi_{n(m)+1}(0)\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}.$$

It implies that $\lambda^1 = \frac{4m\ell\pi^2}{\alpha}$ for any $\ell \in \mathbb{Z}$ and

$$u_{n(m)}(x) = d_0 \sin\left(\ell\pi\frac{x}{\alpha}\right) \text{ and } u_{n(m)+1}(x) = d_0\cos(\ell\pi\frac{x}{\alpha}),$$

then the two-scale eigenmode is

$$w(x,y) = d_0[\sin\left(\ell\pi\frac{x}{\alpha}\right)\cos(2\pi my) + \cos(\ell\pi\frac{x}{\alpha})\sin(2\pi my)] \text{ for } \ell, m \in \mathbb{Z}.$$

4.5 Neumann boundary conditions

We consider the spectral problem with Neumann boundary conditions

$$P^{\varepsilon}w^{\varepsilon} = \lambda^{\varepsilon}\rho^{\varepsilon}w^{\varepsilon} \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_x w^{\varepsilon} = 0 \quad \text{on} \quad \partial\Omega.$$

The process of homogenization and the results are similar to the case of Dirichlet boundary conditions. The microscopic problem and the internal macroscopic equation are unchanged while the boundary conditions of the latter are

$$\sum_{\sigma\in I^k}\sum_m u_m^{\sigma}(x)\,\partial_y\phi_m^{\sigma}(0)\,e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega$$

where the cases $k \neq 0$ and $k = 0$ are not separated so a general notation is adopted for the sum over m and σ. Their derivation follows the same steps, so we only mention the boundary condition satisfied by the test functions. They are chosen to satisfy $\partial_x v^{\varepsilon}(x) = 0$ on $\partial\Omega$ or equivalently,

$$\sum_{\sigma\in I^k}\sum_m \partial_x\psi_m^{\sigma}(x)\,\phi_m^{\sigma}\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\psi_m^{\sigma}(x)\,\partial_y\phi_m^{\sigma}\left(\frac{x}{\varepsilon}\right) = 0 \text{ on } \partial\Omega.$$

Multiplying by ε,

$$\sum_{\sigma\in I^k}\sum_m \psi_m^{\sigma}(x)\,\partial_y\phi_m^{\sigma}\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) = 0 \text{ on } \partial\Omega, \tag{55}$$

then using the $\sigma-$quasi-periodicity of ϕ_m^{σ} and passing to the limit

$$\sum_{\sigma\in I^k}\sum_m \psi_m^{\sigma}(x)\,\partial_y\phi_m^{\sigma}(0)\,e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega.$$

5 Homogenization based on a first order formulation

In this section, the homogenized model is derived based on a first order formulation. The calculations are less detailed than in Section 4, only the main results and the proof principles are given.

5.1 Reformulation of the spectral problem and the main result

We start by setting

$$U^{\varepsilon} = \left(\frac{\sqrt{a^{\varepsilon}}\partial_x w^{\varepsilon}}{i\sqrt{\lambda^{\varepsilon}}}, \sqrt{\rho^{\varepsilon}}w^{\varepsilon}\right),\ \mu^{\varepsilon} = \sqrt{\lambda^{\varepsilon}},$$

$$A^{\varepsilon} = \begin{pmatrix} 0 & \sqrt{a^{\varepsilon}}\partial_x\left(\frac{1}{\sqrt{\rho^{\varepsilon}}}\cdot\right) \\ \frac{1}{\sqrt{\rho^{\varepsilon}}}\partial_x\left(\sqrt{a^{\varepsilon}}\cdot\right) & 0 \end{pmatrix},\ n_{A^{\varepsilon}} = \frac{1}{\sqrt{\rho^{\varepsilon}}}\begin{pmatrix} 0 & \sqrt{a^{\varepsilon}}n_{\Omega} \\ \sqrt{a^{\varepsilon}}n_{\Omega} & 0 \end{pmatrix}$$

with the domain of the operator A^{ε},

$$D(A^{\varepsilon}) := \left\{(\varphi,\phi) \in L^2(\Omega)\times L^2(\Omega)\,\middle|\,\sqrt{a^{\varepsilon}}\varphi \in H^1(\Omega),\phi \in H_0^1(\Omega)\right\} \subset L^2(\Omega)^2,$$

so that iA^{ε} is self-adjoint on $L^2(\Omega)^2$ as proved in [8]. The spectral equation (8) can be recasted as a first-order system

$$A^{\varepsilon}U^{\varepsilon} = i\mu^{\varepsilon}U^{\varepsilon} \text{ in } \Omega \ \text{ and } U_2^{\varepsilon} = 0 \text{ on } \partial\Omega, \tag{56}$$

where U_2^ε is the second component of U^ε. We observe that $\|\sqrt{\rho^\varepsilon} w^\varepsilon\|_{L^2(\Omega)} \leq \|\sqrt{\rho^\varepsilon}\|_{L^\infty(\Omega)}$ and that $\left\|\frac{\sqrt{a^\varepsilon}\partial_x w^\varepsilon}{i\sqrt{\lambda^\varepsilon}}\right\|_{L^2(\Omega)} \leq M_0$ can be deduced from the weak formulation (11), therefore U^ε is uniformly bounded,

$$\|U^\varepsilon\|_{L^2(\Omega)}^2 \leq M_1. \tag{57}$$

We start our analysis from the system expressed in a distributional sense,

$$\int_\Omega U^\varepsilon \cdot (i\mu^\varepsilon - A^\varepsilon)\,\Psi\; dx = 0, \tag{58}$$

for all admissible test functions $\Psi = (\varphi, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$. We choose $\mu_0 = \sqrt{\lambda^0}$ and $\mu_1 = \frac{\lambda^1}{2\mu_0}$, so μ^ε can be decomposed as

$$\mu^\varepsilon = \frac{\mu_0}{\varepsilon} + \mu_1 + O(\varepsilon). \tag{59}$$

