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Optimal Control

Diffusive realization of operator solutions of certain operational partial differential equations

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Abstract

This Note is focused on the derivation of state-realizations of diffusive type for linear operator solutions of some linear partial differential operational equations. It allows the implementation of a large class of linear operators on semi-decentralized architectures. The practical interest of this work relates, for example, to the realization of optimal control law for linear partial differential equations. ***To cite this article: M. Lenczner, G. Montseny, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Réalisation diffusive d'opérateurs solutions de certaines équations aux dérivées partielles opérationnelles. Cette Note concerne la réalisation d'opérateurs linéaires solutions d'équations aux dérivées partielles opérationnelles basée sur la méthode dite des réalisations diffusives. Elle permet d'envisager l'implantation de tels opérateurs sur des calculateurs ayant une architecture semi-décentralisée. L'intérêt pratique du résultat est relatif à la mise en oeuvre de lois de contrôle optimal pour des problèmes régis par des équations aux dérivées partielles. ***Pour citer cet article : M. Lenczner, G. Montseny, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

The optimal control theories, including the LQR, H_2 or H_∞ control, are now well established for linear problems even in infinite-dimension, see [5,6,8]. However, their use for real complex applications, as for real systems governed by partial differential equations, remains a difficult problem due to computation and communication complexity. The field of applications that we have had in mind when we were working on this Note was that of large arrays of coupled microsystems that appears as an unreachable field of application for optimal control theories when using conventional computational means.

A possible way to overcome this difficulty could be the use of computational systems made of a large array of processors connected only between neighbors so that they constitute a semi-decentralized arrayed architecture (SDAA for shortness). The Cellular Neural Networks stands as the more popular example of SDAA that we have in mind, see [2] for further details. Let us mention the papers [1,3,4], where the question of approximating an optimal control law

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on a SDAA was already addressed in much more restricted cases. SDAAs operate on arrays of data and we consider that the allowed elementary operations are the usual algebraic operations, application of finite differences operators and of their inverse.

In an optimal control problem related to partial differential equations, the determination of a control law requires the computation of realizations Pu of P solution of a Riccati equation which is a nonlinear partial differential operatorial equation. The operator P is linear and acts on functions belonging to an infinite-dimensional space. In the case of a one-dimensional domain ω and of a causal integral operator P, the realization Pu may be formulated thanks to the diffusive representation, see [7], under the form $(Pu)(x) = \int \mu(x,\xi)\psi(x,\xi)\,\mathrm{d}\xi$, where ψ is the unique solution of the Cauchy problem parameterized by ξ, $\partial_x\psi(x,\xi) + \gamma(\xi)\psi(x,\xi) = u(x)$ for $x \in \omega$ and $\psi(0,\xi) = 0$ when μ and γ are respectively a function and a path of integration in $\mathbb{C}$ depending on P only. Using this formula, it is easy to build an approximate realization of P that is implementable on a SDAA. This is the approach followed in this paper which also applies to noncausal operators.

The main result of this Note consists in the determination of the equations satisfied by the symbols μ and a sufficient characterization of the admissible path γ provided that P is a linear operator solution of a linear partial differential operatorial equation. This result constitutes a necessary stage before the treatment of the Riccati equations.

2. An example of a Lyapunov equation

Throughout this Note, we shall use the superscripts $+$ or $-$ to refer to causal or anti-causal operators or to the function's domain on which they operate and the convention $\mp = -(\pm)$. We denote $\omega :=]0,1[$, $\Omega := \omega \times \omega$, $\Omega^\pm = \{(x,y) \in \Omega$ so that $\pm y < \pm x\}$ and $\nabla := {}^t(\partial_x, \partial_y)$. The boundary of $\Omega^+ \cup \Omega^-$ is divided in the closure of $\Gamma_y^+ = \{1\} \times \omega$, $\Gamma_y^- = \{0\} \times \omega$, $\Gamma_0 = \{(x,y) \in \Omega$ s.t. $x = y\}$, $\Gamma_x^+ = \omega \times \{0\}$ and $\Gamma_x^- = \omega \times \{1\}$. For a given self-adjoint positive bounded operator $Q \in \mathcal{L}(L^2(\omega))$, consider the solution $P \in \mathcal{L}(H_0^1(\Omega))$ of the Lyapunov equation $\int_\omega \nabla u \nabla(Pv) + \nabla(Pu)\nabla v\,\mathrm{d}x = \int_\omega Quv\,\mathrm{d}x$ for all $u, v \in H_0^1(\Omega)$ that appears in the context of internal stabilization of a system governed by the heat equation with Dirichlet boundary conditions and that may be seen as a simplification of the Riccati equation associated to an optimal control problem. The operator P has an integral representation

