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HOMOGENIZATION OF PERIODIC ELECTRICAL NETWORKS INCLUDING VOLTAGE TO CURRENT AMPLIFIERS*

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Abstract. We derive the homogenized model of a periodic electrical network that includes resistive devices, voltage to current amplifiers, sources of tension, and sources of current. First, a mixed variational formulation is associated with the classical equations of such electrical networks. In an abstract framework, inf-sup conditions are given for the existence and uniqueness of its solution. Second, optimal conditions, based on the network topology, are stated so that the inf-sup conditions are satisfied. Third, the homogenized model of such a periodic network is derived using the two-scale convergence developed for circuits by Lenczner [*C. R. Acad. Sci. Paris Sér.* II *B*, 324 (1997), pp. 537–542]. Finally, numerical comparisons of the solutions of the homogenized model and of the complete one are detailed. It underlines clearly, if necessary, the strong interest of using the homogenized model when the number of periodic cells is large enough.

Key words. homogenization, two-scale convergence, periodic structure, circuit, network

AMS subject classifications. 74Q15, 35B27, 94C15

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1. Introduction.

This paper has been written in view of applications in fields of engineering where simplified models of large periodic electronic networks are required. A famous example can be found in the cellular neural network technology initiated by Chua and Yang [7, 8] for applications to image processing. The electronic circuits of many mechatronic systems also fall in this category. This concerns sensors and/or actuators arrays used in most fields of physics, e.g., acoustics, fluid mechanics, mechanics of vibrations, and electromagnetism, for a very large variety of applications, e.g., detection of moving acoustic sources, control of acoustic noise, vibration damping, control in fluid mechanics, measure of brain activity, radars, convoying systems, and field effect microscopy. Technologies can be conventional or based on microelectromechanical systems (MEMS). See the illustrative papers of Tsao et al. [19] and Mahamane, Lenczner, and Mrcarica [14] for two examples in the field of distributed control.

The simplified modelling of periodic circuits is a very old problem that was first studied for infinitely large R-L-C networks. More recently, Vogelius [20] has derived a model for a class of resistive two-dimensional periodic networks. All these works were limited to very particular network configurations. The limitations were due to the mathematical technique that was employed at this time. In our works [15] and [16], a new technique of two-scale convergence has been introduced which allows the derivation of homogenized models for a very wide class of problems. In particular, it covers the problems posed on periodic $(n-p)$-dimensional manifolds in domains of $\mathbb{R}^n$. So electronic circuits that are one-dimensional manifolds can now be homogenized. This technique also covers the problems that were already treated with previous ones and in particular can recover the homogenized models that were already known in mechanics.

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This is why we hope reasonably that it will allow us to derive homogenized models of general periodic mechatronic systems. Moreover, this technique is so simple and natural that it has been rediscovered independently by Casado-Diaz, Luna-Laynez, and Martin [6] and Cioranescu, Damlamian, and Griso [9]; we have all rediscovered independently the underlying idea of two-scale transform that has been introduced by Argobast, Douglas, and Hornung [2]. We have already applied this technique to the derivation of homogenized models of general periodic resistive circuits including passive sources [15] and voltage to voltage amplifiers [16].

Another specificity of the analysis of electronic circuits is the formulation of conditions related to their architecture that ensure the existence and uniqueness of the solution. Numerous works have been realized on this topic; see, for example, the synthesis book of Recski focused on linear problems [17], as well as the testing techniques that have been implemented in the usual circuit simulators. This question is posed again for the simulation of mechatronic systems and in particular for simulation of complex systems already mentioned. It is in this perspective that we have developed a variational approach which may constitute a common framework for continuous media and electronic circuit equations. With such an approach, the conditions for the existence and uniqueness of the solutions are stated using some classical tools of functional analysis that are generally used for the analysis of partial differential equations. The resulting inf-sup conditions are very abstract and not adapted to an automatic checking. Thus, a first attempt to reformulate them in terms of a graph was presented in [16].

Let us review the contributions of this paper. They complete the techniques developed in [16] in many aspects, in addition to the fact that another class of amplifiers is considered, which constitutes an intermediary step before taking into account general ones.

The abstract variational framework that is considered here is the most general one, which is required for linear circuit modelling in statics. We consider variational formulation of the form

$$
\begin{aligned}
a(u,v) + b_1(v,p) &= \langle f,p\rangle,\\
b_2(u,q) - c(p,q) &= \langle g,q\rangle,
\end{aligned}
$$

where b_1 and b_2 are different. Such equations have been studied in the literature in the context of magnetohydrodynamics and also in the theory of elliptic systems (see, for instance, [10], where spectral properties of such block operators are discussed). They have also been studied in Bernardi, Canuto, and Maday [3], where $c = 0$, and for which the authors gave necessary and sufficient conditions for the existence of a unique solution; meanwhile, the case $b_1 = b_2$ was treated in Brezzi and Fortin [5]. In [16], the bilinear form c was vanishing. Here, we have extended the study of existence and uniqueness to the general case (i.e., the case $c \neq 0$ and $b_1 \neq b_2$), and we have found that proving its well posedness is equivalent to checking four inf-sup conditions.

To know whether or not they are satisfied, for a given network, is not an easy task. Conditions formulated directly on the network graph would be preferable. In [16], conditions related to the graph were introduced to guarantee the well posedness of the problem. Unfortunately, this was only necessary conditions. Here, a deeper understanding leads to graph conditions that are equivalent to the four inf-sup conditions so that they constitute necessary and sufficient conditions for the existence and uniqueness of the solution. Moreover, we think that the optimal technique that is introduced here could be extended for circuits with general amplifiers.

Let us move to the homogenized model. In [16], only the two-scale model was formulated. In the present paper, it is fully detailed in the most general case. So it has the form of the most general second order system of partial differential equations. By another way, the cell problem is transformed into a *modified cell circuit* so that the existence and uniqueness conditions already obtained are applicable to it.

Finally, numerical comparisons of solutions of the complete system and of the homogenized one are reported. The small differences between the electric potentials and between the voltages computed with the two models show clearly the strong interest that can be found in using the homogenized model for periodic circuits with a large number of cells.

The organization of this paper is as follows.

In section 2, the electrical networks including voltage to current amplifiers are described, and their corresponding classical equations are established. The variational formulation of the electrical network is stated in subsection 2.3. The general abstract result for the mixed variational formulation is stated and proved in section 2.4. In particular, the four inf-sup conditions, which ensure the existence and uniqueness of the solution, are stated.

Section 3 is devoted to the statement and proof of the existence and uniqueness result for electric circuit equations. In subsection 3.1, the conditions based on the topology of the network, which ensure the existence and uniqueness of the solution, are stated. Subsection 3.2 is devoted to some illustrations of these conditions. In subsection 3.3, notations and estimates linked with the previous conditions are stated and proved. General technical lemmas are formulated in subsection 3.4. They are the keys of the proof of the existence and uniqueness result stated and proved in subsection 3.5. Finally, in subsection 3.6, the necessity of the graph like conditions for the existence and uniqueness of the solution is studied.

Section 4 deals with homogenization of periodic electrical networks. In subsection 4.1, we state the periodic circuit equations. The homogenized model and the homogenized coefficients are established and computed, respectively, in subsections 4.2 and 4.3. The assumptions and the model derivation are reported in subsection 4.4.

Finally, section 5 is devoted to various examples and in particular to the discussion of some numerical results.

2. Variational formulation of electrical networks. In this section, we state the general variational formulation which is satisfied by the electrical potential and the current in the electrical network. This variational formulation is a basis for the derivation of the two-scale model stated in what follows.

The network includes resistors, current sources, voltage sources, and voltage to current amplifiers. The conditions posed on the network for the existence and uniqueness of the solution are stated. They are based on inf-sup conditions and are interpreted in terms of graph like conditions posed on the electrical network.

2.1. Notations. We use the definitions and the properties relative to electrical networks presented in [21] and [16] (see also Figure 2.1). An electrical network is composed of vertices (or nodes) and edges (or branches). Vertices are linked by edges. The set of edges is denoted by Θ. Mathematically, Θ is a network in $\mathbb{R}^n$, where $n \in \mathbb{N}^*$. We denote by σ_0 the vertices linked to the earth (i.e., where the electrical potential is equal to zero). The network Θ is divided into five disjointed parts Θ_0, Θ_1, Θ_2, Θ_3, and Θ_4. They are occupied, respectively, by the voltage sources, the current sources, the resistors, and the input and the output of the amplifiers. The edges included in these sets are denoted, respectively, by e_0^l, e_1^l, e_2^l, e_3^l, and e_4^l. Here, l is an index varying from one to the number of edges belonging to the respective sets.

The network Θ is assumed to be parameterized. This parameterization defines a positive sense for each edge. We name s_e^+ and s_e^- the vertices belonging to an edge $e \subset \Theta$ such that $s_e^+ \to s_e^-$ is the positive sense. The set of edges arriving in a positive (respectively, negative) sense at a vertex s is denoted by Θ_s^+ (respectively, Θ_s^-). The length of an edge e is denoted by $|e|$. The function L is distributed on Θ. It is constant on each edge, and $L(x) = |e|$ for all $x \in e$. The tangent vector to Θ at point x is denoted by $\tau(x)$.

2.2. Statement of equations. In this subsection, the equations of electrical networks, in their classical form, are recalled. We also introduce the necessary notations in order to write their variational formulation.

Let us define the sets $\mathbb{P}^0(\Theta)$ or $(\mathbb{P}^0(\Theta_k))_{k=0,\dots,4})$ (respectively, $\mathbb{P}^1(\Theta)$) of functions constant on each edge $e \subset \Theta$ or $(e \subset \Theta_k)_{k=0,\dots,4}$ (respectively, affine on each edge $e \subset \Theta$ and continuous on Θ). The current i and the voltage u are some distributed fields belonging to $\mathbb{P}^0(\Theta)$. The electrical potential is also a distributed field, and it belongs to $\mathbb{P}^1(\Theta)$. The tangential derivative of a function ψ defined on Θ is denoted by $\nabla_\tau \psi$. $\|\psi\| = (\int_\Theta (|\nabla_\tau \psi|^2 + |\psi|^2) dl(x))^{\frac{1}{2}}$ will represent a norm on the space $\mathbb{P}^1(\Theta)$, while $|\psi| = (\int_\Theta (|\nabla_\tau \psi|^2) dl(x))^{\frac{1}{2}}$ will be a seminorm.

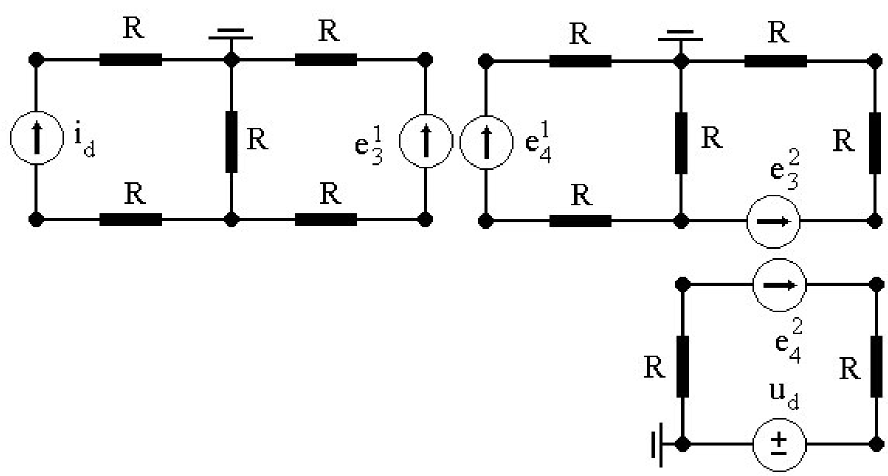


FIG. 2.1. *An example of an electric network.*

An example of the network described below is represented above.

The voltage Kirchhoff law is stated on each edge $e \subset \Theta$ as follows: $u_{|e} = \varphi(s_e^+) - \varphi(s_e^-)$, or, equivalently,

$$L \nabla_\tau \varphi = u \text{ on } \Theta. \tag{2.1}$$

The current Kirchhoff law is stated for each vertex s as $\sum_{e \subset \Theta_s^+} i_{|e} - \sum_{e \subset \Theta_s^-} i_{|e} = 0$. It can be equivalently written under a weak formulation

$$\int_\Theta i(x) \nabla_\tau \psi(x) dl(x) = 0 \quad \forall \psi \in \mathbb{P}^1(\Theta) \text{ such that } \psi = 0 \text{ on } \sigma_0. \tag{2.2}$$

The values of voltage, current, and electrical potential are imposed, respectively, on Θ_0, Θ_1, and σ_0 to be equal to the voltage sources $u_d \in \mathbb{P}^0(\Theta_0)$, $i_d \in \mathbb{P}^0(\Theta_1)$, and 0 on σ_0:

$$u = u_d \text{ on } \Theta_0, \quad i = i_d \text{ on } \Theta_1, \quad \text{and} \quad \varphi = 0 \text{ on } \sigma_0. \tag{2.3}$$

Let us remark that the sign of u_d and of i_d on an edge e depends on the orientation of e.

An impedance $\frac{1}{g} \in \mathbb{P}^0(\Theta_2)$ is associated with Θ_2, which means that u and i are linked by the constitutive linear equation on Θ_2:

$$i = gu \text{ on } \Theta_2. \tag{2.4}$$

We assume that $g \geq g_{\min} > 0$.

We recall that a voltage to current amplifier is a device which imposes two equations between currents and voltages of two edges. The sets Θ_3 and Θ_4 are, respectively, the sets of amplifier inputs and outputs. Each input edge $e_3^l \in \Theta_3$ is associated with a unique output edge $e_4^l \in \Theta_4$, where l varies from one to the number of amplifiers used.

The constitutive relations of the voltage to current amplifier are for each l

$$i_{|e_4^l} - k_l u_{|e_3^l} = 0 \quad \text{and} \quad i_{|e_3^l} = 0, \tag{2.5}$$

where $k_l \in \mathbb{R}$ is the amplification coefficient. The edges e_3^l and e_4^l are, respectively, called the input and the output of the amplifier. Since (2.5) applies to each amplifier, we consider that $k \in \mathbb{P}^0(\Theta_3)$, and we write the amplifier constitutive equations as follows:

$$i_{|\Theta_4} - ku_{|\Theta_3} = 0 \quad \text{and} \quad i_{|\Theta_3} = 0. \tag{2.6}$$

2.3. The variational formulation. In what follows, we give the variational formulation equivalent to the above equations.

For $u_d \in \mathbb{P}^0(\Theta_0)$, let us define the admissible functions set for the variational problem

$$\Psi_{ad}(u_d) = \{\psi \in \mathbb{P}^1(\Theta),\, \psi = 0 \text{ on } \sigma_0 \text{ and } L\, \nabla_\tau \psi = u_d \text{ on } \Theta_0\} \tag{2.7}$$

and the following variational formulation. Consider $(\varphi, i) \in \Psi_{ad}(u_d) \times \mathbb{P}^0(\Theta_4)$ the solution of

$$\begin{cases} \displaystyle\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi dl(x) + \int_{\Theta_4} i\nabla_\tau\psi dl(x) &= \displaystyle -\int_{\Theta_1} i_d\nabla_\tau\psi dl(x), \\ \displaystyle\int_{\Theta_3} kL\nabla_\tau\varphi\, j dl(x) - \int_{\Theta_4} ij dl(x) &= 0 \end{cases} \tag{2.8}$$

for all $(\psi, j) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$.

Let us remark that $j \in \mathbb{P}^0(\Theta_4)$ is used on Θ_3. We adopt the rule that j takes the same value on the input e_3^l and on the output e_4^l of an amplifier.

LEMMA 2.1. *The variational formulation* (2.8) *is equivalent to* (2.1)–(2.6).

Proof. Let us consider (2.2). Since $i = gL\nabla_\tau\varphi$ on Θ_2, $i = 0$ on Θ_3, and $i = i_d$ on Θ_1,

$$\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi dl(x) + \int_{\Theta_0} i\nabla_\tau\psi dl(x) + \int_{\Theta_4} i\nabla_\tau\psi dl(x) = -\int_{\Theta_1} i_d\nabla_\tau\psi dl(x), \tag{2.9}$$

for all $\psi \in \mathbb{P}^1(\Theta)$, such that $\psi = 0$ on σ_0. The variational formulation of (2.5) and the condition $u = u_d$ on Θ_0 are

$$\int_{\Theta_3} kL\nabla_\tau\varphi\, j dl(x) - \int_{\Theta_4} ij dl(x) = 0 \text{ for every } j \in \mathbb{P}^0(\Theta_4) \tag{2.10}$$

and

$$\int_{\Theta_0} L\nabla_\tau\varphi\; j\; dl(x) = \int_{\Theta_0} u_d\; j\; dl(x) \text{ for every } j_0 \in \mathbb{P}^0(\Theta_0), \tag{2.11}$$

where j takes the same value on each e_3^l and e_4^l belonging to the same amplifier. Here i on Θ_0 plays the role of a Lagrange multiplier. Equivalently, $(\varphi, i) \in \Psi_{ad}(u_d) \times \mathbb{P}^0(\Theta_4)$ is the unique solution of

$$\left\{ \begin{array}{lcl} \displaystyle\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi dl(x) + \int_{\Theta_4} i\nabla_\tau\psi dl(x) & = & \displaystyle -\int_{\Theta_1} i_d\nabla_\tau\psi dl(x), \\ \displaystyle\int_{\Theta_3} kL\nabla\varphi\, j dl(x) - \int_{\Theta_4} ij dl(x) & = & 0 \end{array} \right. \tag{2.12}$$

for all $(\psi, j) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$.