The asymptotic spectral problem (20) is also restated as a first order system by setting

$$A_k := \begin{pmatrix} 0 & \sqrt{a}\partial_y\left(\frac{1}{\sqrt{\rho}}.\right) \\ \frac{1}{\sqrt{\rho}}\partial_y\left(\sqrt{a}.\right) & 0 \end{pmatrix} \text{ and } n_{A_k} = \frac{1}{\sqrt{\rho}}\begin{pmatrix} 0 & \sqrt{a}n_Y \\ \sqrt{a}n_Y & 0 \end{pmatrix},$$

and

$$e_n^k := \frac{1}{\sqrt{2}}\begin{pmatrix} -i\frac{s_n}{\sqrt{\lambda_{|n|}^k}}\sqrt{a}\partial_y\left(\phi_{|n|}^k\right) \\ \sqrt{\rho}\phi_{|n|}^k \end{pmatrix} \text{ and } \mu_n^k = s_n\sqrt{\lambda_{|n|}^k} \text{ for all } n \in \mathbb{Z}^*, \tag{60}$$

s_n denoting the sign of n. As proved in [8], iA_k is self-adjoint on the domain

$$D(A_k) := \left\{(\varphi, \phi) \in L^2(Y)^2 \,|\, \sqrt{a}\varphi \in H_k^1(Y), \frac{\phi}{\sqrt{\rho}} \in H_k^1(Y)\right\} \subset L^2(Y)^2.$$

The Bloch wave spectral problem $\mathcal{P}(k)$ is equivalent to finding pairs $\left(\mu_n^k, e_n^k\right)$ indexed by $n \in \mathbb{Z}^*$ solution to

$$\mathcal{Q}(k) : A_k e_n^k = i\mu_n^k e_n^k \text{ in } Y \text{ with } e_n^k \in H_k^1(Y)^2. \tag{61}$$

The corresponding weak formulation is

$$\int_Y e_n^k \cdot \left(A_k - i\mu_n^k\right)\Psi\, dy = 0 \text{ for all } \Psi \in D(A_k). \tag{62}$$

The relation between the operator A^ε and the scaled operator A_k is obtained by considering any regular vector $\psi = \psi(x, y)$ depending on both space scales,

$$A^\varepsilon\left(\psi\left(x, \frac{x}{\varepsilon}\right)\right) = \left(\left(\frac{1}{\varepsilon}A_k + B\right)\psi\right)\left(x, \frac{x}{\varepsilon}\right), \tag{63}$$

where the operator B is defined as the result of the formal substitution of $x-$derivatives by $y-$derivatives in A_k, i.e.

$$B := \begin{pmatrix} 0 & \sqrt{a}\partial_x\left(\frac{1}{\sqrt{\rho}}.\right) \\ \frac{1}{\sqrt{\rho}}\partial_x\left(\sqrt{a}.\right) & 0 \end{pmatrix}.$$

For any $n \in \mathbb{Z}^*$ and $k \in Y^*$, $M_n^k := \left\{i \in \mathbb{Z}^* \mid \mu_i^k = \mu_n^k\right\}$ is the set of indices of eigenvectors related to the same eigenvalue μ_n^k. For all $k \in Y^* \setminus \{0\}$, since $\mu_n^k = \mu_n^{-k}$ then $M_n^k = M_n^{-k}$.

Remark 15 *From now on, we shall assume that the weak limit of $S_k^\varepsilon U^\varepsilon$ in $L^2(\Omega\times Y)$ is not vanishing to avoid eigenmodes related to the boundary spectrum (see Proposition 7.7 in [6]).*

Theorem 16 *For $k\in Y^*$, let $(\mu^\varepsilon, U^\varepsilon)$ be solution of (56) then $\sum_{\sigma\in I^k} S_\sigma^\varepsilon U^\varepsilon$ is bounded in $L^2(\Omega\times Y)$. For $\varepsilon\in E_k$, assuming that the renormalized sequence $\varepsilon\mu^\varepsilon$ satisfies the decomposition (59) with $\mu_0=\mu_n^k$ an eigenvalue of the Bloch wave spectrum, any weak limit G_k of $\sum_{\sigma\in I^k} S_\sigma^\varepsilon U^\varepsilon$ in $L^2(\Omega\times Y)$ has the form*

$$G_k(x,y)=\sum_{\sigma\in I^k}\sum_{m\in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma(y), \tag{64}$$

where $(u_m^\sigma)_{m,\sigma}$ are the solutions of the macroscopic equations (25, 26) or (27, 28).

Therefore, the physical solution U^ε can be approximated by

$$U^\varepsilon(x)\approx\sum_{\sigma\in I^k}\sum_{m\in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma\left(\frac{x}{\varepsilon}\right). \tag{65}$$

Proof. For a given $k\in Y^*$, let U^ε be solution of (56) which is bounded in $L^2(\Omega)$, the property (15) yields the boundness of $\|S_\sigma^\varepsilon U^\varepsilon\|_{L^2(\Omega\times Y)}$. So there exist $U^\sigma\in L^2(\Omega\times Y)^2$ such that, up the extraction of a subsequence, $S_\sigma^\varepsilon U^\varepsilon$ tends weakly to U^σ in $L^2(\Omega\times Y)^2$ and hence, $\sum\limits_{\sigma\in I^k} S_\sigma^\varepsilon U^\varepsilon$ converges to $G_k(x,y)=\sum\limits_{\sigma\in I^k} U^\sigma(x,y)$. Using the decomposition (66) of U^σ in the forthcoming Lemma 17,

$$G_k(x,y)=\sum_{\sigma\in I^k}\sum_{m\in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma(y)$$

The macroscopic problem solved by the coefficients $(u_m^\sigma)_{\sigma,m}$ is derived in Section 5.2.2. ■

5.2 Model derivation

5.2.1 Modal decomposition on the Bloch modes

Lemma 17 *Let a sequence $(\mu^\varepsilon, U^\varepsilon)$ be solution of (56) and satisfies (59) with $\mu_0=\mu_n^k$ for given $n\in\mathbb{Z}^*$ and $k\in Y^*$, we extract a subsequence of ε, still denoted by ε, such that $S_k^\varepsilon U^\varepsilon$ converges weakly to U^k in $L^2(\Omega\times Y)^2$. If $U^k\in D(A_k)$ then (μ_n^k, U^k) is solution of the Bloch wave equation (61) and U^k admits the modal decomposition*

$$U^k(x,y)=\sum_{m\in M_n^k} u_m^k(x)\, e_m^k(y)\ \text{ with } u_m^k\in L^2(\Omega). \tag{66}$$