$$(Pu)(x) = \int\limits_\omega p(x,y)\,u(y)\,\mathrm{d}y \tag{1}$$

and its kernel p is the unique solution of the two boundary value problems, $p_{|\Omega^\pm} \in H^1(\Omega^\pm)$,

$$-\Delta p = q \quad \text{in } \Omega^\pm, \quad \partial_n p_{|\Omega^\pm} = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad p(x,y) = 0 \quad \text{on } \partial\Omega^\pm - \Gamma_0,$$

where q is the kernel of Q.

3. Diffusive realization of integral operators

Consider an operator P in $L^2(\omega)$ defined by its integral form (1) with $p \in L^2(\omega; L^1(\omega))$. This framework is chosen for simplicity, but may be enlarged so as to take into account unbounded operators. Let us start by defining the concept of the diffusive realization of such an operator which requires some preliminary definitions.

Causal and anti-causal parts: An operator P is said to be causal (respectively anti-causal) if $p(x,y) = 0$ for $y > x$ (respectively for $y < x$). Diffusive realizations of P are based on its unique decomposition into causal and anti-causal parts: $P = P^+ + P^-$, where $(P^+u)(x) = \int_0^x p(x,y)u(y)\,\mathrm{d}y$ and $(P^-u)(x) = \int_x^1 p(x,y)u(y)\,\mathrm{d}y$.

Impulse response: The so-called impulse responses $\tilde{p}^\pm$ are defined by $\tilde{p}^\pm(x,y) = p \circ \phi^\pm(x,y)$ for $\phi^\pm(x,y) \in \Omega^\pm$ where $\phi^\pm(x,y) = (x, x \mp y)$. The variables x and y are treated on an unequal footing, assuming that $y \mapsto \tilde{p}^\pm(x,y)$ is analytic with respect to y, with a locally integrable analytic extension to $\mathbb{R}_y^{+*}$ and that for each y the function $x \mapsto p(x,y)$ belongs to $L^2(\omega)$.

Integration paths $\gamma^\pm$: For given $a^\pm \in \mathbb{R}$, we consider $\xi \mapsto \gamma^\pm(\xi)$ two complex Lipschitz functions from $\mathbb{R}$ to $[a^\pm, +\infty[+ \mathrm{i}\mathbb{R} \subset \mathbb{C}$ such that $|\gamma^{\pm\prime}| \geqslant b > 0$ almost everywhere which define simple arcs closed at infinity. Moreover we assume that they are included in some sector $k + \mathrm{e}^{\mathrm{i}[-\alpha,+\alpha]}\mathbb{R}^+$ with $0 \leqslant \alpha < \frac{\pi}{2}$.

Diffusive representation $\psi^\pm$ *of* u: Consider $\psi^\pm(u)$ defined as the unique solutions of the following direct and backward Cauchy problems, parameterized by $\xi \in \mathbb{R}$, of diffusive type thanks to the sector condition on $\gamma^\pm$:

$$\partial_x \psi^+(x,\xi) = -\gamma^+(\xi)\psi^+(x,\xi) + u(x) \quad \forall x \in \omega, \quad \psi^+(0,\xi) = 0 \quad \text{and}$$
$$\partial_x \psi^-(x,\xi) = \gamma^-(\xi)\psi^-(x,\xi) + u(x) \quad \forall x \in \omega, \quad \psi^-(1,\xi) = 0.$$

From now on, we use the convenient notation $\langle \mu, \psi \rangle := \int_{\mathbb{R}} \mu(\xi)\psi(\xi)\,\mathrm{d}\xi$.

Definition 3.1. (i) We shall say that a causal operator P^+ (resp. anti-causal operator P^-) admits a γ^+-diffusive realization (resp. γ^--diffusive realization) if there exists a so-called diffusive symbol $\mu^+(x,\xi)$ (resp. $\mu^-(x,\xi)$) so that $P^+u(x) = \langle \mu^+, \psi^+(u) \rangle$ (resp. $P^-u(x) = \langle \mu^-, \psi^-(u) \rangle$). (ii) We shall say that an operator P admits a $\gamma^\pm$-diffusive realization if both its causal and anti-causal parts P^+ and P^- admit a diffusive realization associated respectively to γ^+ and γ^-.