The proof of the converse is straightforward. $\square$

2.4. An abstract variational system. The analysis of the problem (2.8) involves a variational formulation with the form (2.14) hereafter. This section is devoted to the study of this abstract variational formulation with inf-sup conditions which are necessary and sufficient for the unique solvability of the correspondent problem. The equations (2.8) can be written as follows: Find $(\varphi, i) \in \Psi_{ad}(u_d) \times \mathbb{P}^0(\Theta_4)$ satisfying

$$\left\{ \begin{array}{lcl} a(\varphi, \psi) + b_1(i, \psi) & = & l(\psi), \\ b_2(\varphi, j) - c(i, j) & = & 0 \end{array} \right. \tag{2.13}$$

for all $(\psi, j) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$. Here, $b_1(.,.)$ and $b_2(.,.)$ are different, and $c \not\equiv 0$.

The general abstract variational formulation is now stated. Let X_i and M_i ($i = 1, 2$) be real reflexive Banach spaces. We assume we are given four continuous bilinear forms, $a : X_2 \times X_1 \to \mathbb{R}$, $b_i : X_i \times M_i \to \mathbb{R}$ ($i = 1, 2$), and $c : M_1 \times M_2 \to \mathbb{R}$. For any given f in X_1' and g in M_2', we consider the following problem: Find $(u, p) \in X_2 \times M_1$ such that

$$\left\{ \begin{array}{c} a(u, v) + b_1(v, p) = \langle f, v\rangle, \\ b_2(u, q) - c(p, q) = \langle g, q\rangle \end{array} \right. \tag{2.14}$$

for all $(v, q) \in X_1 \times M_2$.

Remark 2.2. *In our case $X_i = X_j$ and $M_i = M_j$.*

In order to study this problem, let us introduce the linear operators $A \in \mathcal{L}(X_2, X_1')$, $B_i \in \mathcal{L}(X_i, M_1')$ ($i = 1, 2$), and $C \in \mathcal{L}(M_1, M_2')$, associated with the forms a, b_i ($i = 1, 2$), and c, by the relations

$$\forall u \in X_2,\ \forall v \in X_1,\ \langle Au, v\rangle = a(u, v), \tag{2.15}$$

$$\forall u \in X_i,\ \forall q \in M_i,\ \langle B_i u, q\rangle = b_i(u, q), \tag{2.16}$$

$$\forall p \in M_1,\ \forall q \in M_2,\ \langle Cp, q\rangle = c(p, q). \tag{2.17}$$

We denote by $B_1^t \in \mathcal{L}(M_1, X_1')$ the adjoint operator of B_1. The problem (2.14) is equivalent to the following one: Find $(u,p) \in X_2 \times M_1$ such that

$$\left\{ \begin{array}{l} Au + B_1^t p = f, \\ B_2 u - Cp = g. \end{array} \right. \tag{2.18}$$

We define the Banach subspaces

$$W_1 = \{(u,p) \in X_2 \times M_1 : B_2 u - Cp = 0\}, \tag{2.19}$$

$$W_2 = \{(u,p) \in X_2 \times M_1 : Au + B_1^t p = 0\} \tag{2.20}$$

and give four conditions C_i $(i = 1, \dots, 4)$:

Condition C_1.

$$\forall (u,p) \in W_1 - \{0\}, \ \sup_{v \in X_1} (a(u,v) + b_1(v,p)) > 0.$$

Condition C_2. There exists a constant $\alpha > 0$ such that

$$\forall v \in X_1, \ \sup_{(u,p) \in W_1 - \{0\}} \frac{a(u,v) + b_1(v,p)}{\|(u,p)\|_{X_2 \times M_1}} \geq \alpha \|v\|_{X_1}.$$

Condition C_3.

$$\forall (u,p) \in W_2 - \{0\}, \ \sup_{q \in M_2} (b_2(u,q) - c(p,q)) > 0.$$

Condition C_4. There exists a constant $\beta > 0$ such that

$$\forall q \in M_2, \ \sup_{(u,p) \in W_2 - \{0\}} \frac{b_2(u,q) - c(p,q)}{\|(u,p)\|_{X_2 \times M_1}} \geq \beta \|q\|_{M_2}.$$

Let us remark that the operator C, according to (2.12), is continuously invertible. In our case the second equation of (2.18) can be easily solved for p and its solution inserted into the first equation. Exploiting the particularly simple structure of the form C would simplify the assumptions C_3 and C_4. Nevertheless, we will not take into account this simplification voluntarily because we have in mind later to extend this technique for networks with general amplifiers; we think that the simplification of C_3 and C_4 will be impossible, and one aim of the present paper is to provide a tool for more general networks.

We can prove the following theorem.

THEOREM 2.3. *The problem* (2.14) *admits a unique solution* $(u,p) \in X_2 \times M_1$ *if and only if the four previous conditions* C_i $(i = 1, \dots, 4)$ *are satisfied. Moreover, the following inequality is satisfied:*

$$\|(u,p)\|_{X_2 \times M_1} \leq \frac{1}{\alpha} \|f\|_{X_1'} + \frac{1}{\beta} \|g\|_{M_2'}. \tag{2.21}$$

Proof. Let us consider the two following problems (2.22)–(2.23): Find $(u,p) \in W_1$ such that

$$Au + B_1^t p = f, \tag{2.22}$$

and find $(u,p) \in W_2$ such that

$$B_2 u - Cp = g. \tag{2.23}$$

The problem (2.18) admits a unique solution if and only if each of the problems (2.22)–(2.23) admits a unique solution. Moreover, the solution of (2.18) is the sum of the solutions of the problems (2.22)–(2.23). Consider now the linear operator M:

$$\left\{ \begin{array}{l} M : W_1 \longrightarrow X_1' \\ \qquad (u,p) \mapsto Au + B_1^t p. \end{array} \right. \tag{2.24}$$

Since B_2 and C are continuous, it is clear that W_1 is a closed subspace of the Banach space $X_2 \times M_1$; thus W_1 is a Banach space. The bilinear form m defined by $m(x,v) = a(u,v) + b_1(v,p)$ (here $x = (u,p)$) is the bilinear form associated with the operator M. Now, M is one to one if and only if

$$\forall x \in W_1 - \{0\}, \quad \sup_{v \in X_1} m(x,v) > 0. \tag{2.25}$$

Equation (2.25) is Condition C_1.

Using Theorem II.19 of [4], M is onto if and only if there exists $\alpha > 0$ such that

$$\alpha \|v\|_{X_1} \leq \|M^* v\|_{W_1'} \quad \forall v \in D(M^*). \tag{2.26}$$

Since M is bounded its domain is W_1, and $D(M^*)$ (the domain of M^*) is equal to the space X_1. Now, we have $\|M^* v\| = \sup_{\|x\| \leq 1} \langle x, M^* v \rangle = \sup_{\|x\| \leq 1} m(x,v) = \sup_{x \in W_1} \frac{m(x,v)}{\|x\|}$. The previous equality and (2.25) give C_2.

Now, let us assume that M is an isomorphism; then thanks to (2.26) we obtain an estimation of the norm of M^{-1}: $\|M^{-1}\| \leq \frac{1}{\alpha}$, and the solution of $Mx = f$ satisfies the estimation

$$\|x\| \leq \frac{1}{\alpha} \|f\|. \tag{2.27}$$

Using the same arguments as previously for the problem (2.23), we obtain Conditions C_3 and C_4, and the solution x' satisfies the estimate

$$\|x'\| \leq \frac{1}{\beta} \|g\|. \tag{2.28}$$

The estimate (2.21) comes from (2.27)–(2.28). ☐

REMARK 2.4. *In the particular case $C = 0$ (typically the Stokes equations), the previous proof says that Conditions C_i, $i = 1, \ldots, 4$, are equivalent to the isomorphism of the operator Λ:*

$$\left\{ \begin{array}{ll} \Lambda : & X_2 \times M_1 \to X_1' \times M_2' \\ & (u,p) \mapsto (Au + B_1^t p, B_2 u). \end{array} \right.$$

Consequently, from Theorem 2.1 *of* [3], *Conditions C_i, $i = 1, \ldots, 4$, are reduced to the conditions C_i', $i = 1, 2$, and C_i'', $i = 1, 2$, hereafter:*

Condition C_1'.

$$\forall u \in \operatorname{Ker} B_2 - \{0\}, \quad \sup_{v \in \operatorname{Ker} B_1} a(u,v) > 0.$$

Condition C_2'. There exists a constant $\alpha > 0$ such that

$$\forall v \in \operatorname{Ker} B_1, \quad \sup_{u \in \operatorname{Ker} B_2 - \{0\}} \frac{a(u,v)}{\|u\|_{\operatorname{Ker} B_2}} \geq \alpha \|v\|_{X_1}.$$

Condition C_i'', $i = 1,2$. There exists a constant $\beta_i > 0$ such that

$$\forall q \in M_i, \quad \sup_{u \in X_i - \{0\}} \frac{b_i(u,q)}{\|u\|_{X_i}} \geq \beta_i \|q\|_{M_i}.$$

Also let us note that in [5] *the case $B_1 = B_2$, $C = 0$ is also studied. It is a particular case of the equations considered in* [3]. *We can find there some results when $C \neq 0$ but with some hypothesis of coercivity for the bilinear forms a and c.*

3. Existence and uniqueness under graph like assumptions.

3.1. Statement of the result. In this subsection, we introduce assumptions based on the graph theory which ensure that Conditions C_i ($i = 1, \ldots, 4$) are satisfied. This means that it is an interpretation in terms of the location of the various devices such as resistors, amplifier inputs and outputs, current and voltage sources, and earth. At the end of this subsection, the theorem for the existence and uniqueness of the solution is stated under these graph like assumptions.

First, let us recall the definitions of path and circuit (see [17] for instance).

DEFINITION 3.1. (i) *A path is a sequence of edges where the end of an edge is connected to the beginning of the following one.*

(ii) *A circuit is a path where the beginning of the first edge is connected to the end of the last one. The circuits are denoted by the letter β.*

For these definitions, all vertices belonging to σ_0 (the earth) are considered as one.

Statements of the assumptions.

(H1) There is no circuit solely made up of edges belonging to Θ_0.

It is intuitively clear that two or more voltage sources cannot form a circuit in a network graph since the tensions are independently given on such a circuit, and the voltage Kirchhoff equations on such a circuit might lead to a contradiction with assumption (H1).

Assumption (H1) implies that, for all $u_d \in \mathbb{P}^0(\Theta_0)$, $\psi_{ad}(u_d) \neq \emptyset$.

DEFINITION 3.2 (of $\overline{\Theta_2}$). *Let us consider the subgraph $\Theta_0 \cup \Theta_2$ of Θ. Thanks to* (H1)*, there exists at least a subset of edges X of Θ_2 such that $\Theta_0 \cup \Theta_2 - X$ is a subgraph without a circuit of $\Theta_0 \cup \Theta_2$ (i.e., a forest of $\Theta_0 \cup \Theta_2$). If, for all $X' \subset X$, $\Theta_0 \cup \Theta_2 - X'$ contains at least one circuit, we say that X is minimal.*

REMARK 3.3. *In that case $\Theta_0 \cup \Theta_2 - X$ is a spanning forest of $\Theta_0 \cup \Theta_2$ (for this definition and the following remarks see* [12, 13, 14, 15, 16, 17]*).*

REMARK 3.4. *Every X minimal in the previous definition contains the same number of edges.*

REMARK 3.5. *There exist algorithms for the construction of such a spanning forest.*

Now, let us fix such a subset X and denote for what follows $\widetilde{\Theta_2} = X$ and $\overline{\Theta_2} = \Theta_2 - X$.

(H2) For every edge $e \in \Theta_1 \cup \Theta_3 \cup \Theta_4$, there exists a circuit β such that $\{e\} \subset \beta \subset \Theta_0 \cup \overline{\Theta_2} \cup \{e\}$.

In order to satisfy the current Kirchhoff law for a graph without amplifiers, a necessary condition for the unique solvability of a network is that the set Θ_1 should not contain any cut set, where a cut set is a minimal set of edges whose deletion will increase the number connected components by one (see [17, Chapter 2]). Since the current is given independently on each edge of Θ_3 and Θ_4, physically we extend this condition for all the edges of $\Theta_1 \cup \Theta_3 \cup \Theta_4$. Using the definition of $\overline{\Theta_2}$, we can show that the interpretation of this last condition is (H2); nevertheless, we leave the details to the reader since the justification is quite technical.

Lemma 3.6. *Hypothesis* (H2) *does not depend on the choice of* X.

Proof. If it is not true, there exist X' minimal and a branch $e \in \Theta_1 \cup \Theta_3 \cup \Theta_4$ such that there is no circuit in $\Theta_0 \cup (\Theta_2 - X') \cup \{e\}$.

Since $\Theta_0 \cup \overline{\Theta_2}$ is a spanning forest of $\Theta_0 \cup \Theta_2 \cup \{e\}$, if we add any edge of $\widetilde{\Theta_2} \cup \{e\}$ we will obtain a circuit. On the other hand, $\Theta_0 \cup (\Theta_2 - X')$ is not a spanning forest of $\Theta_0 \cup \Theta_2 \cup \{e\}$ since there is no circuit in $\Theta_0 \cup (\Theta_2 - X') \cup \{e\}$. Therefore we can get a spanning forest containing $\Theta_0 \cup (\Theta_2 - X') \cup \{e\}$. But X and X' have the same cardinal, so we have obtained two spanning forests of $\Theta_0 \cup \Theta_2 \cup \{e\}$ with a different number of edges. By section 1.2 of [17], this is not possible. ◻

Definition 3.7. *For* $e \in \Theta_1 \cup \widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$, *there exists a unique circuit* β *such that* $\{e\} \subset \beta \subset \{e\} \cup \Theta_0 \cup \overline{\Theta}_2$ *(see section* 1.2 *of* [17], *definition of a fundamental system of circuits associated with* $\Theta_0 \cup \overline{\Theta}_2$*). We denote this circuit by* $Z(e)$.

The definition of $Z(e)$ implies $\nabla_\tau \psi_{|e}$ is a linear combination of $(\nabla_\tau \psi_{|e'})_{e' \in Z(e) - \{e\}}$. Therefore the following result holds.

Lemma 3.8. *If* (H1), (H2) *are fulfilled, there exists* $C > 0$ *such that for all* $e \in \Theta_1 \cup \widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$ *we have the estimate*

$$|\nabla_\tau \psi|_e \leq C |\nabla_\tau \psi|_{Z(e)-\{e\}} \quad \forall \psi \in \Psi_{ad}(u_d). \tag{3.1}$$

Corollary 3.9. *If* (H1), (H2) *are fulfilled, there exists* $C > 0$ *such that for all* $e \in \Theta_1 \cup \widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$ *we have the estimates*

$$|\nabla_\tau \psi|_e \leq C |\nabla_\tau \psi|_{\Theta_0 \cup \overline{\Theta}_2} \quad \forall \psi \in \Psi_{ad}(u_d) \tag{3.2}$$

and

$$|\nabla_\tau \psi|_e \leq C |\nabla_\tau \psi|_{\overline{\Theta}_2} \quad \forall \psi \in \Psi_{ad}(0). \tag{3.3}$$

Now, we give some definitions and immediate consequences useful in what follows. These definitions are illustrated with examples in the following subsection.

Definition 3.10 (of a relation r in the set $\widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$). *For* e *and* e' *in* $\widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$, *we say that* $e \; r \; e'$ *if and only if* $Z(e) \cap Z(e') \cap \overline{\Theta_2} \neq \emptyset$.

Remark 3.11. *The previous definition implies that if* $e \; r \; e'$, *then the set* $Y(e, e') = Z(e) \cup Z(e') - (Z(e) \cap Z(e'))$ *is included in* $\overline{\Theta_2} \cup \Theta_0 \cup \{e, e'\}$ *and is a circuit.*

Lemma 3.12. *If* $e \; r \; e'$, *then*

$$|\nabla_\tau \psi|_{e'} \leq C |\nabla_\tau \psi|_{Y(e,e')-\{e'\}} \leq C |\nabla_\tau \psi|_{\overline{\Theta}_2 \cup \{e\}} \quad \forall \psi \in \Psi_{ad}(0).$$

Definition 3.13. *A sequence* $c(e, e')$ *associated with the relation* r *is a set* $\{e, e_1, \ldots, e_k, e'\}$ *of elements in* $\widetilde{\Theta_2} \cup \Theta_3 \cup \Theta_4$ *such that* $e \; r \; e_1$, $e_1 \; r \; e_2$, $\ldots$, $e_k \; r \; e'$.

Definition 3.14. *The set-valued function* R *from* Θ_4 *to* $\mathcal{P}(\Theta_3)$ *is defined by*

$$e_4 \in \Theta_4 \quad \longmapsto \quad R(e_4) = \{e_3 \in \Theta_3 : \textit{there exists a sequence which satisfies } c(e_3, e_4) - \{e_3, e_4\} \subset \widetilde{\Theta_2}\}.$$

Now, we assume that the network Θ contains $N \in \mathbb{N}$ amplifiers denoted by $A_i = \{e_3^i, e_4^i\}$, $i = 1, \ldots, N$. The corresponding coefficients will be denoted by k_i. Let us associate with the function R the directed graph $G = (V, E)$, where the set of points V are the amplifiers A_i ($i = 1, \ldots, N$) and the set of directed edges are the pairs (A_i, A_j) when $e_3^j \in R(e_4^i)$. The following assumption is formulated as an hypothesis on the graph G.

(H3) G is without a circuit.

REMARK 3.15. *One can use classical algorithms to check* (H3) *(see* [12, 17]*).*

Roughly speaking, a path in the directed graph G shows the propagation of the information through the amplifiers. So assumption (H3) does not allow any circuit of amplification which would lead to a infinite amplification.

(H4) In each connected component of $\Theta - \Theta_1$ there is a vertex belonging to σ_0.

Assumption (H4) eliminates the constant potential.

REMARK 3.16. *Assumption* (H2) *implies that there exists a positive constant C such that for every $\psi \in \Psi_{ad}(0)$ we have $|\nabla_\tau \psi|_{\Theta_1} \leq C|\nabla_\tau \psi|_{\Theta - \Theta_1}$. It implies the continuity of the linear form $l(\psi) = -\int_{\Theta_1} i_d \nabla_\tau \psi dl(x)$ with respect to the seminorm $|\nabla_\tau \psi|_\Theta$. Taking account of* (H4) *we obtain that $|\nabla_\tau \psi|_\Theta$ is a norm on $\Psi_{ad}(0)$.*

Now we are ready to state the theorem of existence and uniqueness.