Proof. For each $k\in Y^*$, taking $\Psi(x,y):=\theta(x)\phi(y)$ with $\theta(x)\in C_c^\infty(\Omega)$ and $\phi(y)\in C^\infty(Y)^2$ $k-$quasi-periodic in y, considering $\Re\Psi$ as a test functions in (58), and using (63,59),

$$\int_\Omega U^\varepsilon\cdot\Re\left(i\frac{\mu_0}{\varepsilon}+i\mu_1-\frac{A_k}{\varepsilon}-B\right)\Psi\, dx+O(\varepsilon)=0.$$

Multiplying by ε

$$\int_\Omega U^\varepsilon\cdot\Re\left(i\mu_0-A_k\right)\Psi\, dx+O(\varepsilon)=0,$$

and passing to the limit thanks to Corollary 4,

$$\frac{1}{|Y|}\int_{\Omega\times Y} U^k\cdot(i\mu_0 - A_k)\,\Psi\ dxdy = 0$$

which is the weak formulation of the Bloch wave equations. If in addition $U^k \in D(A_k)$, integrating by parts yields

$$\frac{1}{|Y|}\int_{\Omega\times Y}(A_k - i\mu_0)\,U^k\cdot\Psi\ dxdy - \frac{1}{|Y|}\int_{\Omega\times\partial Y} U^k\cdot n_{A_k}\Psi\ dxdy = 0 \tag{67}$$

providing in turn the strong formulation,

$$A_k U^k = i\mu_0 U^k \quad \text{in} \quad \Omega\times Y. \tag{68}$$

Since the product of a periodic function by a $k-$quasi-periodic function is $k-$quasi-periodic then $n_{A_k}\Psi$ is $k-$quasi-periodic in y. Therefore, U^k is $k-$quasi-periodic in y and finally is a Bloch eigenvector in y. By projection, it can be decomposed as

$$U^k(x,y) = \sum_{m\in M_n^k} u_m^k(x)\,e_m^k(y) \ \text{ with } u_m^k = \frac{1}{b(k,m,m)}\int_Y U^k\cdot e_m^k\ dy \in L^2(\Omega).$$

■

5.2.2 Derivation of the macroscopic equation

The macroscopic equation is stated for each $k\in Y^*$ and each eigenvalue μ_n^k of the Bloch wave spectral problem $\mathcal{Q}(k)$. We pose

$$\kappa(k,n,m) = \frac{-ic(k,n,m)}{2\mu_0} \text{ for } m\in M_n^k \tag{69}$$

where $c(k,n,m)$ is defined in (22) and notice that

$$\kappa(k,n,m) = -\kappa(-k,m,n),\ \ \kappa(k,n,m) = -\overline{\kappa(-k,n,m)},$$
$$\kappa(k,n,m) = -\overline{\kappa(k,m,n)}, \text{ and } \kappa(0,n,n) = 0.$$

For the sake of simplicity, we do the proof for $n\in\mathbb{Z}^{*+}$ only and denote by $\kappa(k,n) = \kappa(k,n,n)$ and $\kappa(n,m) = \kappa(0,n,m)$. For general n, the proof is the same but ϕ_n^k is replaced by $\phi_{|n|}^k$.

Case $k\neq 0$ The pairs $\left(\mu_n^k, e_n^k\right)$ and $\left(\mu_n^{-k}, e_n^{-k}\right)$ are the eigenmodes of the spectral equations $\mathcal{Q}(\pm k)$ in (61) corresponding to the eigenvalue $\mu_0 = \mu_n^k = \mu_n^{-k}$. We pose $\Psi^\varepsilon = \Re\left(\Psi^k + \Psi^{-k}\right) \in H^1(\Omega)\times H_0^1(\Omega)$ as a test function in the weak formulation (58), with each $\Psi^\sigma(x,y) = \psi^\sigma(x)\,e_n^\sigma(y)$ where $\psi^\sigma\in H^1(\Omega)$ and satisfies the boundary conditions,

$$\sum_\sigma \psi^\sigma(x)\,\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) = 0 \text{ on } \partial\Omega.$$

Notice that this condition is related to the second component of Ψ^ε only. Proceeding as in Section 4.3.1 yields (41). Since $(i\mu_0 - A_\sigma)\,\Psi^\varepsilon = 0$ for all σ, applying (59, 63), then Equation (58) yields

$$\sum_\sigma\int_\Omega U^\varepsilon\cdot\Re\,(i\mu_1 - B)\,\Psi^\sigma\ dx + O(\varepsilon) = 0. \tag{70}$$

But $(i\mu_1 - B)\,\Psi^\sigma$ is $\sigma-$quasi-periodic so passing to the limit thanks to Corollary 4,

$$\frac{1}{|Y|}\sum_{\sigma}\int_{\Omega\times Y} U^\sigma \cdot (i\mu_1 - B)\,\Psi^\sigma dxdy = 0. \tag{71}$$

From Lemma 17, U^σ is decomposed as

$$U^\sigma(x,y) = u_n^\sigma(x)\, e_n^\sigma(y)\,.$$

After replacement,

$$\sum_\sigma \int_\Omega \left(-i\mu_1 b(\sigma,n)\, u_n^\sigma \cdot \psi^\sigma + \kappa(\sigma,n)\, u_n^\sigma \cdot \partial_x \psi^\sigma\right) dx = 0$$

for all $\psi^\sigma \in H^1(\Omega)$ fulfilling (41). Moreover, if $u_n^\sigma \in H^1(\Omega)$ it satisfies the strong form of the internal equations

$$\kappa(\sigma,n)\,\partial_x u_n^\sigma - i\mu_1 b(\sigma,n)\, u_n^\sigma = 0 \text{ in } \Omega \text{ for all } \sigma \in I^k, \tag{72}$$

and the boundary conditions

$$\sum_\sigma \kappa(\sigma,n)\, u_n^\sigma \cdot \psi^\sigma = 0 \text{ on } \partial\Omega.$$

Following the same calculations as in Section 4.3.1, with the matrices $C_1 = diag(\kappa(\sigma,n))$, $C_2 = diag(b(\sigma,n))$ and the vectors $u = (u_n^\sigma)_\sigma$, $\psi = (\psi^\sigma)_\sigma$, $\varphi = \left(\phi^\sigma(0)\, e^{sign(\sigma)2i\pi x\frac{l^k}{\alpha}}\right)_\sigma$, (72) is written on the matrix form

$$C_1 \partial_x u = i\mu_1 C_2 u \text{ in } \Omega\,,$$

with boundary condition

$$C_1 u(x)\,.\overline{\psi}(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \overline{\varphi}(x,0)\,.\overline{\psi}(x) = 0 \text{ on } \partial\Omega.$$

Equivalently, $Cu(x)$ is collinear with $\overline{\varphi}(x,0)$ yielding the boundary conditions

$$u_n^k(x)\,\phi_n^k(0)\, e^{2i\pi\frac{l^k x}{\alpha}} + u_n^{-k}(x)\,\phi_n^{-k}(0)\, e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega \tag{73}$$

after remarking that $\kappa(\sigma,n) \neq 0$. Finally, with (69) and $\lambda^1 = 2\mu_0\mu_1$ the macroscopic problem (25, 26) is recovered.