Let us state some sufficient conditions for the existence of the so-called canonical diffusive realization of an operator P for general paths $\gamma^\pm$. They pertain to the Laplace transforms with respect to y of the impulse responses $\mathcal{P}^\pm(x,\cdot) = \mathcal{L}(\tilde{p}^\pm(x,\cdot))$. Their holomorphic extension to the left of the half plane where the Laplace transform is defined is still denoted by $\mathcal{P}^\pm(x,\cdot)$.

Theorem 3.2. *For a given path γ^+ (resp. γ^-), a causal (resp. anti-causal) operator P^+ (resp. P^-) admits a $\gamma^\pm$-diffusive realization if the two following conditions are fulfilled*:

(i) $\lambda \mapsto \mathcal{P}^+(x,\lambda)$ (*resp.* $\lambda \mapsto \mathcal{P}^-(x,\lambda)$) *is holomorphic in a domain D^+ (resp. D^-) that contains the closed set located at right of the arc $-\gamma^+$ (resp. of the arc $-\gamma^-$)*;
(ii) $\mathcal{P}^\pm(x,\lambda)$ *vanish when $|\lambda| \to \infty$ uniformly with respect to* $\arg\lambda$.

Then the so-called canonical symbols are given by

$$\mu^\pm(x,\xi) = -\frac{\gamma^{\pm\prime}(\xi)}{2i\pi}\mathcal{P}^\pm\big(x, -\gamma^\pm(\xi)\big)$$

and have the same regularity as $\gamma^{\pm\prime}$.

4. Diffusive symbolic formulation of linear partial differential operational equations

If P solves a partial differential operatorial equation as a Lyapunov equation then its symbol solves a boundary value problem. For the sake of shortness, we start directly from the boundary value problem satisfied by the kernel and we derive the equations satisfied by the symbols $\mu^\pm$. To avoid too much complexity, we restrict the presentation to the case where the two problems related to the causal and the anti-causal parts are discoupled as it was the case in the example of the Lyapunov equation in Section 2:

$$A(x,\nabla)p(x,y) = q(x,y) \quad \text{in } \Omega^+ \cup \Omega^- \tag{2}$$

with a number of boundary conditions depending on the order of A,

$$B(x,\nabla)p(x,y) = r(x,y) \quad \text{on } \partial\Omega^+ \cup \partial\Omega^-, \tag{3}$$

where q is the kernel of a given operator Q with diffusive symbol $\nu^\pm$. The restrictions of r to the boundaries $\Gamma_y^\pm$ are assumed to be the kernels of a causal operator R^+ and an anti-causal operator R^- with diffusive symbols ρ^+ and ρ^-. The partial differential equation solved by $\tilde{p}^\pm$ is

$$\widetilde{A}^\pm(x,\nabla)\tilde{p}^\pm = \tilde{q}^\pm, \tag{4}$$

where $\widetilde{A}^\pm(x,\nabla) = A(x, K^\pm\nabla)$ and $K^\pm = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & \mp 1 \end{smallmatrix}\right)$. Consider that $\tilde{p}^\pm$ has an analytic continuation with respect to y on $\mathbb{R}^{+*}$ and let us extend it by 0 in $\mathbb{R}^-$. From (4) formulated on $\tilde{p}^\pm_{|\omega}$, one deduces the equation on the extension $\tilde{p}^\pm$ in the sense of distributions

$$\widetilde{A}^\pm(x,\nabla)\tilde{p}^\pm + \sum_k \widetilde{A}_k^\pm(x,\nabla)\tilde{p}^\pm\delta_0^{(k)} = \tilde{q}^\pm \quad \text{in } \mathcal{D}'_+(\mathbb{R}),$$

where $\widetilde{A}_k^{\pm}(x,\nabla)$ are suitable partial differential operators and $\delta_0^{(k)}$ is the kth derivative of the Dirac distribution at point $y=0$. We are now in position to introduce two differential operators associated to A and $\gamma^{\pm}$:

$$A^{\pm}(x,\partial_x,\lambda)=A\big(x,K^{\pm t}(\partial_x,-\lambda)\big)\quad\text{and}\quad A_0^{\pm}\big(x,\nabla,\gamma^{\pm}(\xi)\big)=-\frac{\gamma^{\pm\prime}(\xi)}{2\mathrm{i}\pi}\sum_k\big(-\gamma^{\pm}(\xi)\big)^k\,\widetilde{A}_k^{\pm}\big(x,(K^{\pm})^{-1}\nabla\big).$$

In the same way operator $B^{\pm}$ can be derived from B on $\partial\Omega^+\cup\partial\Omega^-$ and $B_0^{\pm}$ from B on $\Gamma_y^{\pm}$.