THEOREM 3.17. *If assumptions* (H1)–(H4) *are fulfilled, then the variational formulation* (2.8) *has a unique solution.*

3.2. Illustration of the assumptions. We consider two similar examples described in Figure 3.1 for which we determine the graph G.

For both examples, assumptions (H1), (H4) are obviously satisfied. Removing the set $\{\widetilde{e}_2\} \cup \Theta_1 \cup \Theta_3 \cup \Theta_4$ from these two networks we obtain the same spanning forest $\Theta_0 \cup \widetilde{\Theta_2}$ described in Figure 3.1.

So $\widetilde{\Theta_2} = \{\widetilde{e}_2\}$, and it is clear that assumption (H2) is satisfied.

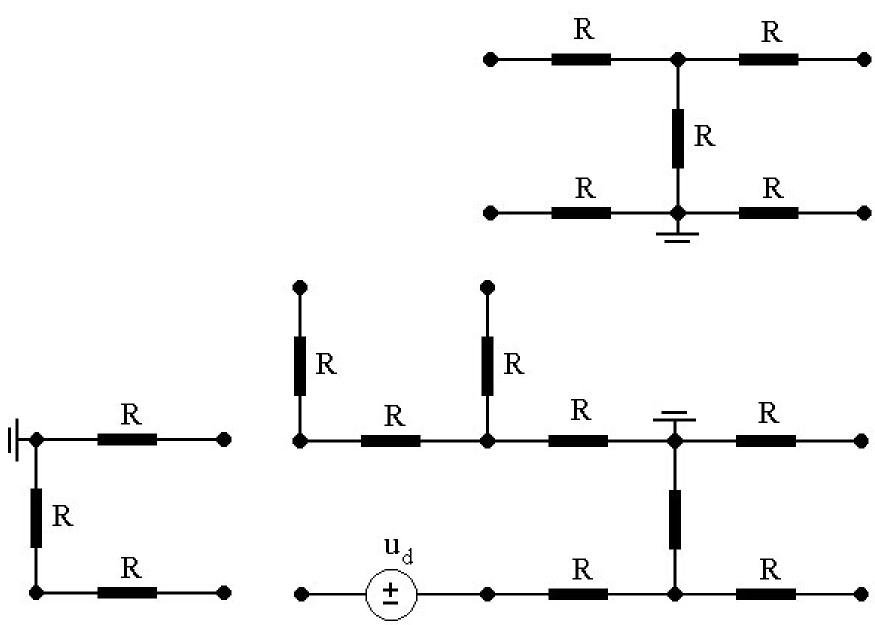


FIG. 3.1. *The spanning forest for Examples* 1 *and* 2.

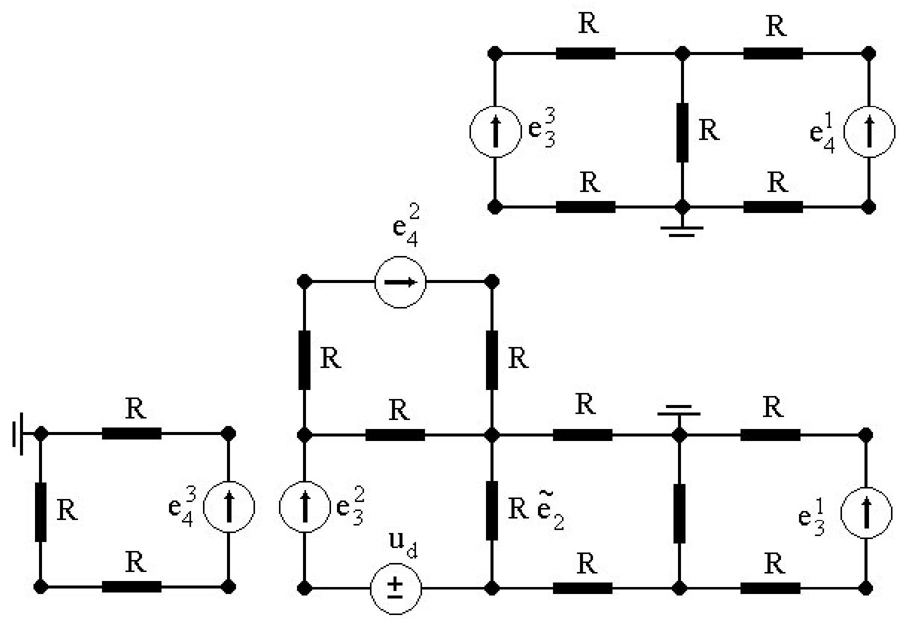


Fig. 3.2. *Example* 1.

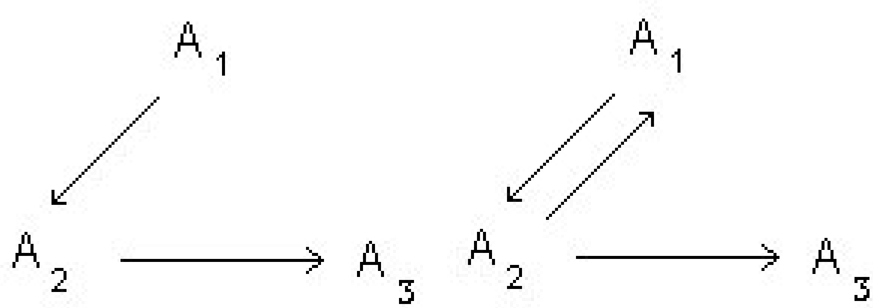


Fig. 3.3. *Graphs* G_1 *and* G_2.

Example 1 (see Figure 3.2). The amplifiers $A_i = \{e_3^i, e_4^i\}$, $i = 1, 2, 3$, are such that $Z(e_4^1) \cap Z(e_3^2) \cap \overline{\Theta_2} \neq \emptyset$ and $Z(e_4^2) \cap Z(e_3^3) \cap \overline{\Theta_2} \neq \emptyset$, so

$$e_4^1 \; r \; e_3^2 \quad \text{and} \quad e_4^2 \; r \; e_3^3. \tag{3.4}$$

Thus $c(e_4^1, e_3^2) = \{e_4^1, e_3^2\}$ and $c(e_4^2, e_3^3) = \{e_4^2, e_3^3\}$. Hence, $c(e_4^1, e_3^2) - \{e_4^1, e_3^2\} = \emptyset \subset \widetilde{\Theta_2}$ and $c(e_4^2, e_3^3) - \{e_4^2, e_3^3\} = \emptyset \subset \widetilde{\Theta_2}$. Therefore the function R is defined by

$$R(e_4^1) = \{e_3^2\}, \quad R(e_4^2) = \{e_3^3\}, \quad \text{and} \quad R(e_4^3) = \emptyset.$$

The directed graph G_1 is given on the left side of Figure 3.3.

There is no circuit in G_1: assumption (H3) holds.

Example 2 (see Figure 3.4). The amplifiers $A_i = \{e_3^i, e_4^i\}$, $i = 1, 2, 3$, satisfy (3.4) and

$$e_4^2 \; r \; \widetilde{e}_2; \;\; \widetilde{e}_2 \; r \; e_3^1. \tag{3.5}$$

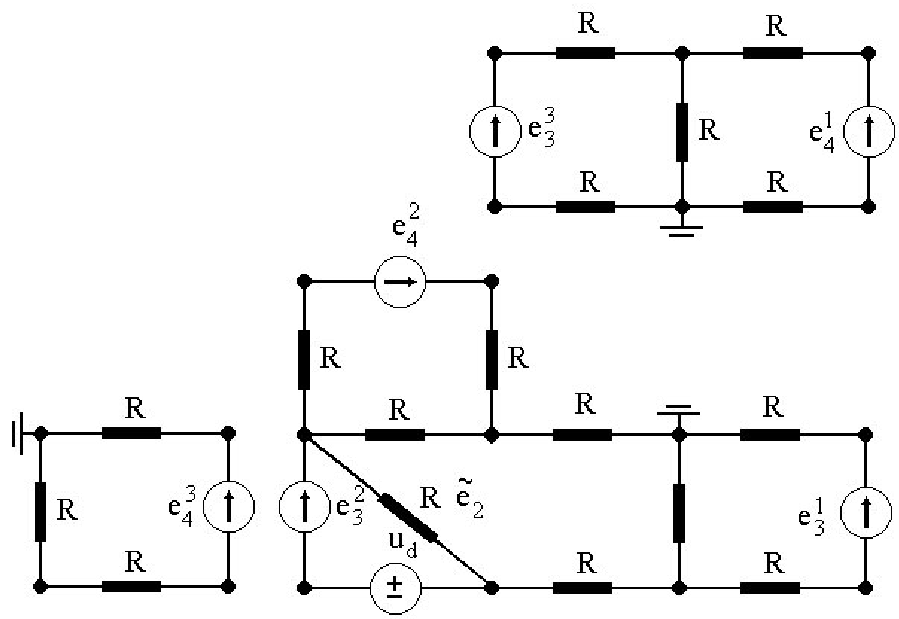


Fig. 3.4. *Example* 2.

From (3.5), there exists an additional sequence $c(e_4^2, e_3^1) = \{e_4^2, \widetilde{e_2}, e_3^1\}$ such that $c(e_4^2, e_3^1) - \{e_4^2, \widetilde{e_2}, e_3^1\} = \{\widetilde{e_2}\} \subset \widetilde{\Theta_2}$. Thus the function R is defined by

$$R(e_4^1) = \{e_3^2\}, \quad R(e_4^2) = \{e_3^1, e_3^3\}, \quad \text{and} \quad R(e_4^3) = \emptyset.$$

The directed graph G_2 is given on the right side of Figure 3.3. The circuit $\{A_1, A_2\}$ belongs to G_2, so assumption (H3) does not hold.

3.3. Notations for the proof. Some additional notations and properties are introduced in this section. They are of constant use in the proof of the existence and uniqueness theorem.

First, we define a subset of $\widetilde{\Theta_2}$ associated with each edges in $\Theta_3 \cup \Theta_4$.

Definition 3.18. *Under* (H1), (H2)*, if* $e \in \Theta_3 \cup \Theta_4$*,* $X(e)$ *is the set of elements* e_2 *of* $\widetilde{\Theta_2}$ *such that there exists a sequence* $c(e, e_2)$ *which satisfies*

$$c(e, e_2) - \{e\} \subset \widetilde{\Theta_2}.$$

We also define

$$Z'(e) = \left(Z(e) \bigcup_{e_2 \in X(e)} Z(e_2) \right) \cap \overline{\Theta_2}.$$

From Definition 3.18 and Lemma 3.8 we have the following.

Lemma 3.19. *Under* (H1), (H2)*, there exists* $C > 0$ *such that for all* $e \in \Theta_3 \cup \Theta_4$ *and* $e' \in X(e) \cup \{e\}$ *the following estimate holds:*

$$|\nabla_\tau \psi|_{e'} \leq C |\nabla_\tau \psi|_{Z'(e)} \quad \forall \psi \in \Psi_{ad}(0).$$

Now we give two lemmas useful in what follows.

LEMMA 3.20. *Under* (H1), (H2), *there exists* $C > 0$ *such that for all* $e \in \Theta_3 \cup \Theta_4$ *and* $\widetilde{e} \in \widetilde{\Theta_2} - X(e)$ *we have*

$$|\nabla_\tau \psi|_{\widetilde{e}} \leq C |\nabla_\tau \psi|_{\overline{\Theta_2} - Z'(e)} \quad \forall \psi \in \Psi_{ad}(0).$$

Proof. An immediate consequence of Definition 3.18 is that if $\widetilde{e} \in \widetilde{\Theta_2} - X(e)$, then $Z(\widetilde{e}) \cap Z'(e) = \emptyset$. We conclude with the help of Lemma 3.8. ∎

LEMMA 3.21. *Under* (H1), (H2), *there exists* $C > 0$ *such that for all* $e_3 \in \Theta_3$ *and* $e_4 \in \Theta_4$ *which satisfy* $e_3 \notin R(e_4)$ *we have the estimate*

$$|\nabla_\tau \psi|_{e_4} \leq C |\nabla_\tau \psi|_{\overline{\Theta_2} - Z'(e_3)} \quad \forall \psi \in \Psi_{ad}(0). \tag{3.6}$$

Proof. By the definitions of R (see Definition 3.14) and $Z'(e_3)$ (see Definition 3.18) we have $Z'(e_3) \cap Z(e_4) = \emptyset$. With the help of Lemma 3.8 we conclude the proof. ∎

Now, we define a partition of Θ_4 based on assumption (H3).

LEMMA 3.22. *Under* (H1)–(H3) *there exists a partition* $\Theta_4^1 \cup \Theta_4^2 \cup \cdots \cup \Theta_4^p$ *of* Θ_4 *such that* (i) *and* (ii) *hereafter are satisfied:*

(i) $\Theta_4^1 = \{e_4^l \in \Theta_4 \mid \forall e_4 \in \Theta_4,\ e_3^l \notin R(e_4)\}$.

(ii) $$\Theta_4^j = \left\{ e_4^l \in \Theta_4 \;\middle|\; \begin{array}{l} \forall e_4 \in \Theta_4 \textit{ such that } e_3^l \in R(e_4)\textit{; then} \\ e_4 \subset \displaystyle\bigcup_{i<j} \Theta_4^i, \textit{ and there exists } e_4 \in \Theta_4^{j-1} \textit{ such that } e_3^l \in R(e_4) \end{array} \right\},\ j = 2, \ldots, p.$$

Proof. From (H3), G is without a circuit. The set of vertices of V without a predecessor is not empty (see, for instance, [12, Chapter 2]). Then we can associate with each vertex A_i one and only one level $j = 1, \ldots, p$ in the following way:

(i) A_i is of level 1 if and only if A_i is without a predecessor.

(ii) For $j = 2, \ldots, p$, A_i is of level j if and only if its predecessors are of level strictly lower than j and at least one of them is exactly of level $j-1$.

Now we consider the partition $\Theta_4 = \bigcup_{j=1}^p \Theta_4^j$ such that $e_4^i \in \Theta_4^j$ if A_i is of level j. From Definition 3.14 and G, it is clear that this partition satisfies (i) and (ii). ∎

DEFINITION 3.23. *Under* (H1)–(H3), *we define the partition* $\Theta_3^1 \cup \Theta_3^2 \cup \cdots \cup \Theta_3^p$ *of* Θ_3 *such that* $e_3^l \in \Theta_3^j$ *if and only* $e_4^l \in \Theta_4^j$, $j = 1, \ldots, p$.

3.4. Technical results. The proof of Theorem 3.17 is based on the following general technical results.

LEMMA 3.24. *If* (H1), (H2), (H4) *are fulfilled, then the seminorm* $|\nabla_\tau \psi|_{\overline{\Theta_2}} = (\int_{\overline{\Theta_2}} |\nabla_\tau \psi|^2)^{\frac{1}{2}} dl(x)$ *is a norm on* $\Psi_{ad}(0)$.

Proof. It is a direct consequence of Corollary 3.9 and Remark 3.16. ∎

LEMMA 3.25. *If* (H1), (H2), (H4) *are fulfilled, then the bilinear form* $a(.,.)$ *defined by*

$$a(\varphi, \psi) = \int_{\Theta_2} g L \nabla_\tau \varphi \nabla_\tau \psi \, dl(x) \quad \forall \varphi, \psi \in \Psi_{ad}(0)$$

is continuous and coercive on $\Psi_{ad}(0)$. *Moreover, the problem*

$$a(\varphi, \psi) = l(\psi) \quad \forall \psi \in \Psi_{ad}(0),$$

where l is a linear continuous form on $\Psi_{ad}(0)$, admits a unique solution $\varphi \in \Psi_{ad}(0)$. There exists a constant C, $C > 0$, such that the solution φ follows the estimation

$$\|\varphi\| \leq \frac{1}{C} \|l\|_{(\Psi_{ad}(0))'}.$$

Proof. The continuity follows immediately. For all ψ in $\Psi_{ad}(0)$, $a(\psi, \psi) = \int_{\Theta_2} Lg \left|\nabla_\tau \psi\right|^2 dl(x) \geq L_{\min} g_{\min} \left|\nabla_\tau \psi\right|^2_{\Theta_2}$. From Lemma 3.24 there exists $C' > 0$ such that $\left|\nabla_\tau \psi\right|^2_{\Theta_2} \geq C' \|\psi\|^2$. Setting $C = g_{\min} L_{\min} C'$, the bilinear form a is coercive on $\Psi_{ad}(0)$ since for all ψ in $\Psi_{ad}(0)$

$$a(\psi, \psi) \geq C \|\psi\|^2.$$

We conclude with the Lax–Milgram theorem. $\square$

LEMMA 3.26. *Let us consider $K_1 \subset \overline{\Theta_2}$ and $K_2 = \overline{\Theta_2} - K_1$ and set*

$$\Psi_i = \{\psi \in \Psi_{ad}(0) : \nabla_\tau \psi_{|e} = 0 \ \forall e \in K_i\}, \ i = 1, 2. \tag{3.7}$$

Under (H1), (H2), (H4), $\Psi_{ad}(0) = \Psi_1 \oplus \Psi_2$.