Case $k = 0$ We adopt the same simplifications of notations that in Section 4.3.2. Let e_n and e_m be the Bloch eigenmodes of $\mathcal{Q}(0)$ in (61) regarding the double eigenvalue $\mu_0 = \mu_n = \mu_m$. In this case $M_n^0 = \{n,m\}$. Taking $\Psi^\varepsilon = \sum_{p\in M_n^0} \Re(\Psi_p) \in H^1(\Omega)\times H_0^1(\Omega)$ as a test function with $\Psi_p(x,y) = \psi_p(x)\, e_p(y)$ and $\psi_p \in H^1(\Omega)$. Due to the periodicity of ϕ_p, the second component of Ψ^ε satisfies the boundary conditions

$$\sum_{p\in M_n^0} \psi_p(x)\,\phi_p(0) = 0 \text{ on } \partial\Omega. \tag{74}$$

Following similar calculations as for the case $k \neq 0$, the weak limit U^0 of $S_0^\varepsilon U^\varepsilon$ in $L^2(\Omega\times Y)^2$ is

$$U^0(x,y) = \sum_{p\in M_n^0} u_p(x)\, e_p(y)$$

and u_p is solution to the weak formulation

$$\sum_{q\in M_n^0}\int_\Omega -i\mu_1 b\left(p,q\right)u_q\cdot\psi_p+\ \kappa\left(p,q\right)u_q\cdot\partial_x\psi_p\ dx=0$$

for all $\psi_p\in H^1\left(\Omega\right)$ with $p\in M_n^0$. If $u_q\in H^1\left(\Omega\right)$ it is a solution to the internal equations

$$\sum_{q\in M_n^0}\kappa\left(p,q\right)\partial_x u_q-i\mu_1\ b\left(p,q\right)u_q=0\ \text{ in }\Omega\ \text{ for }p\in M_n^0, \tag{75}$$

and to the boundary conditions

$$\int_{\partial\Omega}\sum_{p,q\in M_n^0}\kappa\left(p,q\right)u_q\cdot\psi_p\ dx=0.$$

Here, with $C_1=\left(\ \kappa\left(p,q\right)\right)_{p,q}$, $C_2=\left(\ b\left(p,q\right)\right)_{p,q}$, $u=\left(u_p\right)_p,\ \psi=\left(\psi_p\right)_p,\ \phi=\left(\phi_p\right)_p$,

$$C_1\partial_x u=i\mu_1 C_2 u\ \text{ in }\Omega\,,$$

$$\text{and } Cu\left(x\right).\overline{\psi}\left(x\right)=0\ \text{ on }\partial\Omega\text{ for all }\psi\text{ such that }\phi\left(0\right).\overline{\psi}\left(x\right)=0\text{ on }\partial\Omega.$$

But $\kappa\left(p,p\right)=0$, therefore

$$u_n\left(x\right)\phi_n\left(0\right)+u_m\left(x\right)\phi_m\left(0\right)=0\text{ on }\partial\Omega. \tag{76}$$

As for $k\neq 0$, these macroscopic equations are equivalent to (27, 28).

6 Numerical simulations

We report simulations regarding comparisons of physical eigenmodes and their approximation by two-scale modes for $\rho=1$. In Subsection 6.2, for each given high frequency physical eigenelement a two-scale eigenelement realizing a good approximation is identified. This shows that the two-scale model can actually be used as an approximation of the complete high-frequency spectra. Conversely, Subsection 6.3 addresses the modeling problem i.e. it introduces a way to generate approximations of high-frequency spectra from the two-scale model only. Finally, in 6.4 the order of convergence with respect to ε is analyzed. The next section describes the main simulation parameters.

6.1 Simulation methods and conditions

Both, the physical spectral problem and the Bloch wave spectral problem are discretized by a quadratic finite element method. The number of elements are respectively denoted N_{phys} and N_{bloch}. The implementation of the $k-$quasi-periodic boundary condition is achieved by elimination of the last degree of freedom. More precisely, for $n\in\{1,...,2N_{bloch}+1\}$ the node indices, ϕ_n a degree of freedom of ϕ a Bloch eigenmode and φ_n the corresponding quadratic Lagrange interpolation function,

$$\phi\left(y\right)\simeq\sum_{n=2}^{2N_{bloch}}\phi_n\varphi_n+\phi_1\varphi_1+\phi_{2N_{bloch}+1}\varphi_{2N_{bloch}+1}.$$

Using the relation $\phi(1) = e^{2i\pi k}\phi(0)$ and taking $\varphi_1 + e^{2i\pi k}\varphi_{2N_{bloch}+1}$ as the first base function allows to eliminate $\phi_{2N_{bloch}+1}$,

$$\phi(y) \simeq \sum_{n=2}^{2N_{bloch}} \phi_n \varphi_n + \phi_1 \left(\varphi_1 + e^{2i\pi k}\varphi_{2N_{bloch}+1}\right).$$

The sets of indices considered in the simulations of high frequency physical modes and Bloch modes are denoted by $\mathcal{J}^\varepsilon$ and J^k, the former being generally included in $(\alpha/2\varepsilon, N_{phys}/2)$. The Bloch modes are calculated for $k \geq 0$ only, and the other cases can be deduced by conjugation. For each Bloch eigenmode $\left(\lambda_n^k, \phi_n^k\right)$, the macroscopic solutions $\left(\lambda^{1,\ell}, u_{m,\ell}^k\right)_{m,\ell}$ are given in Section 4.4.1 with $\delta = 1$ and $d_2 = \phi_m^0(0)$ for any m such that $\lambda_m^k = \lambda_n^k$ and $\ell \in \mathbb{Z}$. In fact, according to Remark 14 the index ℓ should vary in $J_n^k = \left[\frac{2k}{\varepsilon}\right] + \{-r, ..., r\}$, for a small integer r, so that only the first macroscopic eigenmodes be taken into account. In the next discussions, we use the following notations for the two-scale approximations of the eigenvalues and eigenmodes exhibiting clearly their parameters ε, k, n and ℓ,

$$\gamma_{n,\ell}^{\varepsilon,k} := \lambda_n^k + \varepsilon\lambda^{1,\ell} \text{ and } \psi_{n,\ell}^{\varepsilon,k}(x) := \sum_{\sigma\in I^k}\sum_m u_{m,\ell}^\sigma(x)\,\phi_m^\sigma\left(\frac{x}{\varepsilon}\right) \text{ for } \ell \in J_n^k,\ n \in J^k. \tag{77}$$