Theorem 4.1. *Assuming that P, Q and $R^{\pm}$ fulfill the assumptions of Theorem* 3.2*, the kernel p is solution of the boundary value problem* (2), (3) *iff its canonical $\gamma^{\pm}$-symbols are solution of*:

$$A^{\pm}\big(x,\partial_x,\gamma^+(\xi)\big)\mu^{\pm}(x,\xi)+A_0^{\pm}\big(x,\nabla,\gamma^{\pm}(\xi)\big)p(x,x)=\nu^{\pm}(x,\xi)\quad\forall(x,\xi)\in\omega\times\mathbb{R}^+,$$
$$B^{\pm}\big(x,\partial_x,\gamma^+(\xi)\big)\mu^{\pm}(x,\xi)+B_0^{\pm}\big(x,\nabla,\gamma^{\pm}(\xi)\big)p(x,x)=\rho^{\pm}(x,\xi)\quad\forall(x,\xi)\in\big(\{1\}\text{ or }\{0\}\big)\times\mathbb{R}^+,$$
$$\big\langle B^{\pm}\big(x,\partial_x,\gamma^{\pm}(\xi)\big)\mu^{\pm}(x,\xi),\mathrm{e}^{\mp\gamma^{\pm}(\xi)(x-y_0(x))}\big\rangle=r\big(x,y_0(x)\big)\quad\text{on }\Gamma_x^{\pm}\cup\Gamma_0$$

with $y_0(x)=x$, 0 or 1 on Γ_0, Γ_x^+ or Γ_x^-.

Finally, we state some sufficient conditions on the operators A and B which insure that P satisfies the assumption (i) of Theorem 3.2. The differential operators $A^{\pm}$ and $B^{\pm}$ can be expanded with respect to the derivatives: $A^{\pm}(x,\partial_x,\lambda)=\sum_m a_m^{\pm}(x,-\lambda)\partial_x^m$ and $B^{\pm}(x,\partial_x,\lambda)=\sum_m b_m^{\pm}(x,-\lambda)\partial_x^m$, which allows us to define the union of zeros of the analytic functions $\lambda\mapsto a_m^{\pm}(x,-\lambda)$ and $\lambda\mapsto b_m^{\pm}(x,-\lambda)$ over all x and m:

$$W_A^{\pm}:=\bigcup_{x,m}\big[a_m^{\pm}(x,\cdot)\big]^{-1}(0)\quad\text{and}\quad W_B^{\pm}:=\bigcup_{x,m}\big[b_m^{\pm}(x,\cdot)\big]^{-1}(0).$$

Theorem 4.2. *If $D^{\pm}$ is such that $W_A^{\pm}\cup W_B^{\pm}\subset\mathbb{C}-D^{\pm}$ and if Q and $R^{\pm}$ fulfill the assumption* (i) *of Theorem* 3.2 *then P fulfills it also.*

4.1. Application to the Lyapunov equation

The application of the above results to the example of Section 2 leads to $A^{\pm}=(\partial_{xx}^2\mp2\gamma(\xi)\partial_x+2\gamma^2(\xi))$, $B^{\pm}=(\partial_x\mp2\gamma(\xi))$ on $\Gamma_y^{\pm}$ and $B^{\pm}=Id$ on $\Gamma_x^{\pm}\cup\Gamma_0$. Therefore, $W_A^{\pm}=\{0\}$ and $W_B^{\pm}=\emptyset$, which says that any paths $-\gamma^{\pm}$ enlacing 0 and the singularities of $\nu^{\pm}$ by the right is admissible. The assumption (ii) of Theorem 3.2 is established using an elementary spectral argument. Numerical computation of $\mu^{\pm}$ may be conducted from their equations or from the kernel's equations.

5. Concluding remarks

These results can be extended in three various directions without much effort. First when impulse responses are analytic in $y\in\omega$ and singular at $y=0$, second for multi-dimensional domains ω as products of intervals third for coupled kernel equations on Ω^+ and Ω^-. Finally, we think that extensions to operator P solutions of some nonlinear equations will also be possible.

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