Proof. For any $\psi \in \Psi_{ad}(0)$, let ψ_1 be a function in Ψ_1 such that

$$\begin{aligned} \nabla_\tau \psi_1 &= \nabla_\tau \psi \text{ on } K_1 \\ &= 0 \text{ on } K_2. \end{aligned}$$

ψ_1 exists because there is no circuit in $\overline{\Theta}_2$. Thus from Lemma 3.24 ψ_1 is defined on Θ. It is the same for $\psi_2 \in \Psi_2$ such that

$$\begin{aligned} \nabla_\tau \psi_2 &= \nabla_\tau \psi \text{ on } K_2 \\ &= 0 \text{ on } K_1. \end{aligned}$$

Hence $\nabla_\tau \psi = \nabla_\tau \psi_1 + \nabla_\tau \psi_2$ on $\overline{\Theta}_2$; thus from Lemma 3.24 $\psi = \psi_1 + \psi_2$ on Θ. If $\psi \in \Psi_1 \cap \Psi_2$, then $\nabla_\tau \psi = 0$ on $\overline{\Theta}_2$, and from Lemma 3.24 $\psi = 0$. So the sum $\Psi_1 \oplus \Psi_2$ is direct. $\square$

LEMMA 3.27. *Under* (H1), (H2), *we have the following:*

(i) *For all $e \in \Theta_3 \cup \Theta_4$ and all $e' \in \widetilde{\Theta}_2$ we have*

$$e' \in X(e) \textit{ if and only if } Z(e') \cap Z'(e) \neq \emptyset.$$

(ii) *For all $e \in \Theta_3$ and all $e' \in \Theta_4$ we have*

$$e \in R(e') \textit{ if and only if } Z(e') \cap Z'(e) \neq \emptyset.$$

Proof. (i) If $e' \in X(e)$, then obviously $Z(e') \subset Z'(e)$. Reciprocally, if $Z(e') \cap Z'(e) \neq \emptyset$ there exists a sequence $c(e, e')$, and thus $e' \in X(e)$.

(ii) It uses analogous arguments as in (i). $\square$

COROLLARY 3.28. *Under* (H1), (H2), (H4), *for $e \in \Theta_3 \cup \Theta_4$, let $K_1 = Z'(e)$, $K_2 = \overline{\Theta}_2 - K_1$, and Ψ_1 and Ψ_2 be defined as in Lemma* 3.26. *Then, for all $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$,*

$$\int_{\widetilde{\Theta}_2} gL \nabla_\tau \psi_1 \nabla_\tau \psi_2 dl(x) = 0.$$

Proof. Since $\psi_1 \in \Psi_1$, then $\nabla_\tau \psi_1 = 0$ on $X(e) \subset Z'(e)$, and, consequently,

$$\int_{\widetilde{\Theta}_2} gL\nabla_\tau\psi_1\nabla_\tau\psi_2 dl(x) = \int_{\widetilde{\Theta}_2 - X(e)} gL\nabla_\tau\psi_1\nabla_\tau\psi_2 dl(x). \tag{3.8}$$

For all $e' \in \widetilde{\Theta}_2 - X(e)$, $Z(e') \cap Z'(e) = \emptyset$ by Lemma 3.27. Thus $Z(e') \cap \overline{\Theta}_2 \subset K_2$, and this implies $\nabla_\tau\psi_2 = 0$ on $Z(e')$. Thus, $\nabla_\tau\psi_{2|e'} = 0$ for all $e' \in \widetilde{\Theta}_2 - X(e)$. With (3.8), this leads to the result. $\square$

COROLLARY 3.29. *Let us assume that* (H1), (H2), (H4) *hold. Let* $e \in \Theta_3 \cup \Theta_4$, $K_1 = Z'(e)$, $K_2 = \overline{\Theta}_2 - K_1$, *and* $\varphi \in \Psi_{ad}(0)$ *be the unique solution of*

$$a(\varphi, \psi) = \int_e \nabla_\tau\psi dl(x) \quad \forall\psi \in \Psi_{ad}(0).$$

Then $\varphi \in \Psi_2$ *(*Ψ_2 *defined as in Lemma* 3.26*); that is,* $\nabla_\tau\varphi = 0$ *on* $\overline{\Theta}_2 - Z'(e)$.

Proof. By Lemma 3.26, $\Psi_1 \oplus \Psi_2 = \Psi_{ad}(0)$. Let $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in \Psi_1$, and $\varphi_2 \in \Psi_2$; then φ_2 is the unique solution of

$$a(\varphi_1, \psi) = -a(\varphi_2, \psi) \quad \forall\psi \in \Psi_1,$$

since $\int_e \nabla_\tau\psi = 0$ for all $\psi \in \Psi_1$. Therefore for all $\psi \in \Psi_1$, $a(\varphi_2, \psi) = \int_{\Theta_2} g\nabla_\tau\varphi_2\nabla_\tau\psi = \int_{\widetilde{\Theta}_2} g\nabla_\tau\varphi_2\nabla_\tau\psi = 0$ by Corollary 3.28. Hence $a(\varphi_1, \psi) = 0$; thus $\int_{\widetilde{\Theta}_2 \cup Z'(e)} g\nabla_\tau\varphi_1\nabla_\tau\psi = 0$ for all $\psi \in \Psi_1$. This implies $\nabla_\tau\varphi_1 = 0$ on $Z'(e)$ and thus on $\overline{\Theta}_2$, and, finally, $\varphi_1 = 0$ on Θ. Consequently, $\varphi = \varphi_2$; i.e., $\nabla_\tau\varphi = 0$ on $\overline{\Theta}_2 - Z'(e)$. $\square$

LEMMA 3.30. *Let us assume that* (H1)–(H4) *hold. Let* $j \in \{1, \dots, p\}$ *(here and in what follows* p *is given by Lemma* 3.22*),* $e_4 \in \Theta_4^j$, $K_1 = Z'(e_4)$, $K_2 = \overline{\Theta}_2 - K_1$, *and* $\varphi \in \Psi_{ad}(0)$ *be the unique solution of*

$$a(\varphi, \psi) = \int_{e_4} \nabla_\tau\psi dl(x) \quad \forall\psi \in \Psi_{ad}(0).$$

Then $\nabla_\tau\varphi = 0$ *on* $\bigcup_{1\le i\le j}\bigcup_{e_3\in\Theta_3^i} Z(e_3)$, *where* Θ_3^i *is given in Definition* 3.23.

Proof. By Corollary 3.29, we know that $\nabla_\tau\varphi = 0$ on $\overline{\Theta}_2 - Z'(e_4)$. It remains to prove that $\overline{\Theta}_2 \cap Z(e_3) \subset \overline{\Theta}_2 - Z'(e_4)$ for all $e_3 \in \Theta_3^i$ and all $1 \le i \le j$. However, if $Z(e_3) \cap Z'(e_4) \neq \emptyset$, then $e_3 \in R(e_4)$ and is a contradiction with the fact that $i \le j$. $\square$

COROLLARY 3.31. *Let us assume that* (H1)–(H4) *hold. For all* $\Theta' \subset \Theta_4^j$ *(*$j \in \{1, \dots, p\}$*),* $\alpha \in \mathbb{P}^0(\Theta')$*, let* l *be the linear form defined by* $l(\psi) = \int_{\Theta'} \alpha\nabla_\tau\psi dl(x)$ *and* $\varphi \in \Psi_{ad}(0)$ *be the unique solution of*

$$a(\varphi, \psi) = l(\psi) \quad \forall\psi \in \Psi_{ad}(0). \tag{3.9}$$

Then $\nabla_\tau\varphi = 0$ *on* $\bigcup_{1\le i\le j}\bigcup_{e_3\in\Theta_3^i} Z(e_3)$.

Proof. For $e_4 \in \Theta_4^j$, we denote by φ_{e_4} the solution given by Corollary 3.29. Hence $\varphi = \sum_{e_4\in\Theta_4^j} \alpha_{|e_4}\varphi_{e_4}$. Then φ is the solution of problem (3.9). Therefore, by Lemma 3.30, each $\nabla_\tau\varphi_{|e_4} = 0$ on $Z(e_3)$ for all $e_3 \in \Theta_3^i$ and all $i \in \{1, \dots, j\}$. So this is also true for φ. $\square$

LEMMA 3.32. *Let us assume that* (H1), (H2) *hold. Let* $e \in \Theta_3$ *and* $\varphi \in \Psi_{ad}(0)$ *be given. Let us consider* $\psi \in \Psi_{ad}(0)$ *such that*

$$\nabla_\tau\psi = 0 \text{ on } \overline{\Theta}_2 - Z'(e) \tag{3.10}$$

$$= \nabla_\tau\varphi \text{ on } Z'(e); \tag{3.11}$$

then

$$\nabla_\tau \psi = 0 \text{ on } (\Theta_4 - R^{-1}(e)) \cup (\widetilde{\Theta}_2 - X(e))$$
$$= \nabla_\tau \varphi \text{ on } R^{-1}(e) \cup X(e),$$

where $R^{-1}(e) = \{e_4 \in \Theta_4 : e \in R(e_4)\}$.

Proof. Since $\nabla_\tau \psi = \nabla_\tau \varphi$ on $Z'(e)$ and for all $e' \in X(e)$, $Z(e') - \{e'\} \subset Z'(e)$, then $\nabla_\tau \psi = \nabla_\tau \varphi$ on $Z(e') - \{e'\}$, and thus $\nabla_\tau \psi_{|e'} = \nabla_\tau \varphi_{|e'}$. The same proof holds for $e' \in R^{-1}(e)$.

If $e' \in \widetilde{\Theta}_2 \cup \Theta_4 - (R^{-1}(e) \cup X(e))$, then by Lemma 3.27 $Z(e') \cap Z'(e) = \emptyset$, and thus $\nabla_\tau \psi = 0$ on $Z(e') - \{e'\}$, and, consequently, $\nabla_\tau \psi_{|e'} = 0$. ◻

3.5. Proof of Theorem 3.17. Since (H1) is satisfied, there does not exist any circuit included in Θ_0. Therefore, for all $u_d \in \mathbb{P}^0(\Theta_0)$, $\Psi_{ad}(u_d) \neq \emptyset$. Let us consider $\widetilde{\varphi} \in \Psi_{ad}(u_d)$ and the problem satisfied by $(\overline{\varphi}, \overline{i}) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$:

$$\text{(3.12)} \qquad \begin{cases} a(\overline{\varphi}, \psi) + b_1(\overline{i}, \psi) &= l(\psi) - a(\widetilde{\varphi}, \psi) - b_1(\widetilde{i}, \psi), \\ b_2(\overline{\varphi}, j) - c(\overline{i}, j) &= -b_2(\widetilde{\varphi}, j) + c(\widetilde{i}, j) \end{cases}$$

for all $(\psi, j) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$, where we have set $\overline{\varphi} = \varphi - \widetilde{\varphi}$, $\widetilde{i}_{|\Theta_4} = k\nabla_\tau \widetilde{\varphi}_{|\Theta_3}$, and $\overline{i} = i - \widetilde{i}$.

The problems of existence and uniqueness of (φ, i) or $(\overline{\varphi}, \overline{i})$ are equivalent. Thus, in what follows, we consider only the case where $u_d = 0$.

Now, let us write more simply the problem (3.12) as follows: Find $(\varphi, i) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$ the solution of

$$\text{(3.13)} \qquad \begin{cases} a(\varphi, \psi) + b_1(i, \psi) &= l_1(\psi), \\ b_2(\varphi, j) - c(i, j) &= l_2(j) \end{cases}$$

for all $(\psi, j) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)$ (here l_1 and l_2 are linear continuous forms).

We will check the four conditions C_i $(i = 1, \dots, 4)$ successively.

Condition C_1.

$$\forall (\varphi, i) \in W_1 - \{(0,0)\}, \quad \sup_{\Psi_{ad}(0) - \{0\}} (a(\varphi, \psi) + b_1(i, \psi)) > 0,$$

where $W_1 = \{(\varphi, i) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4) : i_{|\Theta_4} = k(L\nabla_\tau \varphi)_{|\Theta_3}\}$.

If C_1 does not hold, there exists $\varphi \in \Psi_{ad}(0)$, $\varphi \neq 0$, such that for all $\psi \in \Psi_{ad}(0)$

$$\text{(3.14)} \qquad \int_{\Theta_2} gL\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi dl(x) = 0.$$

For all $e \in \Theta_3$, we choose $\psi \in \Psi_{ad}(0)$ satisfying (3.10)–(3.11). Using Lemma 3.32, (3.14) leads to

$$\text{(3.15)} \qquad \int_{Z'(e) \cup X(e)} gL \left|\nabla_\tau \varphi\right|^2 dl(x) + \int_{R^{-1}(e)} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \varphi_{|\Theta_4} dl(x) = 0.$$

We shall show by iteration that $\nabla_\tau \varphi_{|e_3} = 0$ for all $e_3 \in \Theta_3^i$.

If $e \in \Theta_3^1$, $R^{-1}(e) = \emptyset$, then (3.15) leads to $\nabla_\tau \varphi = 0$ on $Z'(e)$; thus $\nabla_\tau \varphi_{|e} = 0$. For $j > 1$ and $e \in \Theta_3^j$, every $e_4^l \in R^{-1}(e)$ is in one of the subsets Θ_4^i with $i < j$, and hence $\nabla_\tau \varphi_{|e_3^l} = 0$; consequently, we also get $\nabla_\tau \varphi_{|e} = 0$.

Now, with $\psi = \varphi$ in (3.14), it remains that $\int_{\Theta_2} gL \left|\nabla_\tau \varphi\right|^2 dl(x) = 0$. This leads to $\varphi = 0$ and proves C_1.

Condition C_2. We have to show that there exists $\alpha > 0$ such that, for all $\psi \in \Psi_{ad}(0)$, we can find $(\varphi, i) \in W_1 - \{0\}$ satisfying

$$\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi dl(x) + \int_{\Theta_4} i\nabla_\tau\psi dl(x) \geq \alpha \left\|(\varphi, i)\right\| \left\|\psi\right\|$$

or, equivalently, find $\varphi \in \Psi_{ad}(0) - \{0\}$ such that

$$\int_{\Theta_2} gL\nabla_\tau\varphi\nabla_\tau\psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau\varphi)_{|\Theta_3}\nabla_\tau\psi dl(x) \geq \alpha \left\|\varphi\right\| \left\|\psi\right\|. \tag{3.16}$$

For all $\psi \in \Psi_{ad}(0)$, let us suppose for the moment that there exists $\varphi \neq 0$, so that

$$a(\varphi, \psi) + \int_{\Theta_4} k(L\nabla_\tau\varphi)_{|\Theta_3}\nabla_\tau\psi dl(x) = a(\psi, \psi), \tag{3.17}$$

with the following estimate:

$$\left\|\varphi\right\| \leq C \left\|\psi\right\|, \tag{3.18}$$

where $C > 0$. Then from (3.17) and (3.18)

$$a(\varphi, \psi) + \int_{\Theta_4} k(L\nabla_\tau\varphi)_{|\Theta_3}\nabla_\tau\psi dl(x) = a(\psi, \psi) \geq L_{\min} g_{\min} C \left\|\varphi\right\| \left\|\psi\right\|.$$

That proves C_2. It remains to find φ.

Let us denote $\psi = \psi_0$. By iteration on $j = 1, \ldots, p$, with the help of Lemma 3.25, let us consider the solution ψ_j of

$$a(\psi_j, \psi') = l_j(\psi') \quad \forall \psi' \in \Psi_{ad}(0), \tag{3.19}$$

where l_j is the linear form defined on $\Psi_{ad}(0)$ by

$$l_j(\psi') = -\sum_{e_3^l \in \Theta_3^j} \int_{e_4^l} k_{e_3^l} \left(L\nabla_\tau \left(\sum_{i=0}^{j-1} \psi_i \right) \right)_{|e_3^l} \nabla_\tau \psi' dl(x). \tag{3.20}$$

We set

$$\varphi = \sum_{j=0}^{p} \psi_j. \tag{3.21}$$

It is easy to show that (3.18) holds. Applying Corollary 3.31 we see that $\nabla_\tau \psi_j = 0$ on $\bigcup_{1 \leq i \leq j} \bigcup_{e_3 \in \Theta_3^j} Z(e_3)$. Therefore, for each $j = 1, \ldots, p$, we can write

$$\begin{aligned} &-\sum_{e_3^l \in \Theta_3^j} \int_{e_4^l} k_{e_3^l} \left(L\nabla_\tau \left(\sum_{i=0}^{j-1} \psi_i \right) \right)_{|e_3^l} \nabla_\tau \psi' dl(x) \\ &\quad = -\sum_{e_3^l \in \Theta_3^j} \int_{e_4^l} k_{e_3^l} \left(L\nabla_\tau \left(\sum_{i=0}^{p} \psi_i \right) \right)_{|e_3^l} \nabla_\tau \psi' dl(x) \end{aligned} \tag{3.22}$$

for all $\psi' \in \Psi_{ad}(0)$. Using (3.19) and (3.22) we have the following equalities:

$$\begin{aligned}
a(\varphi, \psi) &+ \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi dl(x) \\
&= a\left(\sum_{i=0}^{p} \psi_j, \psi\right) + \int_{\Theta_4} k\left(L\nabla_\tau\left(\sum_{i=0}^{p} \psi_j\right)\right)_{|\Theta_3} \nabla_\tau \psi dl(x) \\
&= a(\psi, \psi) + \sum_{j=1}^{p} a(\psi_j, \psi) + \sum_{j=1}^{p} \sum_{e_3^l \in \Theta_3^j} \int_{e_4^l} k_{e_3^l}\left(L\nabla_\tau\left(\sum_{i=0}^{p} \psi_i\right)\right)_{|e_3^l} \nabla_\tau \psi dl(x) \\
&= a(\psi, \psi).
\end{aligned}$$

Thus (3.17) holds.

Condition C_3.

$$\forall(\varphi, i) \in W_2 - \{0\}, \quad \sup_{j \in \mathbb{P}^0(\Theta_4)} (b_2(\varphi, j) - c(i, j)) > 0,$$

where

$$W_2 = \{(\varphi, i) \in \Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4) : \forall \psi \in \Psi_{ad}(0),\ a(\varphi, \psi) + b_1(i, \psi) = 0\}.$$

Suppose that we do not have C_3: there exists $(\varphi, i) \in W_2 - \{0\}$ satisfying, for all $j \in \mathbb{P}^0(\Theta_4)$,

$$\int_{\Theta_3} k_{|e_3^l} (L\nabla_\tau \varphi)_{|e_3^l}\, j_{|e_4^l} dl(x) - \int_{\Theta_4} i_{|e_4^l} j_{|e_4^l} dl(x) = 0. \tag{3.23}$$

It is clear that (3.23) means $i_{|\Theta_4} = k(L\nabla_\tau \varphi)_{|\Theta_3}$ and, consequently, implies

$$\int_{\Theta_2} gL\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi dl(x) = 0$$

for all $\psi \in \psi_{ad}(0)$. Since C_1 holds, we obtain $\varphi = 0$, and therefore $i = 0$.