In the simulations reported in Sections 6.2 and 6.3 only one physical problem is used, namely $\Omega = (0,1)$, $a^\varepsilon(x) = \sin(2\pi x/\varepsilon) + 2$, 50 cells (i.e. $\varepsilon = 1/50$), and $N_{phys} = 2,000$. Other number of cells are used in Section 6.4 for the convergence analysis. Consequently, the coefficient of the Bloch wave spectral problem is $a(y) = \sin(2\pi y) + 2$. The set Y^* of positive wave numbers in Y^* is discretized by $L_{125}^{*+} = \{0, ..., 62/125\}$ with step $\Delta_k = 1/125$ and $N_{bloch} = 50$. The subset of macroscopic eigenvalues is restricted by $r = 15$.

The first ten graphs $(k \mapsto \lambda_n^k)_{n=1,...,10}$ of Bloch eigenvalues are described in Figure 1. The graphs are symmetric about the axis $k = 0$ which confirms that $\lambda_n^k = \lambda_n^{-k}$ as remarked in Notation 5. Moreover, all eigenvalues λ_n^k are simple for $k \neq 0$ and double for $k \in \left\{0, \pm\frac{1}{2}\right\}$.

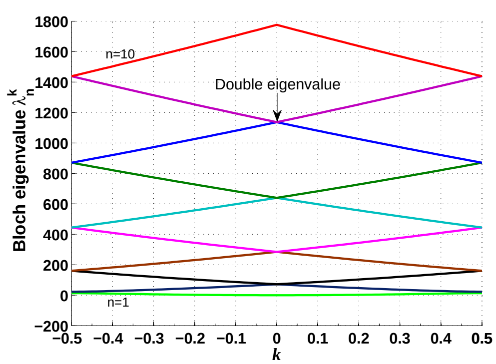

Figure 1: First ten eigenvalues of the Bloch wave spectral problem.

6.2 Approximation of physical modes by two-scale modes

We discuss the approximation of a given solution $\left(\lambda_p^\varepsilon, w_p^\varepsilon\right)$ of Equation (8) for a given value of ε. From Remark 11 we expect to show numerically that there exists a suitable pair (k, n) such that the equality $\left(\lambda_p^\varepsilon, w_p^\varepsilon\right) = (\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ is exact with $(\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ defined in (77) and $\lambda^{1,\ell} = 0$. Moreover, in the perspective of Remark 12, k varies in L_{125}^{*+} only and approximations with $\lambda^{1,\ell} \neq 0$ are expected. Whatever if $\lambda^{1,\ell}$ vanishes or not, we expect to search approximations for both eigenvalues and eigenvectors which turns to be an multi-objective optimization problem that might be solved by a dedicated method. However, to reduce the computational cost, we propose an alternate approach consisting in minimizing the error on eigenvalues in the approximation (10),

$$er_{value}\left(k\right) = \min_{n\in\mathbb{N},\ \ell\in J_n^k} \left| \frac{\varepsilon^2\lambda_p^\varepsilon - \gamma_{n,\ell}^{\varepsilon,k}}{\varepsilon^2\lambda_p^\varepsilon} \right|, \tag{78}$$

for each $k \in L_{125}^{*+}$, and then in finding which one minimizes

$$er_{vector}\left(k\right) = \frac{\left\| w_p^\varepsilon - \psi_{n_k,\ell_k}^{\varepsilon,k} \right\|_{L^2(\Omega)}}{\left\| w_p^\varepsilon \right\|_{L^\infty(\Omega)}}$$

the error on eigenvectors in the approximation (30) where ℓ_k, n_k are the optimal arguments in (78). The optimal error on eigenvectors is then

$$er_{vector} = \min_{k\in L_{125}^{*+}} er_{vector}\left(k\right). \tag{79}$$

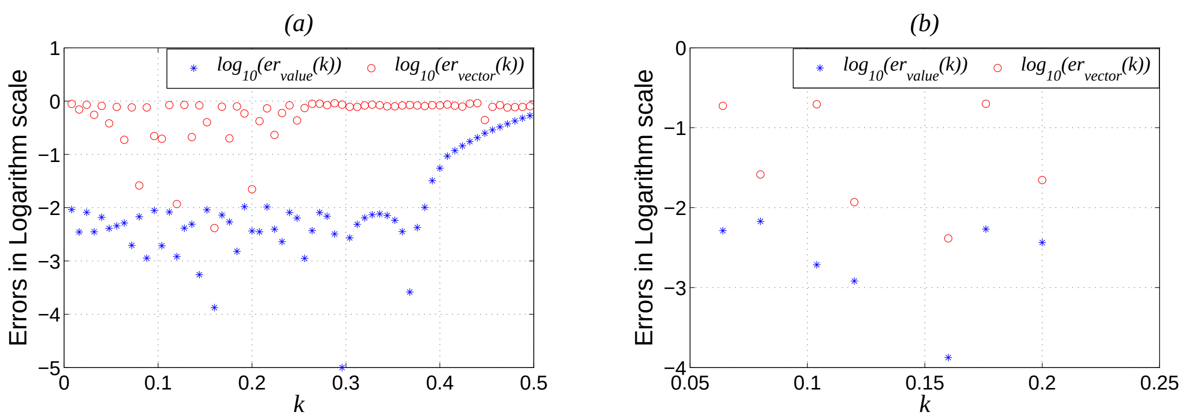

Figure 2: (a) Errors for $p = 85$ and $k \in L_{125}^{*+}$. (b) Errors for a selection of k s.t. $er_{vector}(k) \leq 0.2$.