Condition C_4. There exists a constant $\beta > 0$ such that

$$\forall j \in \mathbb{P}^0(\Theta_4), \quad \sup_{(\varphi, i) \in W_2 - \{0\}} \frac{b_2(\varphi, j) - c(i, j)}{\|(\varphi, i)\|_{\Psi_{ad}(0) \times \mathbb{P}^0(\Theta_4)}} \geq \beta \|j\|_{\mathbb{P}^0(\Theta_4)}.$$

Let $j \in \mathbb{P}^0(\Theta_4)$, $j \neq 0$. Let us suppose that there exists $\varphi \in \Psi_{ad}(0)$ such that

$$a(\varphi, \psi) = \int_{\Theta_4} (j - (kL\nabla_\tau \varphi)_{|\Theta_3}) \nabla_\tau \psi dl(x) \quad \forall \psi \in \Psi_{ad}(0), \tag{3.24}$$

with the estimate

$$\|\varphi\| \leq C' \|j\|_{\mathbb{P}^0(\Theta_4)}, \tag{3.25}$$

$C' > 0$. If we set

$$i = -j + k(L\nabla_\tau \varphi)_{|\Theta_3} \in \mathbb{P}^0(\Theta_4), \tag{3.26}$$

(3.24) implies that $(\varphi, i) \in W_2 - \{0\}$ with the estimate $\|i\| \leq C'' \|\varphi\|$. Then, from the previous estimate, the estimate (3.25), and (3.26), we have

$$\begin{aligned} b_2(\varphi, j) - c(i,j) &= \int_{\Theta_3} kL\nabla_\tau \varphi \, j dl(x) - \int_{\Theta_4} i \, j dl(x) \\ &= \int_{\Theta_4} j \; j \; dl(x) = \|j\|^2 \\ &\geq \frac{1}{C'} \|\varphi\| \, \|j\| \geq \frac{C'''}{C'} \|(\varphi, i)\| \, \|j\| \, . \end{aligned}$$

That will prove C_4. It remains to find φ: let us denote $\psi_0 = 0$. Let $\psi_k \in \Psi_{ad}(0)$ be the solution of

$$a(\psi_k, \psi') = \int_{\Theta_4^j} \left(j - k \left(L\nabla_\tau \sum_{j=0}^{k-1} \psi_j \right)_{|\Theta_3^j} \right) \nabla_\tau \psi dl(x) \quad \forall \psi \in \Psi_{ad}(0), \tag{3.27}$$

$k = 1, \ldots, p$. Using the same argument as in the proof of C_2, the function φ defined by (3.21) is the solution of (3.24) with the estimate $\|\varphi\| \leq C' \|j\|_{\mathbb{P}^0(\Theta_4)}$. Now, we set $i = -j + k(L\nabla_\tau \varphi)_{|\Theta_3} \in \mathbb{P}^0(\Theta_4)$; hence (3.24) implies that $(\varphi, i) \in W_2 - \{0\}$ with the estimate (3.25).

3.6. Optimality of conditions (H1)–(H4). Theorem 3.17 gave sufficient conditions for the unique solvability of networks, containing voltage and current sources, resistors, and voltage to current amplifiers. Since negative resistors or negative amplifiers are useful to model certain physical devices, we do not wish to exclude them. In what follows, we consider networks which do not satisfy one of assumptions (H1)–(H4), and we prove the existence of some coefficients g and k such that the unique solvability does not occur. Indeed, we have already seen the usefulness of conditions (H1), (H2) (for $e \in \Theta_1$), and (H4). In subsections 3.6.1 and 3.6.2 we prove, respectively, that (H2) (for $e \in \Theta_3 \cup \Theta_4$) and (H3) are necessary for the unique solvability of the networks considered in this paper.

3.6.1. (H2) is necessary. First, we show that if our problem admits a unique solution, then each $e \in \Theta_3 \cup \Theta_4$ is included in a circuit.

If an edge $e_3 \in \Theta_3$ does not belong to a circuit, Condition C_2 is not satisfied; indeed, we do not have (3.16) with $\psi \in \Psi_{ad}(0)$, $\nabla_\tau \psi_{|\Theta_2 \cup \Theta_4} = 0$, and $\nabla_\tau \psi_{|e_3} \neq 0$. Whereas, if an edge $e_4 \in \Theta_4$ does not belong to a circuit, Condition C_3 is not satisfied. To see that, we consider φ such that $\nabla_\tau \varphi_{|\Theta_2 \cup \Theta_3} = 0$, $i = 0$, and $\nabla_\tau \varphi_{|e_4} \neq 0$. Hence $(\varphi, i) \in W_2 - \{0\}$, and (3.23) holds for all $j \in \mathbb{P}^0(\Theta_4)$.

Now, we assume that (H2) (for $e \in \Theta_3 \cup \Theta_4$) does not hold. Then there exists $e \in \Theta_3 \cup \Theta_4$ such that no circuit is included in $\Theta_0 \cup \overline{\Theta_2} \cup \{e\}$. Thus for every circuit β which contains e the following assertion holds:

(P) There exists $e' \in \Theta_3 \cup \Theta_4$ such that $\{e, e'\} \subset \beta \subset \Theta_0 \cup \overline{\Theta_2} \cup \{e, e'\}$.

Three cases for such e and e' can be distinguished.

First case. $e \in \Theta_3$ and $e' \in \Theta_3$. There exists $\psi \neq 0$ such that $\nabla_\tau \psi = 0$ on $\Theta_2 \cup \Theta_4 - \{e, e'\}$ and $(L\nabla_\tau \psi)_{|e} = -(L\nabla_\tau \psi)_{|e'}$, and we get

$$\int_{\Theta_2} gL\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi dl(x) = 0$$

for all φ in $\psi_{ad}(0)$. So C_2 is not satisfied.

Second case. $e \in \Theta_4$ and $e' \in \Theta_4$. There exists $\varphi \neq 0$ such that $\nabla_\tau \varphi = 0$ on $\Theta_2 \cup \Theta_3 - \{e, e'\}$ and $(L\nabla_\tau \varphi)_{|e} = -(L\nabla_\tau \varphi)_{|e'}$, and we get

$$\int_{\Theta_2} gL\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi dl(x) = 0$$

for all ψ in $\psi_{ad}(0)$. Thus, C_1 does not hold.

Third case. $e \in \Theta_3$ and $e' \in \Theta_4$. Consider $\varphi \in \Psi_{ad}(0) - \{0\}$ such that $\nabla_\tau \varphi \neq 0$ and $\nabla_\tau \varphi = 0$ on $\overline{\Theta_2}$. Thus $\nabla_\tau \varphi = 0$ on Θ_2 and, for all $\psi \in \Psi_{ad}(0)$,

(3.28)
$$\int_{\Theta_2} gL\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi_{|\Theta_4} dl(x) = \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi_{|\Theta_4} dl(x).$$

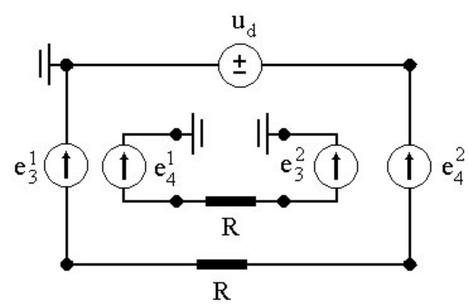
For each $e_3 \in \Theta_3$ such that $\nabla_\tau \varphi \neq 0$, setting $k_{|e_3} = 0$ we get, for all $\psi \in \Psi_{ad}(0)$,

$$\int_{\Theta_2} Lg\nabla_\tau \varphi \nabla_\tau \psi dl(x) + \int_{\Theta_4} k(L\nabla_\tau \varphi)_{|\Theta_3} \nabla_\tau \psi_{|\Theta_4} dl(x) = 0. \tag{3.29}$$

So C_1 does not hold.

Hence, for every coefficient L, g, k assumption (H2) is necessary for the existence and uniqueness of the solution.

REMARK 3.33. *In the last case, the existence and uniqueness does not occur when some coefficients $k_{|e}$ vanish, i.e., when the input and output of the correspondent amplifiers are some passive current sources. For such networks, it can occur that the problem remains well posed when $k \neq 0$ for every amplifier. Two such cases are represented in Figure* 3.5. *Nevertheless, in both cases the output current $i_{|e_4^2}$ is independent of the corresponding coefficient amplifier k_2; that is, the amplifier does not do its work. Those are degenerate situations.*



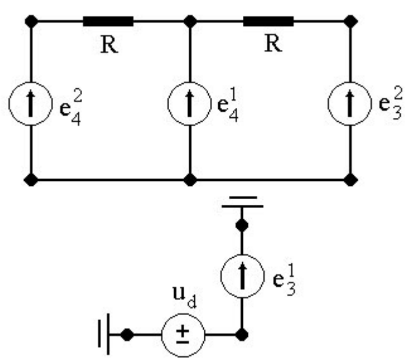


FIG. 3.5. *Two degenerate cases.*

3.6.2. Optimality of (H3). Here, we assume (H2) but not (H3), so it turns out that C_1 cannot be satisfied. Let us consider all the fundamental circuits in G, denoted by

$$A_1^i, \dots, A_{N_i}^i,$$

where i is the index of the circuit number i. We transform the A^i_N in the following way:

$$e_3^{N_i} \text{ is replaced by a current source } i_d = 0,$$
$$e_4^{N_i} \text{ is replaced by a current source } i_d = i_d^{N_i} \neq 0.$$

Hence, all the circuits are removed in G. Assumptions (H1)–(H4) are still satisfied by this modified network. Thus we can apply Theorem 3.17, and we deduce that there exists a unique solution φ for this network. Then, choosing the coefficients k^{N_i} so that the amplification relations (2.5) are satisfied, $\varphi \neq 0$ and fulfills the equality (3.14). Consequently, Condition C_1 does not hold, and this proves the optimality of assumption (H3).

4. Homogenization of periodic electrical networks. The major emphasis of the previous section was the derivation of the variational formulation for an electrical circuit. The cornerstone of the simplified modelling of periodic mechatronic systems is the possibility of dealing with periodically distributed electrical circuits. This section is devoted to this point and make use of the well-known homogenization techniques. It consists in passing to the limit in the model when the period size goes to zero. The limit model derived with this asymptotic method is referred to as the homogenized problem.

4.1. Statement of the periodic circuit equations. Let us first define the standard unit cell $Y =]-\frac{1}{2}, \frac{1}{2}[^n$ which, upon rescaling to size $\varepsilon = \frac{1}{N}$ ($N \in \mathbb{N}^*$), becomes the period in the periodic circuit. The network is now denoted by Θ^ε to stress that it is εY-periodic. We restrict our study to the case where Θ^ε has N cells in each of the n directions. So Θ^ε is included in $\overline{\Omega} = [0,1]^n$; see Figure 4.1. The N^n identical cells of the square Ω and of the circuit Θ^ε are denoted, respectively, by Y_i^ε and T_i^ε when their center is x_i^ε. A translation and an expansion by a factor N map Y_i^ε and T_i^ε to Y and $T \subset \overline{Y}$. The multi-integer $i = (i_1, \dots, i_n)$ which enumerates all cells in Ω takes its values in $\mathbb{N}^n$. We assume that the spatial distribution of the electrical devices is also periodic. The set of nodes σ^ε connected to the ground is an exception; it is divided into two subsets, σ_0^ε and $\sigma_\Gamma^\varepsilon$, that are periodically distributed, respectively, in Ω and, as detailed later, on each of its faces constituting its boundary Γ.

When the values of the resistors $g^\varepsilon \in \mathbb{P}^0(\Theta_2^\varepsilon)$, the admittances $g^\varepsilon \in \mathbb{P}^0(\Theta_2^\varepsilon)$, the amplifier coefficients $k^\varepsilon \in \mathbb{P}^0(\Theta_3^\varepsilon)$, the voltage sources $u_d^\varepsilon \in \mathbb{P}^0(\Theta_0^\varepsilon)$, the current sources $i_d^\varepsilon \in \mathbb{P}^0(\Theta_1^\varepsilon)$, and the branch lengths $L^\varepsilon \in \mathbb{P}^0(\Theta^\varepsilon)$ are chosen, one rewrites the variational formulation (2.8) of the circuit equation as follows: Find $(\varphi^\varepsilon, i^\varepsilon) \in \Psi_{ad}^\varepsilon(u_d^\varepsilon) \times \mathbb{P}^0(\Theta_4^\varepsilon)$ the solution of

$$\text{(4.1)} \quad \begin{cases} \displaystyle\int_{\Theta_2^\varepsilon} g^\varepsilon L^\varepsilon \nabla_\tau \varphi^\varepsilon \nabla_\tau \psi \, dl(x) + \int_{\Theta_4^\varepsilon} i^\varepsilon \nabla_\tau \psi \, dl(x) &= -\displaystyle\int_{\Theta_1^\varepsilon} i_d^\varepsilon \nabla \psi \, dl(x), \\ \displaystyle\int_{\Theta_3^\varepsilon} L^\varepsilon k^\varepsilon \nabla_\tau \varphi^\varepsilon \, j \, dl(x) - \int_{\Theta_4^\varepsilon} L^\varepsilon i^\varepsilon \, j \, dl(x) &= 0, \end{cases}$$

satisfied for all $(\psi, j) \in \Psi_{ad}^\varepsilon(0) \times \mathbb{P}^0(\Theta_4^\varepsilon)$, where the set of the admissible functions is

$$\Psi_{ad}^\varepsilon(u_d^\varepsilon) = \{\psi \in \mathbb{P}^1(\Theta^\varepsilon),\ \psi = 0 \text{ on } \sigma^\varepsilon, \text{ and } L^\varepsilon\ \nabla_\tau \psi = u_d^\varepsilon \text{ on } \Theta_0^\varepsilon\}.$$

4.2. Homogenized model. The final result of this article is the homogenized model whose solution φ^0 can be seen as an approximation of the solution φ^ε of the

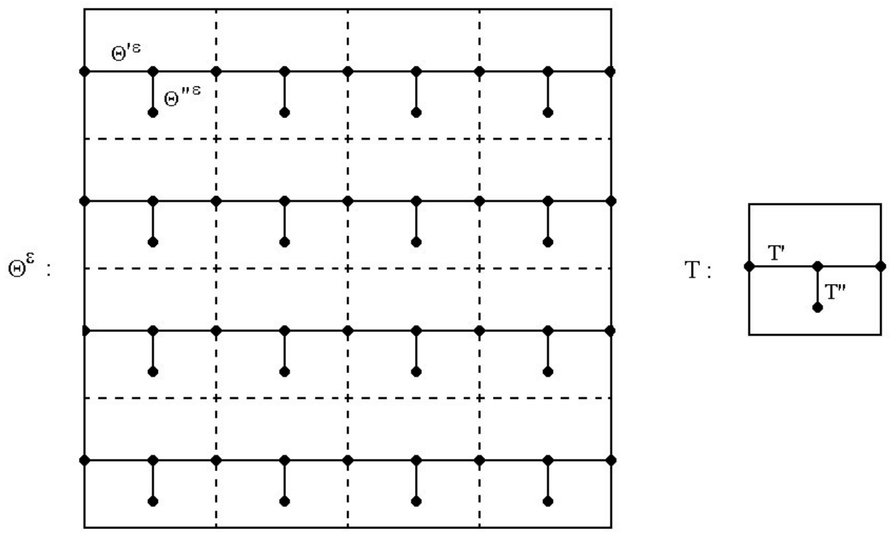


FIG. 4.1. *A periodic network and its reference cell.*

original problem when the number $N = \frac{1}{\varepsilon}$ of cells tends to infinity. The justification of this approximation relies upon the convergence of φ^ε in a sense that is fully detailed in section 4.4.1. Before stating the homogenized model, let us introduce a partition of T in crossing and no crossing branches.

NOTATION 4.1. (i) *The subcircuit $T' \subset T$ is constituted of paths which are traversing the cell Y whose tips are located periodically on its boundary ∂Y.*

(ii) *The complement of T' in T is T''.*

(iii) *The subcircuit T' of crossing paths has n_c connected components denoted by $T^{q\prime}$, where q takes its values in $\{1, \dots, n_c\}$. In other words, $T' = \bigcup_{q=1}^{n_c} T^{q\prime}$.*

(iv) *$J^q \subset \{1, \dots, n\}$ is the set of the directions of the paths which are traversing the cell Y and belonging to the connected component $T^{q\prime}$.*

(v) *For each $q \in \{1, \dots, n_c\}$ all vectors $x \in \mathbb{R}^n$ are divided into two blocks $\overline{x}^q = (x_i)_{i\in J^q}$ and $\widetilde{y}^q = (y_i)_{i\notin J^q}$.*

Inverse translations and expansion mappings of section 4.1 applied to $T^{q\prime}$, T', and T'' define the subnetworks $T_i^{q\varepsilon\prime}$, $T_i^{\varepsilon\prime}$, and $T_i^{\varepsilon\prime\prime}$ of T_i^ε and then the subnetworks $\Theta^{q\varepsilon\prime} = \bigcup_i T_i^{q\varepsilon\prime}$, $\Theta^{\varepsilon\prime} = \bigcup_i T_i^{\varepsilon\prime}$, and $\Theta^{\varepsilon\prime\prime} = \bigcup_i T_i^{\varepsilon\prime\prime}$ of Θ^ε. Let us remark that the definitions of $\Theta^{q\varepsilon\prime}$ and $\Theta^{\varepsilon\prime\prime}$ of section 4.4.1 which are based on the two-scale transform are equivalent to the last one. Figure 4.1 illustrates the definition of crossing subcircuits $\Theta^{\varepsilon\prime}$ and no crossing subcircuits $\Theta^{\varepsilon\prime\prime}$ of Θ^ε. In this case, the direction which goes from the left to the right is a direction of crossing, and the direction which goes from the bottom to the top is a direction of no crossing.

For the moment, we are unable to treat the most general case; hence we make the three assumptions $\sigma_0^\varepsilon \subset \Theta^{\varepsilon\prime\prime}$, $\sigma_\Gamma^\varepsilon \subset \Theta^{\varepsilon\prime}$, and $T'' \cap \partial Y = \varnothing$. The first two are natural; hence it does not seem necessary to weaken them. Let us emphasize that the third one could be weakened in some sense but not totally omitted. A typical case that does not fulfill the third assumption and therefore does not enter into our theory is represented in Figure 4.2. The periodic network Θ^ε is made of crossing paths in a direction that is not orthogonal to any coordinate axis. So there is no crossing path

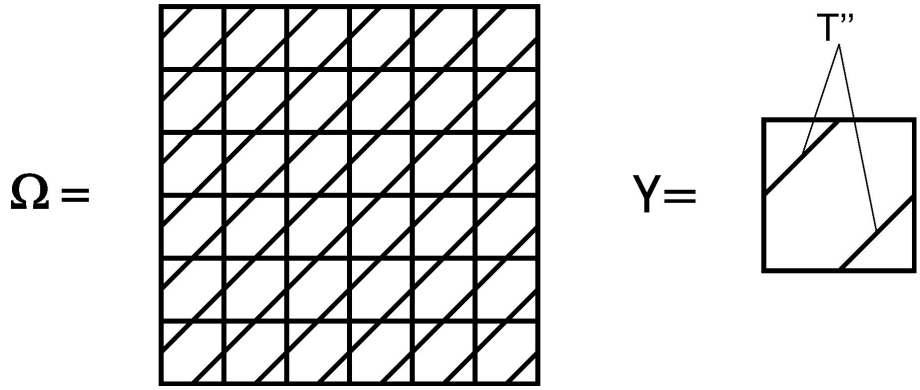


FIG. 4.2. *Typical example of T'' that is excluded.*

with periodic tips in T and T'' which is equal to T meets ∂Y. As mentioned, we are unable to treat, in a general manner, such an example, but a great part of our program goes through, and we expect the method, suitably modified, to work in this case also.

Since $\sigma_\Gamma^\varepsilon$ is included in $\Theta^{\varepsilon\prime}$, it is divided into n_c subsets $(\sigma_\Gamma^\varepsilon \cap \Theta^{q\varepsilon\prime})_{q\in\{1,\dots,n_c\}}$, which are assumed, for the sake of simplicity, to be periodic on each face of the domain Ω. For each index q enumerating the connected components, Γ may be separated into Γ_D^q, which meets $\sigma_\Gamma^\varepsilon \cap \Theta^{q\varepsilon\prime}$, and its complementary set Γ_N^q.

The restrictions $(\varphi^{q\varepsilon})_{q\in\{1,\dots,n_c\}}$ of the electric potential φ^ε to $\Theta^{q\varepsilon\prime}$ converge, in a sense that is explained in the material of section 4.4, toward the solutions $\varphi^0 = (\varphi^{0q}(\overline{z}^q))_{q\in\{1,\dots,n_c\}}$ of the homogenized model. This last is a system with n_c partial differential equations enumerated by p varying from 1 to the number of connected components n_c,

$$\begin{aligned}
&-\sum_{i\in J^p} \partial_{z_i}\left(\sum_{(q,j)\in\mathcal{J}} G^{11}_{pqij}\ \partial_{z_j}\varphi^{0q} + \sum_{q=1}^{n_c} G^{10}_{pqi}\varphi^{0q}\right) + \sum_{(q,j)\in\mathcal{J}} G^{01}_{pqj}\partial_{z_j}\varphi^{0q} \\
&+\sum_{q=1}^{n_c} G^{00}_{pq}\varphi^{0q} = \sum_{i\in J^p}\partial_{z_i} H^1_{pi} + H^0_p \text{ in } \Omega,
\end{aligned} \tag{4.2}$$

and with boundary conditions

$$\begin{aligned}
&\sum_{i\in J^p}\left(\sum_{(q,j)\in\mathcal{J}} G^{11}_{pqij}\ \partial_{z_j}\varphi^{0q} + \sum_{q=1}^{n_c} G^{10}_{pqi}\varphi^{0q}\right) n_{\Omega i} \\
&\quad = -\sum_{i\in J^p} H^1_{pi} n_{\Omega i} \text{ on } \Gamma^p_N \text{ and } \varphi^{0p} = 0 \text{ on } \Gamma^p_D.
\end{aligned} \tag{4.3}$$

The set $\mathcal{J}$ of multi-integers enumerates all the crossing directions for each reference axis

$$\mathcal{J} = \{(p,i) \in \{1,\dots,n_c\} \times \{1,\dots,n\} \text{ so that } i \in J^p\}$$

and the coefficients $(G^{11}_{pqij})_{((p,i),(q,j))\in\mathcal{J}^2}$, $(G^{10}_{pqi})_{(p,i)\in\mathcal{J},\, q\in\{1,\dots,n_c\}}$, $(G^{01}_{pqj})_{p\in\{1,\dots,n_c\}\times(q,j)\in\mathcal{J}}$, and $(G^{00}_{pq})_{p,q\in\{1,\dots,n_c\}}$, and the right-hand sides $(H^1_{pi}(z))_{(p,i)\in\mathcal{J}}$, $(H^0_p(z))_{p\in\{1,\dots,n_c\}}$ are defined in (4.6) and (4.7). The parts Γ_D^p and Γ_N^p of the boundary Γ are defined in section 4.4.1.

It is useful to say that $\varphi^0 \in H^1(\Omega)^{n_c}$ is the solution of the strong formulation (4.2)–(4.3) if and only if it is the solution of the following subsequent variational formulation: Find $\varphi^0 \in \Psi_{ad}^0$ such that

$$a^0(\varphi^0, \psi^0) = l^0(\psi^0) \quad \forall \psi^0 \in \Psi_{ad}^0, \tag{4.4}$$

where the set of admissible functions is

$$\Psi_{ad}^0 = \{\psi^0 = (\psi^{0p})_{p\in\{1,\dots,n_c\}} \in H^1(\Omega)^{n_c} \text{ so that } \nabla_{\tilde{z}^p}\psi^{0p} = 0 \text{ and } \psi^{0p} = 0 \text{ on } \Gamma_D^p\ \forall p\}.$$

The bilinear form and the linear form are

$$\begin{aligned} a^0(\varphi^0, \psi^0) = \int_\Omega \sum_{(p,i)\in\mathcal{J}} & \left(\sum_{(q,j)\in\mathcal{J}} G_{pqij}^{11}\ \partial_{z_j}\varphi^{0q} + \sum_{q=1}^{n_c} G_{pqi}^{10}\varphi^{0q} \right) \partial_{z_i}\psi^{0p} \\ & + \sum_{p=1}^{n_c} \left(\sum_{(q,j)\in\mathcal{J}} G_{pqj}^{01}\partial_{z_j}\varphi^{0q} + G_{pq}^{00}\varphi^{0q} \right) \psi^{0p}\ dz \end{aligned}$$

and

$$l^0(\psi^0) = -\int_\Omega \sum_{(p,i)\in\mathcal{J}} H_{pi}^1(z)\ \partial_{z_i}\psi^{0p}(z) + \sum_{p=1}^{n_c} H_p^0(z)\ \psi^{0p}(z)\ dz.$$

As usual, existence and uniqueness of φ^0 can be established indifferently with any of the two above-mentioned formulations.

The scalar case, $n_c = 1$, is well documented in Chapter 8 of Gilbard and Truginger [11] for Dirichlet boundary conditions. Other boundary conditions are studied in articles referred to in this treatise.

4.3. Homogenized coefficients and right-hand sides. The aim of this part concerns the determination of the coefficients G and the right-hand sides H of the equations of the homogenized model. The usual method consists in the use of the problem micro or *cell problem*, stated in section 4.4.4, in order to express the fields micro as functions of the fields macro and then to plug these expressions into the two-scale model stated in section 4.4.3. This yields a set of equations having only macroscopic fields as unknowns, which is precisely the homogenized model. We will follow this method, except that we make use of the so-called *modified cell problem* in place of the cell problem. There are at least two drawbacks in using the original cell problem. The first one is that it includes some nonstandard devices that are not implemented in classical electronic circuit simulators. Thus the modified cell problem has been designed so that on one hand it produces an equivalent solution to that of the original problem and on the other hand it is directly implementable with usual simulators. The second point is that a new theorem for the existence and uniqueness of the solution is required when nonstandard devices are in the circuit cell. This is evidently avoided when a reformulation with standard devices is possible.

The definition of the modified cell problem is presented in the first subsequent subsection when the expression for G and H that make use of certain of its solutions are fully stated in the next one.

4.3.1. The unit circuit cell $\mathcal{T}$. Accordingly to the partition of Θ^ε, the unit circuit cell $T \subset Y$ is partitioned in voltage sources T_0 with voltages u_d, current sources T_1 with currents i_d, resistors T_2 with admittances g, and inputs T_3 and outputs T_4 of amplifiers with coefficients k. The nodes that are connected to the earth are denoted by S_0. The so-called *modified circuit cell* $\mathcal{T} = \bigcup_{i=0}^4 \mathcal{T}_i$, which will be built in what follows, is generated, starting from the circuit $T = \bigcup_{i=0}^4 T_i$, by two transformations. The voltages, the currents, the electric potential, and the voltage sources in $\mathcal{T}$ are denoted, respectively, by u, i, φ, and $\mathcal{U}_d$. The current sources are the same as in the original circuit, so their notation i_d remains unchanged.

In this section, one makes use of the vectors $\eta = ((\eta_i^q)_{i \in J^q})_{q \in \{1,\ldots,n_c\}}$ and $\phi = (\phi^q)_{q \in \{1,\ldots,n_c\}}$ that will be replaced by the macroscopic fields of electric potential $\varphi^0 = (\varphi^{0q})_{q \in \{1,\ldots,n_c\}}$ and by their derivatives $((\partial_{z_i}\varphi^{0q})_{i \in J^q})_{q \in \{1,\ldots,n_c\}}$ for the derivation of the homogenized model. At this stage ϕ and η are some given vectors of real numbers.

Now, we depict the two nonstandard features of the original cell problem which is stated in section 4.4.4 and the required transformations. Using a standard simulator, the way to impose the periodicity conditions to the electrical potential φ^1 is to link the concerned nodes by a zero voltage source. In addition, let us mention that the restriction of the electric potential φ^0 to each subcircuit $T^{q\prime}$ is constant and is viewed as a macroscopic field when its restriction to the subcircuits T'' is a microscopic field. Thus, at every node, where the subcircuits T'' meet the subcircuits $T^{q\prime}$, the potential is imposed by the common value $\varphi^0_{|T^{q\prime}}$. The standard realization of these constraints consists in making a link between all these nodes and a common voltage source. The two transformations are summarized as follows.

1. One adds zero voltage sources $\mathcal{T}_0^{per}$ between the nodes which belong to the same crossing path and which are located on opposite edges of the unit circuit cell in a periodic way. In brief, $\mathcal{U}_d = 0$ on $\mathcal{T}_0^{per}$.

2. One cuts the connections between T' and T'', and one duplicates the nodes being located at the cutting points. Then one connects together all the so-created tips of the side T'' to a common voltage source, denoted by $\mathcal{T}_0^q$, which imposes a voltage equal to $\mathcal{U}_d = \phi^q$.

In addition to the above-mentioned modifications, we have to take into account the contributions of the macroscopic electrical potential which plays the role of given data in the circuit cell problem. This is achieved by adding *macroscopic voltage sources* u_d^{0q} which are equal to $u_d^{0q} = L\sum_{i \in J^q} \eta_i^q \tau_i$ parallel to the original voltage sources, the resistors, and the amplifiers' inputs.

3. One replaces the imposed voltages u_d on the voltage sources $T_0 \cap T^{q\prime}$ by the adequately modified imposed voltages $\mathcal{U}_d = u_d - (u_d^{0q})_{|T_0}$. To each resistor $e_2 \subset T_2 \cap T^{q\prime}$, one connects in series a voltage source e_0 that imposes a voltage of $\mathcal{U}_d = -(u_d^{0q})_{|e_2}$. The set of these voltage sources is denoted by $\mathcal{T}_{02}$. Similarly, the amplification relation of each amplifier $(e_3, e_4) \subset (T_3 \cap T^{q\prime}) \times T_4$ is replaced by $i_{|e_4} = k(u_{|e_3} + (u_d^{0q})_{|e_3})$.

The other circuit equations remain unchanged:

$$\begin{aligned} &u = \mathcal{U}_d = u_d \text{ on } T_0'',\ i = i_d \text{ on } T_1,\ i = gu \text{ on } T_2'',\\ &i = 0 \text{ on } T_3 \text{ and } i_{|e_4} = ku_{|e_3} \text{ when } e_3 \in T_3'',\\ &\varphi = 0 \text{ on } S_0, \end{aligned}$$

as well as the usual relation between the voltages and the electric potential, $u = \varphi(s^+) - \varphi(s^-)$.

Furthermore, we assume that the modified unit circuit cell fulfills assumptions (H1)–(H3), so that the cell circuit problem is well posed in the usual sense for the problems micro, where only the uniqueness of the $\nabla_\tau \varphi$ is required but not that of the φ.

THEOREM 4.2. *If conditions* (H1)–(H3) *relative to the modified network are satisfied, then there exists a unique solution, up to a constant on each connected component, to the modified circuit cell problem.*

Proof. The variational formulation equivalent to the set of the circuit equations is now clear. Find $(\varphi, i) \in \Psi^1_{ad,\mathcal{T}}(\eta, \phi, u_d) \times \mathbb{P}^0(\mathcal{T}_4)$ so that

$$\begin{aligned} a^1(\varphi, \psi) + b^1_1(i, \psi) &= l^1_1(\psi), \\ b^1_2(\varphi, j) - c^1(i, j) &= l^1_2(j) \end{aligned} \tag{4.5}$$

for any $(\psi, j) \in \Psi^1_{ad,\mathcal{T}}(0,0,0) \times \mathbb{P}^0(\mathcal{T}_4)$, where

$$\begin{aligned} a^1(\varphi,\psi) &= \int_{\mathcal{T}_2} Lg\ \nabla_\tau\varphi\nabla_\tau\psi\ dl(y),\ c^1(i,j) = \int_{\mathcal{T}_4} i\ j\ dl(y), \\ b^1_1(i,\psi) &= \int_{\mathcal{T}_4} i\ \nabla_\tau\psi\ dl(y),\ b^1_2(\varphi,j) = \int_{\mathcal{T}_3} k\ L\nabla_\tau\varphi\ j_{|\mathcal{T}_4}\ dl(y), \\ l^1_1(\psi) &= -\int_{\mathcal{T}_1} i^1_d\ \nabla_\tau\psi\ dl(y) - \sum_{q=1}^{n_c}\int_{\mathcal{T}_2\cap\mathcal{T}^{q\prime}} g\ u^{0q}_d\ \nabla_\tau\psi\ dl(y), \\ l^1_2(j) &= -\sum_{q=1}^{n_c}\ \sum_{e_3\in\mathcal{T}_3\cap\mathcal{T}^{q\prime}} \int_{e_4} k(u^{0q}_d)_{|e_3}\ j\ dl(y),\ e_4 \text{ being the output of } e_3. \end{aligned}$$

The space of admissible electric potentials and currents is

$$\Psi^1_{ad,\mathcal{T}}(\eta,\phi,u_d) = \{\psi \in \mathbb{P}^1(\mathcal{T}) \text{ so that } L\nabla_\tau\psi = \mathcal{U}_d \text{ on } \mathcal{T}_0 \text{ and } \psi^0 = 0 \text{ on } S_0\}.$$

Except for the presence of the right-hand side in the second equation, this system has exactly the same form as the system (2.14), which admits a unique solution, up to a constant on each connected component, under assumptions (H1)–(H3). Thus, it remains to prove the continuity of the linear form l^1_2, which is evident. ◻

4.3.2. Formulae for coefficients calculation. In this section, the imposed currents u_d and voltages i_d, obtained via the two-scale convergence in section 4.4.3, are considered indifferently as functions $u_d \in L^2(\Omega; \mathbb{P}^0(T_0))$ and $i_d \in L^2(\Omega; \mathbb{P}^0(T_1))$ or as vectors $(u_{di}(z))_{i\in\{1,\dots,|T_0|\}}$ and $(i_{di}(z))_{i\in\{1,\dots,|T_1|\}}$.