Figure 2 (a) shows the distributions of errors $er_{value}(k)$ and $er_{vector}(k)$ in logarithmic scale for the index $p = 85$ of physical eigenmode with respect to k varying in L_{125}^{*+}. The minimal error is reached for $k = 0.16$, $n = 2$, $\ell = 17$, $\lambda_n^k = 51.1$ and $\lambda^{1,\ell} = 58.9$ yielding the errors $er_{value} = 10^{-4}$ and $er_{vector} = 4.10^{-3}$. Figure 2 (b) focuses on values of k such that $er_{vector}(k) \leq 0.2$. In Figure 3 (a) the real (dashed line) and the imaginary (solid line) parts of the Bloch wave ϕ_n^k are shown when Figure 3 (b) presents the real (solid line) and the imaginary (dashed-dotted line) parts of $u_{n,\ell}^k$ and also the real (dotted line) and the imaginary (dashed line) parts of $u_{n,\ell}^{-k}$. In addition, the physical eigenmode w_p^ε and the relative error vector between w_p^ε and $\psi_{n,\ell}^{\varepsilon,k}$ are plotted in Figure 4 (a) and (b).

After presenting a detailed study of the approximation of a given physical mode, i.e. for a single physical mode index p, we report approximation results for the list $\mathcal{J}_0^\varepsilon = \{40, ..., 150\} \setminus \{50\}$ of

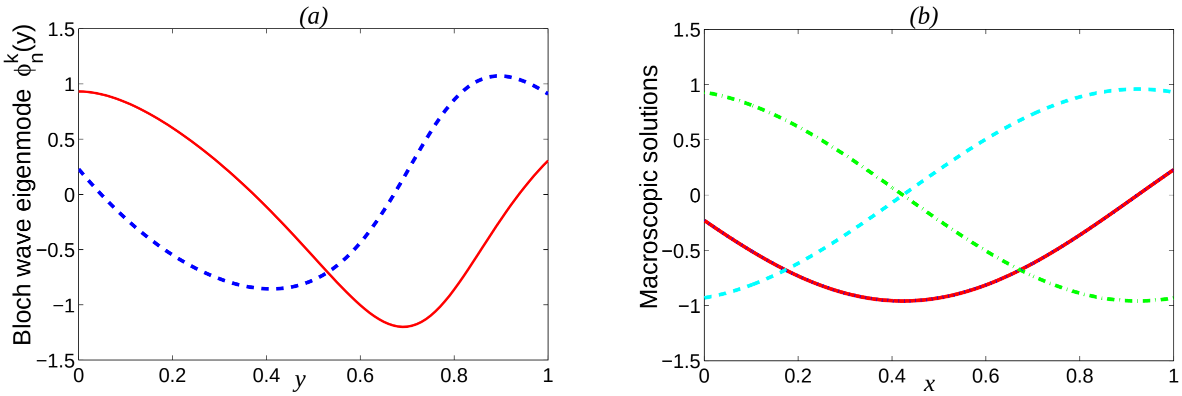

Figure 3: (a) Bloch wave solution ϕ_n^k. (b) Macroscopic solutions $u_{n,\ell}^k$ and $u_{n,\ell}^{-k}$.

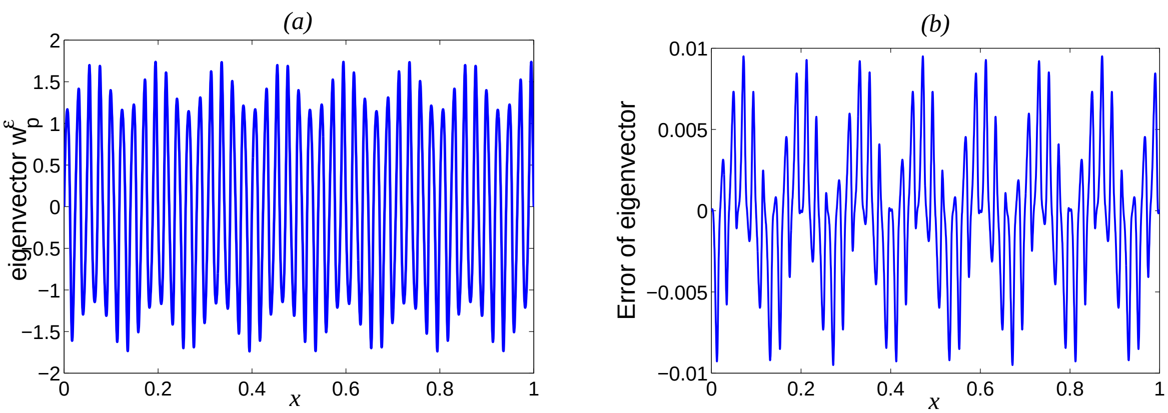

Figure 4: (a) Physical eigenmode w_p^ε. (b) Relative error between between w_p^ε and $\psi_{n,\ell}^{\varepsilon,k}$.

consecutive physical mode indices. The list starts at $p = 40$ corresponding to an intermediary mode between the low frequency modes approximated by the classical homogenized method and the high frequency modes considered in this paper. The index $p = 50$ is excluded from the list since the corresponding eigenvector is evanescent, and as such corresponds to an element of the boundary spectrum. The previous optimization has been applied to each p yielding errors plotted in logarithm scale in Figure 5 (a). The error bounds are $er_{value} \leq 6.10^{-3}$ and $er_{vector} \leq 8.10^{-2}$.

Globally, the errors start by growing before to decrease except around $p = 100$ where they exhibit a peak that we do not explain. Figure 5 (b) reports the corresponding macroscopic eigenvalues $\lambda^{1,\ell}$. Some of them are close to pairs (k, n) such that $\lambda^{1,\ell}$ vanishes as discussed in Remark 11; their relative errors on eigenvalues are in the order of 10^{-5}. A way to answer the question in Remark 11 is to decrease the step Δ_k and see if all error decrease. A detailed presentation is made in the table below for two indices, namely $p = 66$ related to an eigenvalue in the beginning of the high frequency spectrum and $p = 102$ corresponding to one of the large errors. In both cases, the error diminishes as the step Δ_k is reduced from 8e-3 to 3e-3.