The expression of the homogenized coefficients and of the homogenized right-hand sides is summarized hereafter:

$$\begin{aligned} G^{11}_{pqij} &= \int_{\mathcal{T}_2} Lg\widehat{\mathcal{L}}^3_{pi}\widehat{\mathcal{L}}^3_{qj}\ dl(y) + \int_{\mathcal{T}_4} k(L\widehat{\mathcal{L}}^3_{pi})_{|\mathcal{T}_3}\widehat{\mathcal{L}}^3_{qj}\ dl(y), \\ G^{10}_{pqi} &= \int_{\mathcal{T}_2} Lg\widehat{\mathcal{L}}^3_{pi}\mathcal{L}^4_q\ dl(y) + \int_{\mathcal{T}_4} k(L\widehat{\mathcal{L}}^3_{pi})_{|\mathcal{T}_3}\mathcal{L}^4_q\ dl(y), \\ G^{01}_{pqj} &= \int_{\mathcal{T}_2} Lg\mathcal{L}^4_p\widehat{\mathcal{L}}^3_{qj}\ dl(y) + \int_{\mathcal{T}_4} k(L\mathcal{L}^4_p)_{|\mathcal{T}_3}\widehat{\mathcal{L}}^3_{qj}\ dl(y), \\ G^{00}_{pq} &= \int_{\mathcal{T}_2} Lg\mathcal{L}^4_p\mathcal{L}^4_q\ dl(y) + \int_{\mathcal{T}_4} k(L\mathcal{L}^4_p)_{|\mathcal{T}_3}\mathcal{L}^4_q\ dl(y), \end{aligned} \tag{4.6}$$

(4.7)

$$H^1_{pi}(\overline{z}) = \int_{[0,1]^{n_c-|J^p|}} \int_{\mathcal{T}_1} \widehat{\mathcal{L}}^3_{pi} i_d \; dl(y) + \sum_{j=1}^{|T_0|} \int_{\mathcal{T}_4} k(L\mathcal{L}^1_j)_{|\mathcal{T}_3} \widehat{\mathcal{L}}^3_{pi} dl(y) u_{dj}$$
$$+ \sum_{j=1}^{|T_1|} \int_{\mathcal{T}_4} k(L\mathcal{L}^2_j)_{|\mathcal{T}_3} \widehat{\mathcal{L}}^3_{pi} dl(y) i_{dj} + \sum_{j=1}^{|T_0|} \int_{\mathcal{T}_2} Lg\mathcal{L}^1_j \widehat{\mathcal{L}}^3_{pi} dl(y) u_{dj}$$
$$+ \sum_{j=1}^{|T_1|} \int_{\mathcal{T}_2} Lg\mathcal{L}^2_j \widehat{\mathcal{L}}^3_{pi} dl(y) i_{dj} \; d\widetilde{z},$$

$$H^0_p(\overline{z}) = \int_{[0,1]^{n_c-|J^p|}} \int_{\mathcal{T}_1} \mathcal{L}^4_p \; i_d \; dl(y) + \sum_{j=1}^{|T_0|} u_{dj} \int_{\mathcal{T}_4} k(L\mathcal{L}^1_j)_{|\mathcal{T}_3} \mathcal{L}^4_p \; dl(y)$$
$$+ \sum_{j=1}^{|T_1|} \int_{\mathcal{T}_4} k(L\mathcal{L}^2_j)_{|\mathcal{T}_3} \mathcal{L}^4_p \; dl(y) \; i_{dj} + \sum_{j=1}^{|T_0|} \int_{\mathcal{T}_2} Lg\mathcal{L}^1_j \mathcal{L}^4_p \; dl(y) \; u_{dj}$$
$$+ \sum_{j=1}^{|T_1|} \int_{\mathcal{T}_2} Lg\mathcal{L}^2_j \mathcal{L}^4_p \; dl(y) \; i_{dj} \; d\widetilde{z},$$

where $\widehat{\mathcal{L}}^3_{pi} = \chi_{\mathcal{T}^{p\prime}} \tau_i + \mathcal{L}^3_{pi}$. The $\mathcal{L}^k$ are determined from the general formula

$$\frac{u(y)}{L(y)} = \sum_{i=1}^{|T_0|} \mathcal{L}^1_i(y) u_{di} + \sum_{i=1}^{|T_1|} \mathcal{L}^2_i(y) i_{di} + \sum_{q=1}^{n_c} \sum_{i \in J^q} \mathcal{L}^3_{qi}(y) \eta^q_i + \sum_{q=1}^{n_c} \mathcal{L}^4_q \phi^q. \tag{4.8}$$

Here u is a voltage which is solution of the unit circuit cell equations and $\chi_{\mathcal{T}^{p\prime}}$ is the characteristic function of the set $\mathcal{T}^{p\prime}$ which is equal to one on $\mathcal{T}^{p\prime}$ and to zero elsewhere. The calculation of each $\mathcal{L}^k$ is conducted by making particular choices of the imposed data u_d, i_d, ϕ, and η. For example $\mathcal{L}^1_1$ is computed by imposing $u_{d1} = 1$ and the other components of u_d, i_d, ϕ, and η vanishing. Further details of the calculation are safely left to the reader.

4.4. Justification of the homogenized model based on the two-scale convergence. In this section, we apply Theorem 2 of [16], with a little variation, to derive the two-scale model, from which the homogenized model is extracted.

4.4.1. Two-scale convergence. Let us recall the mathematical framework of two-scale transform and convergence, and let us state the convergence theorem that allows us to pass to the limit in the circuit equations.

We begin with the definition of the geometric transform of the domain Ω, the so-called two-scale transform, into the product of the macroscopic domain Ω and the unit circuit cell T.

DEFINITION 4.3. *The two-scale transform of* Θ^ε *is the map defined from* Θ^ε *to* $\Omega \times T$ *that is defined cell by cell as follows:*

$$T^\varepsilon_i \to \overline{Y}^\varepsilon_i \times T$$
$$x \mapsto (z, y) = \overline{Y}^\varepsilon_i \times \left\{ \frac{x - x^\varepsilon_i}{\varepsilon} \right\}.$$

The two-scale transform is onto. Thus, the inverse two-scale transform may be applied to any subset $\Omega \times X$ for any $X \subset T$. This is how the subsets $\Theta^{\varepsilon\prime}$, $\Theta^{q\varepsilon\prime}$, $\Theta^{\varepsilon\prime\prime}$,

$(\Theta_k^\varepsilon)_{k\in\{0,\dots,4\}}$, $(\Theta_{kl}^{\varepsilon\prime})_{k\in\{0,\dots,4\},\,l\in\{1,2\}}$, and σ_0^ε of Θ^ε may be built as the inverse two-scale transform of $\Omega\times X$, where X is successively equal to T', $T^{q\prime}$, T'', $(T_k)_{k\in\{0,\dots,4\}}$, $(T_{kl})_{k\in\{0,\dots,4\},\,l\in\{1,2\}}$ (defined below), and S_0.

DEFINITION 4.4. (i) *The two-scale transform $\widehat{v}^\varepsilon$ of a function $v^\varepsilon\in L^p(\Theta^\varepsilon)$ is defined almost everywhere on $\Omega\times T$ by $\widehat{v}^\varepsilon(z,y)=v^\varepsilon(x)$, where (z,y) is any couple belonging to the two-scale transform of $x\in\Theta^\varepsilon$.*

(ii) *One says that a sequence of functions $(v^\varepsilon)_\varepsilon\in L^p(\Theta^\varepsilon)$ two-scale converges strongly (resp., weakly) toward some function $v\in L^p(\Omega\times T)$ if its two-scale transform $(\widehat{v}^\varepsilon)_\varepsilon$ converges strongly (resp., weakly) toward v in $L^p(\Omega\times T)$.*

By its very definition, the limit v has two arguments, the macroscopic variable z and the microscopic variable y.

The norm for $(\varphi,i)\in H^1(\Theta^\varepsilon)\times L^2(\Theta_4^\varepsilon)$ that plays a key role for the convergence of $(\varphi^\varepsilon,i^\varepsilon)$ is

$$\begin{aligned}\|(\varphi,i)\|_{r,\varepsilon}^2=&\int_{\Theta^{\varepsilon\prime}}|\nabla_\tau\varphi|^2+|\varphi|^2\,dl(x)+\int_{\Theta^{\varepsilon\prime\prime}}|\varepsilon\nabla_\tau\varphi|^2+|\varphi|^2\,dl(x)\\&+\int_{\Theta_{41}^{\varepsilon\prime}\cup\Theta_{41}^{\varepsilon\prime\prime}}|\varepsilon^{-r}i|^2\,dl(x)+\int_{\Theta_{42}^{\varepsilon\prime}\cup\Theta_{42}^{\varepsilon\prime\prime}}|\varepsilon^{-r-1}i|^2\,dl(x),\end{aligned}$$

r being any real number that is chosen regarding the estimates on the data. The subnetworks $\Theta_{kl}^{\varepsilon\prime}$ are defined as the corresponding parts in Θ^ε of the subnetworks T'_{kl} and T''_{kl} of T defined hereafter. Each of the sets T'_3, T''_3, T'_4, and T''_4 is divided into two complementary subsets: T'_{31},T'_{32}, T''_{31},T''_{32}, T'_{41},T'_{42}, T''_{41},T''_{42} so that the input and output of each amplifier belong to one of the subsequent couples: (T'_{31},T'_{41}), (T'_{32},T''_{41}), (T''_{31},T'_{42}), (T''_{32},T''_{42}).

The key theorem that is used for passing to the limit is stated hereafter. It is a reformulation slightly different from that of Theorem 2 in [16].

THEOREM 4.5. (i) *For any sequence $(\varphi^\varepsilon,i^\varepsilon)\in\{\psi\in H^1(\Theta^\varepsilon),\ \psi=0$ on $\sigma^\varepsilon\}\times L^2(\Theta_4^\varepsilon)$ that is uniformly bounded $\|(\varphi^\varepsilon,i^\varepsilon)\|_{r,\varepsilon}^2\le C$, one can extract a subsequence, still denoted by $(\varphi^\varepsilon,i^\varepsilon)$, which satisfies the following weak two-scale convergences in $L^2(\Omega\times T)$:*

$$\begin{aligned}\varphi^\varepsilon_{|\Theta^{\varepsilon\prime}}\rightharpoonup\varphi^0,\ \nabla_\tau\varphi^\varepsilon_{|\Theta^{\varepsilon\prime}}\rightharpoonup D(\varphi^0,\varphi^1)&\ \text{on } \Omega\times T',\\ \varepsilon\nabla_\tau\varphi^\varepsilon_{|\Theta^{\varepsilon\prime\prime}}\rightharpoonup D(\varphi^0,\varphi^1)&\ \text{on } \Omega\times T'',\\ \varepsilon^{-r}i^\varepsilon\rightharpoonup i \text{ on } \Omega\times T'_4 \text{ and } \varepsilon^{-r-1}i^\varepsilon&\rightharpoonup i \text{ on } \Omega\times T''_4,\end{aligned}$$

where the two-scale fields $\varphi^0\in L^2(\Omega;H^1(T))$, $\varphi^1\in L^2(\Omega;H^1_\sharp(T'))$, and $i\in L^2(\Omega\times T_4)$. The restriction of φ^0 to the qth connected component $\Omega\times T^{q\prime}$ is denoted by φ^{0q}. This is the macroscopic electric potential; in other words, it is constant with respect to the microscopic variable y. It has further regularity $\partial_{z_i}\varphi^{0q}(z)\in L^2(\Omega\times T'^q)$ in each crossing direction $i\in J^q$, and it vanishes on part Γ_D^q of the boundary.

(ii) *Moreover, if $(\varphi^\varepsilon,i^\varepsilon)\in\mathbb{P}^1(\Theta^\varepsilon)\times\mathbb{P}^0(\Theta_4^\varepsilon)$, then $\varphi^0\in L^2(\Omega;\mathbb{P}^1(T''))$, $\varphi^1\in L^2(\Omega;\mathbb{P}^1_\sharp(T'))$, and $i\in L^2(\Omega;\mathbb{P}^0(T_4))$.*

The index $\sharp$ refers to Y-periodic functions. Any function ψ^0 defined on $\Omega\times T'$, which is constant with respect to y in each $\Omega\times T^{q\prime}$, may be seen as a vector $\psi^0=(\psi^{0q})_{q\in\{1,\dots,n_c\}}$; so $D(\psi^0,\psi^1)=\chi_{T'}\nabla_{\overline{z}}\psi^0.\overline{\tau}+d(\psi^0,\psi^1)$ on T, where $\nabla_{\overline{z}}\psi^0=((\partial_{z_i}\psi^{0q})_{i\in J^q})_{q\in\{1,\dots,n_c\}}$ and $\chi_{T'}$ is the characteristic function of T'. In other words,

$$D(\psi^0,\psi^1) = \sum_{i\in J^q} \partial_{z_i}\psi^{0q}(z)\tau_i(y) + d(\psi^0,\psi^1) \text{ on } \Omega\times T^{q\prime}$$
$$= d(\psi^0,\psi^1) \text{ on } \Omega\times T'',$$

where

$$d(\psi^0,\psi^1) = \nabla_\tau\psi^1 \text{ on } T'$$
$$= \nabla_\tau\psi^0 \text{ on } T''.$$

The definitions of the variables $\overline{\tau}$, $\overline{z}$, $\widetilde{z}$ as well as the variable $\overline{n}_\Omega$ used hereafter are based on Notation 4.1(v).

Proof. The proof of Theorem 4.5 follows that of Theorem 2 in [16] with some changes that are detailed hereafter. The reader is referred to [16] for further details.

The discussion is carried out for one connected component $T^{q\prime}$ of T'. As explained in [16], there exists an extracted subsequence of $(\varphi^\varepsilon_{|\Theta^{q\varepsilon\prime}})_\varepsilon$ still denoted by $(\varphi^\varepsilon_{|\Theta^{q\varepsilon\prime}})_\varepsilon$, which two-scale converges weakly in L^2 toward a limit $\varphi^{0q}(z,y)$ so that $\nabla_{\boldsymbol{\tau}}\varphi^{0q} = 0$ in $\Omega\times T^{q\prime}$. This implies that φ^{0q} is constant on $\Omega\times T^{q\prime}$.

Let us compute the limit f of $f^\varepsilon = \nabla_\tau\varphi^\varepsilon_{|\Theta^{q\varepsilon\prime}}$. Without loss of generality, we assume that z_1 is a crossing direction in the qth connected component $T^{q\prime}$ of T'. Let us denote by $t'\subset T^{q\prime}$ one of the paths that cross Y in this direction and (s^-,s^+) its left and right tips located on the cell boundary ∂Y. By virtue of formula (18) in [16]

$$\int_{\Omega\times t} \widehat{f}^\varepsilon(z,y)\ \psi(z,y)\ dl(y)dz = -\int_{\Omega\times t'} \frac{1}{\varepsilon}\widehat{\varphi}^\varepsilon(z,y)\ \nabla_{\boldsymbol{\tau}}\psi(z,y)\ dl(y)dz$$
$$+\int_\Omega \frac{1}{\varepsilon}\left[\widehat{\varphi}^\varepsilon(z,y)\ \psi(z,y)\right]_{s^-}^{s^+}\ dz.$$

Lemma 6.2 in [16], for any $\psi\in H^1(\Omega\times t')$ verifying $\nabla_{\boldsymbol{\tau}}\psi = 0$ on t', says that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon}\left[\widehat{\varphi}^\varepsilon(z,s)\ \psi(z,s)\right]_{s^-}^{s^+}\ dz = -\int_{\Omega\times t'} \varphi^{0q}(z,y)\ \tau_1\partial_{z_1}\psi(z,y)\ dl(y)dz$$
$$+\int_{(\Gamma^+\cup\Gamma^-)\times t'} \varphi^{0q}(z,y)\psi(z,y)\ \tau_1 n_{\Omega 1}\ dl(y)ds(z).$$

Choosing any ψ such that $\nabla_{\boldsymbol{\tau}}\psi = 0$ on $\Omega\times t'$ yields

$$\int_{\Omega\times t'} f(z,y)\psi(z,y)\ dl(y)dz = \int_{\Omega\times t'} \partial_{z_1}\varphi^0(z)\tau_1\ \psi(z,y)\ dl(y)dz,$$

from which one extracts the expression of $\partial_{z_1}\varphi^{0q}$:

$$\partial_{z_1}\varphi^{0q}(z) = \frac{1}{\int_{t'}\tau_1 dl(y)}\int_{t'} f(z,y)dl(y) \in L^2(\Omega).$$

Similarly, one may prove the regularity result $\partial_{z_i}\varphi^{0q}(z)\in L^2(\Omega)$ for any crossing direction $i\in J^q$.

Since $\int_{t'}\tau_i\ dl(y) = 0$ for any no crossing direction $i\neq 1$, the integrand in the above limit can be replaced by an expression that is not related to the chosen crossing direction:

$$\int_{\Omega\times t'} f(z,y)\psi(z,y)\ dl(y)dz = \int_{\Omega\times t'} \nabla_{\overline{z}}\varphi^{0q}(z).\overline{\tau}(y)\ \psi(z,y)\ dl(y)dz.$$

This equality holds for any ψ belonging to the kernel of $\nabla_{\boldsymbol{\tau}}$. Thus, by a classical orthogonality argument, one may establish that there exists a function φ^1 such that the formula

$$\int_{\Omega\times t'} f(z,y)\psi(z,y)\ dl(y)dz = \int_{\Omega\times t'} \nabla_{\overline{z}}\varphi^{0q}(z).\overline{\tau}(y)\ \psi(z,y) - \varphi^1(z,y)\nabla_{\boldsymbol{\tau}}\psi(z,y)\ dl(y)dz$$

is valid for every $\psi \in H^1(\Omega; H^1_\sharp(t'))$. The interpretation of this variational equality is

$$\nabla_{\boldsymbol{\tau}}\varphi^1(z,y) + \nabla_{\overline{z}}\varphi^{0q}(z).\overline{\tau}(y) = f(z,y) \text{ in } \Omega\times t',$$
$$\left[\varphi^1(z,y)\right]_{y=s^-}^{y=s^+} = 0 \text{ in } \Omega,$$

where the second equality is a consequence of the periodicity of the test functions ψ. Combination of the regularity result $\varphi^{0q} \in H^1(\Omega)$ and the first equality in the above characterization of the limit f yields the regularity $\varphi^1 \in L^2(\Omega; H^1_\sharp(t'))$.

The construction of φ^1 has been carried out for a given path $t' \subset T^{q\prime}$. It remains to prove that $\nabla_{\boldsymbol{\tau}}\varphi^1$ is uniquely defined on the whole $T^{q\prime}$.

This results immediately from the fact that both limits f and φ^{0q} are defined on the whole connected component $T^{q\prime}$ and that the choice of any crossing path t' would have led to the same limit.