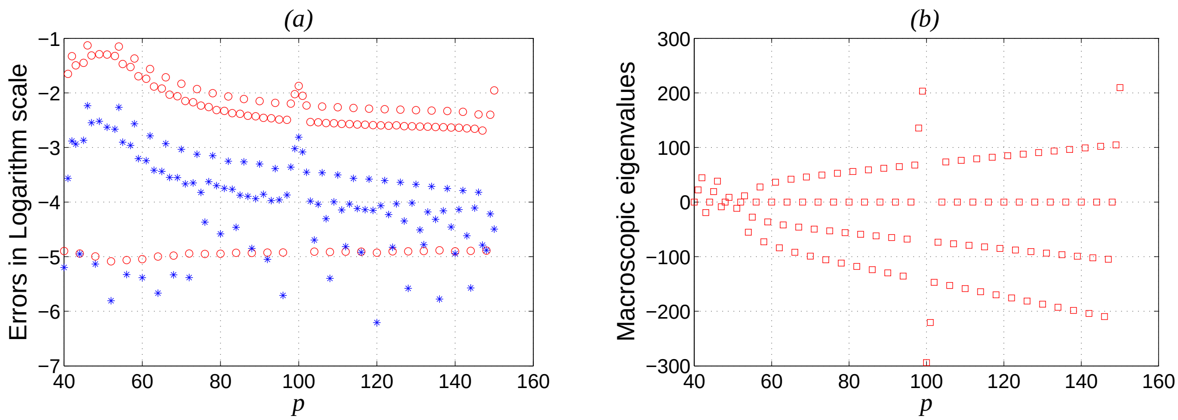

Figure 5: (a) Errors for p varying in $\mathcal{J}_0^{\varepsilon}$. (b) Macroscopic eigenvalues.

Δ_k	p	k	n	$\lambda^{1,\ell}$	er_{value}	er_{vector}
8.0e-3	66	2.16e-1	2	-92	1.2e-3	1.9e-2
3.0e-3	66	3.4e-1	2	21.7	9.0e-5	5.3e-3
8.0e-3	102	4.0e-2	3	-147	4.0e-4	5.8e-3
3.0e-3	102	1.5e-2	3	35.9	3.0e-5	1.4e-3

Table 1: Errors for $\Delta_k = 8.e-3$ and $3e-3$.

Figure 6 (a) is a global view of the errors in logarithm scale when $\Delta_k = 8.e-3$ for $90 \leq p \leq 110$. It shows that for this k-step a large part of the errors on eigenvalues is in the range of 1.0e-5 i.e. almost the roundoff error. A measure of the error reduction is provided in Figure 6 (b) where the two ratios

$$E_{value} = \frac{er_{value}^{\Delta_k=3.e-3}}{er_{value}^{\Delta_k=8.e-3}} \text{ and } E_{vector} = \frac{er_{value}^{\Delta_k=3.e-3}}{er_{vector}^{\Delta_k=8.e-3}}$$

of error reduction are represented in logarithmic scale.

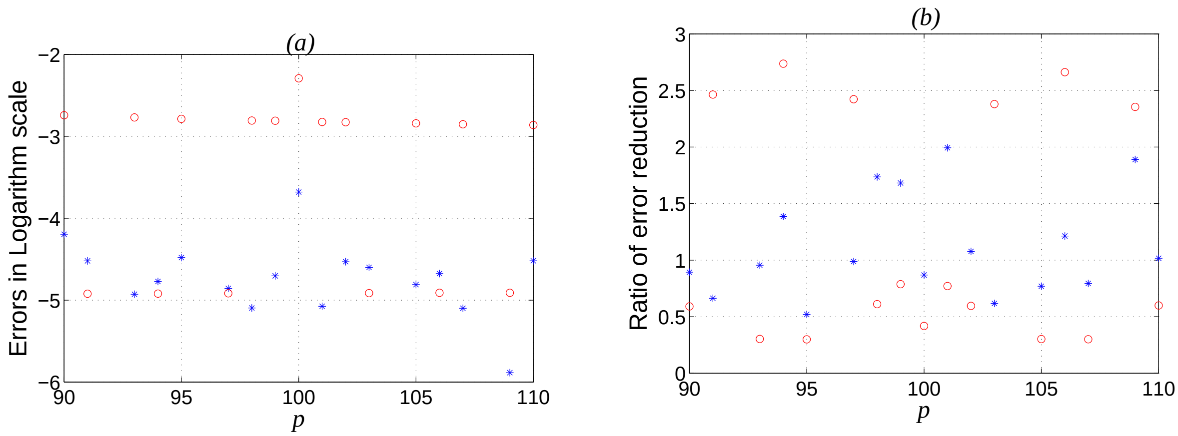

Figure 6: (a) Error of approximation for $\Delta_k = 3.0e-3$. (b) Ratios E_{value} and E_{vector} of error reduction.

6.3 The modeling problem

The modeling problem is reciprocal to the previous one. It consists in fixing a period ε as well as the parameters (k, n) of a Bloch mode and to search if there exists $\ell \in J_n^k$ such that $(\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ is close from a physical mode or in other words if it is almost a solution to the physical spectral problem i.e. if

$$\varepsilon^2 P^\varepsilon \psi_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} = O(\varepsilon) \text{ in } \Omega. \tag{80}$$

Posing for $\ell \in J_n^k$,

$$F_n^{\varepsilon,k}(\ell) = \frac{\left\| \varepsilon^2 P^\varepsilon \psi_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} \right\|_{L^2(\Omega)}}{\left\| \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} \right\|_{L^2(\Omega)}} \tag{81}$$

the modeling problem relies to the minimization problem $F_n^{\varepsilon,k}(\ell_0) = \min_{\ell \in J_n^k} F_n^{\varepsilon,k}(\ell)$. If the minimum is small enough, $(\gamma_{n,\ell_0}^{\varepsilon,k}, \psi_{n,\ell_0}^{\varepsilon,k})$ is close from a physical eigenelement and it is a solution to the modeling problem. A subsequent problem is to identify the corresponding physical eigenelement. This is done be minimizing the errors er_{value} and er_{vector} introduced in the previous section but considered as depending on the parameter $p \in \mathcal{J}^\varepsilon$ instead of k. Two illustrative examples are reported in the table below, one yielding $\lambda^{1,\ell} = 0$ and the other $\lambda^{1,\ell} \neq 0$. The solution $\psi_{n,\ell}^{\varepsilon,k}$ and the relative error between $\psi_{n,\ell}^{\varepsilon,k}$ and w_p^ε are reported in Figures 7 (a) and (b).

k	n	λ_n^k	$F_n^{\varepsilon,k}(\ell)$	$\lambda^{1,\ell}$	p	er_{value}	er_{value}
1.6e-1	2	5.11e1	8.9e-3	0	84	3.4e-5	2.1e-5
3.52e-1	2	3.14e1	4.5e-2	-8.55	65	1.5e-2	4.3e-3

Table 2: Results for the modeling problem

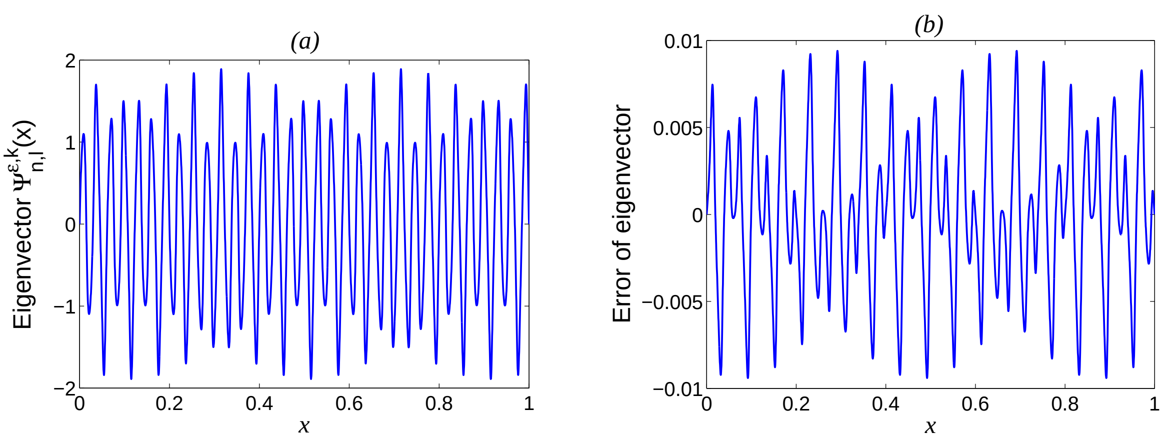

Figure 7: (a) Two-scale eigenmode $\psi_{n,\ell}^{\varepsilon,k}$. (b) Relative error vector between $\psi_{n,\ell}^{\varepsilon,k}$ and w_p^ε.

Additional results for $k = 3.52e-1$ with $n = \{1, ..., 15\}$ are reported in Figures 8 (a) and (b) showing $\lambda^{1,\ell}$ and $\gamma_{n,\ell}^k$ respectively.

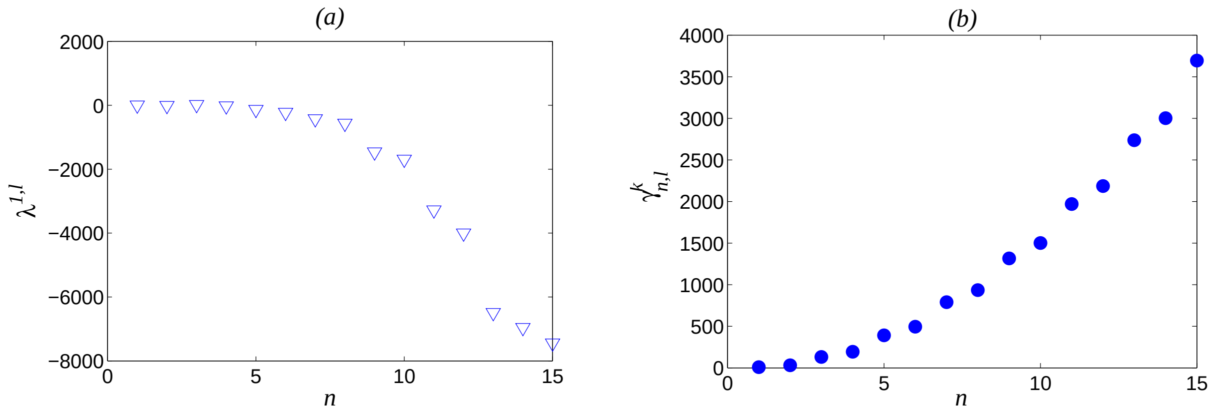

Figure 8: (a) $\lambda^{1,\ell}$ with respect to n. (b) $\gamma^k_{n,\ell}$ with respect to n.

6.4 Order of convergence

For a given pair k and $n \in J^k$, we investigate the order of convergence of the errors er_{value} and er_{vector} when the number of cells increases. To follow the convergence result, the sequence of periods ε is in fact a subsequence ε_h satisfying

$$\frac{1}{\varepsilon_h} = \frac{h+l}{k} \in \mathbb{N}^*$$

with $l \in [0, 1)$ and for a sequence of $h \in \mathbb{N}^*$. Table 3 summarizes the results for $k = 0.3$, $l = 0.6$ and $h \in \{3, 9, 15, 21\}$.

h	ε_h	$er^{h,\ell}_{value}$	$er^{h,l}_{vector}$	p
3	$8.3e-2$	$4.3e-2$	$6.3e-3$	17
9	$3.1e-2$	$1.6e-2$	$2.4e-3$	45
15	$1.9\,1e-2$	$1.0e-2$	$1.5e-3$	73
21	$1.4e-2$	$7.0e-3$	$1.0e-3$	101

Table 3: Errors for a decreasing subsequence ε_h

To evaluate the decay rate of the errors, we pose $er^{h,\ell}_{value} = c_{value}\,(\varepsilon_h)^{q_{value}}$ and $er^{h,\ell}_{vector} = c_{vector}\,(\varepsilon_h)^{q_{vector}}$, so the decay rates satisfy

$$q_{value} = \frac{\log\left(er^{h,\ell}_{value}/er^{h',\ell}_{value}\right)}{\log\left(\varepsilon_h/\varepsilon_{h'}\right)} \text{ and } q_{vector} = \frac{\log\left(er^{h,\ell}_{vector}/er^{h',\ell}_{vector}\right)}{\log\left(\varepsilon_h/\varepsilon_{h'}\right)}.$$

Using successive results for h and h', yields

$$q_{value} = \{0.988,\, 0.995,\, 0.985\} \approx 1 \text{ and } q_{vector} = \{0.985,\, 0.993,\, 0.994\} \approx 1$$

with coefficients

$$c_{value} = \{0.504,\, 0.518,\, 0.497\} \approx 0.5 \text{ and } c_{vector} = \{0.0734,\, 0.0755,\, 0.0757\} \approx 0.07.$$

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