Point (ii) of the theorem is proven as follows. For any subcircuit $X \subset T$, Ξ^ε denotes the subset of the whole periodic circuit Θ^ε, which has $\Omega\times X$ as the inverse two-scale transform. The characteristic functions of X in T and of Ξ^ε in Θ^ε are denoted, respectively, by χ_{Ξ^ε} and χ_X. Consider a sequence $\varphi^\varepsilon(x)$ that is constant on each cell $Y_i^\varepsilon \cap \Xi^\varepsilon$. It may be expressed as $\varphi^\varepsilon(x) = a^\varepsilon(x)\chi_{\Xi^\varepsilon}(x)$; thus its two-scale transform is $\widehat{\varphi}^\varepsilon(z,y) = \widehat{a}^\varepsilon(z)\chi_X(y)$, and $\widehat{a}^\varepsilon$ is independent of y. Assuming that φ^ε is two-scale weakly convergent in L^2, its limit $\varphi(z,y)$ is necessarily a product on the same form $a(z)\chi_X(y)$, which tell us that it remains constant on X with respect to the y variable. Such reasoning applied to any branch $X = e$ of T_4 combined with a linear combination leads to the desired result $i \in L^2(\Omega; \mathbb{P}^0(T_4))$. Obviously, the derivation of $\varphi^0 \in L^2(\Omega; \mathbb{P}^1(T''))$ and $\varphi^1 \in L^2(\Omega; \mathbb{P}^1_\sharp(T'))$ can be done on a similar way. For the sake of brevity, the details are left to the reader. $\square$

4.4.2. Assumptions. For any ε, if assumptions (H1)–(H4) hold, then there exists a unique solution of the circuit equations (4.1). In order to pass to the limit in these equations, the two-scale convergencies of the data are required. Moreover, uniform estimates are required on the solutions for application of Theorem 4.5. This is the spirit of assumptions (H5) and (H6) that we summarize now.

(H5) Let us postpone for a few lines the definition of the precise scaling of each data. Taking into account this scaling, the data i_d^ε, u_d^ε, L^ε, k^ε, and g^ε are two-scale convergent toward some limits $i_d \in L^2(\Omega, \mathbb{P}^0(T_1))$, $u_d \in L^2(\Omega, \mathbb{P}^0(T_0))$, $L \in \mathbb{P}^0(T)$, $g \in \mathbb{P}^0(T_2)$, and $k \in \mathbb{P}^0(T_3)$. Let r be a given real number, which may be chosen arbitrarily; and all the fields that play a role in the problem are scaled in a manner that depends on r.

(H5-1) The current sources and the voltage sources:

$$\varepsilon^{-r} i_d^\varepsilon \to i_d \text{ in } L^2(\Omega\times T_1'),\ \varepsilon^{-r-1} i_d^\varepsilon \to i_d \text{ in } L^2(\Omega\times T_1''),$$
$$\varepsilon^{-1} u_d^\varepsilon \to u_d \text{ in } L^2(\Omega\times T_0'),\ u_d^\varepsilon \to u_d \text{ in } L^2(\Omega\times T_0'').$$

(H5-2) The branch lengths:

$$\varepsilon^{-1} L^\varepsilon \rightharpoonup L \text{ in } L^\infty(\Omega; \mathbb{P}^0(T)) \text{ weak}^*.$$

(H5-3) The admittances:

$$\varepsilon^{1-r}g^\varepsilon \rightharpoonup g \text{ in } L^\infty(\Omega;\mathbb{P}^0(T_2')) \text{ weak}^* \text{ and } \varepsilon^{-1-r}g^\varepsilon \rightharpoonup g \text{ in } L^\infty(\Omega;\mathbb{P}^0(T_2'')) \text{ weak}^*.$$

(H5-4) The amplification coefficients:

$$\varepsilon^{-r+1}k^\varepsilon \rightharpoonup k \text{ in } L^\infty(\Omega;\mathbb{P}^0(T_3')) \text{ weak}^* \text{ and } \varepsilon^{-r-1}k^\varepsilon \rightharpoonup k \text{ in } L^\infty(\Omega;\mathbb{P}^0(T_3'')) \text{ weak}^*.$$

In accordance with the usual way to proceed when one uses an asymptotic method, one would like to complete the graph assumptions (H1)–(H4), which ensure the well posedness of the problem for a fixed ε, with further assumptions which imply uniform estimates of the solution $(\varphi^\varepsilon, i^\varepsilon)$ with respect to ε. Unfortunately, we have not yet discovered these assumptions, and we are obliged to assume, in a direct way, that the solution is uniformly bounded.

(H6) The couple $(\varphi^\varepsilon, i^\varepsilon)$ is uniformly bounded $\|(\varphi^\varepsilon, i^\varepsilon)\|^2_{r,\varepsilon} \le C$.

4.4.3. Convergence toward the two-scale model. Under the above assumptions and Theorem 4.5, one can pass to the limit in the circuit equations and find that the limit $(\varphi^0, \varphi^1, i)$ is the solution of the following weak formulation, the so-called two-scale circuit model: Find $(\varphi^0, \varphi^1, i) \in \Psi^{ts}_{ad}(u_d) \times L^2(\Omega;\mathbb{P}^0(T_4))$ such that

$$\begin{aligned} a^{ts}((\varphi^0,\varphi^1),(\psi^0,\psi^1)) + b_1^{ts}(i,(\psi^0,\psi^1)) &= l_1^{ts}((\psi^0,\psi^1)),\\ b_2^{ts}((\varphi^0,\varphi^1),j) - c^{ts}(i,j) &= l_2^{ts}(j) \end{aligned} \tag{4.9}$$

for any $(\psi^0, \psi^1, j) \in \Psi^{ts}_{ad}(0) \times L^2(\Omega;\mathbb{P}^0(T_4))$, where

$$a^{ts}((\varphi^0,\varphi^1),(\psi^0,\psi^1)) = \int_{\Omega\times T_2} Lg\ D(\varphi^0,\varphi^1)D(\psi^0,\psi^1)\ dl(y),$$

$$c^{ts}(i,j) = \int_{\Omega\times T_4} i\ j\ dl(y),$$

$$b_1^{ts}(i,(\psi^0,\psi^1)) = \int_{\Omega\times T_4} i\ D(\psi^0,\psi^1)\ dl(y),$$

$$b_2^{ts}((\varphi^0,\varphi^1),j) = \int_{\Omega\times T_3} Lk\ D(\varphi^0,\varphi^1)\ j_{|T_4}\ dl(y),$$

$$l_1^{ts}((\psi^0,\psi^1)) = -\int_{\Omega\times T_1} i_d\ D(\psi^0,\psi^1)\ dl(y), \quad l_2^{ts}(j) = 0.$$

The space of admissible electric potential is

$$\begin{aligned}\Psi^{ts}_{ad}(u_d) = \{&(\psi^0,\psi^1) \in L^2(\Omega;\mathbb{P}^1(T)) \times L^2(\Omega;\mathbb{P}^1_\sharp(T')) \text{ so that}\\ &\psi^0_{|\Omega\times T'} \in \Psi^0_{ad},\ \psi^0_{|\Omega\times T''} \in L^2(\Omega;\mathbb{P}^1(T)),\ LD(\psi^0,\psi^1) = u_d,\ \psi^0 = 0 \text{ on } \Omega\times S_0\}.\end{aligned}$$

Proof. The proof consists in dividing the two sides of the first variational formulation by ε^r and in passing to the limit in each term. Let us start by rewriting every term in a form adapted to pass to the limit:

$$\begin{aligned} a^\varepsilon(\varphi^\varepsilon,\psi) = \varepsilon^r &\int_{\Theta_2^{\varepsilon\prime}} (\varepsilon^{-1}L^\varepsilon)(\varepsilon^{1-r}g^\varepsilon)\ \nabla_\tau\varphi^\varepsilon\nabla_\tau\psi\ dl(y)\\ &+ \varepsilon^r \int_{\Theta_2^{\varepsilon\prime\prime}} (\varepsilon^{-1}L^\varepsilon)(\varepsilon^{-1-r}g^\varepsilon)\ (\varepsilon\nabla_\tau\varphi^\varepsilon)(\varepsilon\nabla_\tau\psi)\ dl(y), \end{aligned}$$

$$b_1^\varepsilon(i^\varepsilon,\psi) = \varepsilon^r \int_{\Theta_{41}^{\varepsilon\prime}} (\varepsilon^{-r} i^\varepsilon)(\nabla_\tau \psi)_{|\Theta_{31}^{\varepsilon\prime}}\, dl(y) + \varepsilon^r \int_{\Theta_{42}^{\varepsilon\prime}} (\varepsilon^{-r-1} i^\varepsilon)(\varepsilon \nabla_\tau \psi)_{|\Theta_{31}^{\varepsilon\prime\prime}}\, dl(y)$$
$$+ \varepsilon^r \int_{\Theta_{41}^{\varepsilon\prime\prime}} (\varepsilon^{-r} i^\varepsilon)(\nabla_\tau \psi)_{|\Theta_{32}^{\varepsilon\prime}}\, dl(y) + \varepsilon^r \int_{\Theta_{42}^{\varepsilon\prime\prime}} (\varepsilon^{-r-1} i^\varepsilon)(\varepsilon \nabla_\tau \psi)_{|\Theta_{32}^{\varepsilon\prime\prime}}\, dl(y),$$

$$l_1^\varepsilon(\psi) = -\varepsilon^r \int_{\Theta_1^{\varepsilon\prime}} (\varepsilon^{-r} i_d^\varepsilon)\ \nabla_\tau \psi\ dl(y) - \varepsilon^r \int_{\Theta_1^{\varepsilon\prime\prime}} (\varepsilon^{-1-r} i_d^\varepsilon)\ \varepsilon \nabla_\tau \psi\ dl(y).$$

As a matter of fact, assumptions (H5), (H6) were stated so that the method of derivation of the limit problem, introduced in our previous paper [16], does apply. Instead of using Theorem 2 of [16], one prefers to make use of its modified version, Theorem 4.5. That is the way we get the first equation of the two-scale model.

The second variational formulation of (4.1) is equivalent to $i^\varepsilon_{|\Theta_4^\varepsilon} = L^\varepsilon k^\varepsilon (\nabla_\tau \varphi^\varepsilon)_{|\Theta_3^\varepsilon}$ or to

$$(\varepsilon^{-r} i^\varepsilon)_{|\Theta_{41}^{\varepsilon\prime}} = (\varepsilon^{-1} L^\varepsilon)(\varepsilon^{-r+1} k^\varepsilon)(\nabla_\tau \varphi^\varepsilon)_{|\Theta_{31}^{\varepsilon\prime}},$$
$$(\varepsilon^{-r-1} i^\varepsilon)_{|\Theta_{42}^{\varepsilon\prime}} = (\varepsilon^{-1} L^\varepsilon)(\varepsilon^{-r-1} k^\varepsilon)(\nabla_\tau \varphi^\varepsilon)_{|\Theta_{31}^{\varepsilon\prime\prime}},$$
$$(\varepsilon^{-r} i^\varepsilon)_{|\Theta_{41}^{\varepsilon\prime\prime}} = (\varepsilon^{-1} L^\varepsilon)(\varepsilon^{-r+1} k^\varepsilon)(\nabla_\tau \varphi^\varepsilon)_{|\Theta_{32}^{\varepsilon\prime}},$$

and

$$(\varepsilon^{-r} i^\varepsilon)_{|\Theta_{42}^{\varepsilon\prime\prime}} = (\varepsilon^{-1} L^\varepsilon)(\varepsilon^{-r-1} k^\varepsilon)(\nabla_\tau \varphi^\varepsilon)_{|\Theta_{32}^{\varepsilon\prime\prime}}.$$

By virtue of assumptions (H5), (H6) one can pass to the limit and find that

$$i_{|\Omega\times T_4} = kD(\varphi^0,\varphi^1)_{|\Omega\times T_3},$$

which is equivalent to the variational formulation (4.9$_2$). ☐

REMARK 4.6. *Before closing this section, let us summarize the role which is played by the various assumptions that have been introduced. Assumptions* (H1)–(H4) *imposed on the periodic circuit yield the well posedness of the original problem for all* ε. *Assumptions* (H5), (H6) *make possible the extraction of bounded subsequences of the sequence* $(\varphi^\varepsilon, i^\varepsilon)$*, which converge to one or several solutions of the limit model that is also called the two-scale model. In addition, assumptions* (H1)–(H3) *related to the modified circuit cell* $\mathcal{T}$ *are necessary and sufficient so that the modified problem has a unique solution. One deduces from this that they are probably necessary to ensure the convergence of the solution of the periodic problem toward that of the two-scale model. Nevertheless, we stress that they do not suffice, as shown by Example* 4 *below.*

4.4.4. An equivalent formulation of the cell problem. So far, for the reasons already put forward, the cell problem has been formulated on the modified circuit cell. Its derivation is carried out in two steps. First, we build the cell problem, formulated on the original circuit cell, by using usual arguments (see, e.g., [16]). Then the problem formulated on the modified unit cell is derived by application of the three transformations listed in section 4.3.1. In what follows, we state the cell problem on the original cell T when its derivation is briefly discussed in the next section.

Find $(\varphi^0,\varphi^1,i) \in \Psi^1_{ad}(\eta,\phi,u_d) \times L^2(\Omega;\mathbb{P}^0(T_4))$ such that

$$a^1((\varphi^0,\varphi^1),(\psi^0,\psi^1)) + b_1^1(i,(\psi^0,\psi^1)) = l_1^1((\psi^0,\psi^1)),$$
$$b_2^1((\varphi^0,\varphi^1),j) - c^1(i,j) = l_2^1(j) \tag{4.10}$$

for any $(\psi^0, \psi^1, j) \in \Psi^1_{ad}(0,0,0)$, where

$$a^1((\varphi^0,\varphi^1),(\psi^0,\psi^1)) = \int_{T_2} Lg\ d(\varphi^0,\varphi^1)d(\psi^0,\psi^1)\ dl(y),\ c^1(i,j) = \int_{T_4} i\ j\ dl(y),$$

$$b^1_1(i,(\psi^0,\psi^1)) = \int_{T_4} i\ d(\psi^0,\psi^1)\ dl(y),\ b^1_2((\varphi^0,\varphi^1),j) = \int_{T_3} Lk\ d(\varphi^0,\varphi^1)\ j_{|T_4}\ dl(y),$$

$$l^1_1((\psi^0,\psi^1)) = -\int_{T_1} i^1_d\ d(\psi^0,\psi^1)\ dl(y) - \sum_{q=1}^{n_c}\int_{T_2\cap T^{q\prime}} Lgu^{0q}_d d(\psi^0,\psi^1)\ dl(y),$$

$$l^1_2(j) = -\sum_{q=1}^{n_c}\sum_{e_3\in(T_3\cap T^{q\prime})}\int_{e_4} k(u^{0q}_d)_{|e_3}\ j\ dl(y),\ e_4 \text{ being the output of } e_3,$$

and the space of admissible electric potentials is

$$\Psi^1_{ad}(\eta,\phi,u_d) = \{(\psi^0,\psi^1) \in \mathbb{P}^1(T'') \times \mathbb{P}^1_\sharp(T') \text{ so that}$$
$$Ld(\psi^0,\psi^1) = \mathcal{U}_d \text{ on } T_0 \text{ and } \psi^0 = \phi \text{ on } T''\cap T',\ \psi^0 = 0 \text{ on } S_0\}.$$

The subsequent proposition summarizes the equivalence between the two cell problems stated, respectively, on T and on $\mathcal{T}$.

PROPOSITION 4.7. *When posing $\varphi = \varphi^0$ on T'' and $\varphi = \varphi^1$ on T', the two cell problems are equivalent.*

Proof. As a matter of fact, the proof is a straightforward consequence of the construction of $\mathcal{T}$ detailed in section 4.3.1. □

4.4.5. Derivation of the homogenized model. For the construction of the homogenized model, one follows the usual method. We first have to establish the cell problem (also called the problem micro), then observing that its solution is a linear form of the fields macro and replacing it in the two-scale model yields the homogenized model that concerns only fields macro.

The cell problem on its variational form (4.10) is derived from the two-scale model by making a particular choice for ψ^0 : $\psi^0_{|\Omega\times T'} = 0$; hence $D(\psi^0,\psi^1) = d(\psi^0,\psi^1)$. One deduces that $u = L\ d(\varphi^0,\varphi^1)$ depends linearly on the fields macro u_d, i_d, $(\nabla_{\overline{z}}\varphi^0)_{|\Omega\times T'}$, and $(\varphi^0)_{|\Omega\times T'}$, which is said in other words in the formula (4.8) with $\eta = \nabla_{\overline{z}}\varphi^0$ and $\phi = \varphi^0$:

$$d(\varphi^0,\varphi^1) = \sum_{i=1}^{|T_0|}\mathcal{L}^1_i(y)u_{di} + \sum_{i=1}^{|T_1|}\mathcal{L}^2_i(y)i_{di} + \sum_{q=1}^{n_c}\sum_{i\in J^q}\mathcal{L}^3_{qi}(y)\partial_{z_i}\varphi^{0q} + \sum_{q=1}^{n_c}\mathcal{L}^4_q\varphi^{0q}.$$

Then, making another choice for (ψ^0,ψ^1) so that

$$d(\psi^0,\psi^1) = \sum_{q=1}^{n_c}\sum_{i\in J^q}\mathcal{L}^3_{qi}(y)\partial_{z_i}\psi^{0q} + \sum_{q=1}^{n_c}\mathcal{L}^4_q\psi^{0q}$$

and replacing the current i by $k\ D(\varphi^0,\varphi^1)$ leads to the weak formulation (4.4) of the homogenized model.

5. Examples.

5.1. Discussion of the assumptions on $\mathcal{T}$. The topic of this subsection is to mention some simple examples for which one of the assumptions (H1)–(H3) is satisfied for the ε-periodic circuit but is violated for the modified unit circuit cell $\mathcal{T}$.

Example 1. The first example is that of a one-dimensional electric network constituted of a series of voltage sources. The electric potential is imposed to vanish on the left side. The three circuits are represented in Figure 5.1. Assumption (H1) is clearly satisfied for the complete network but does not hold for the modified circuit cell. As a matter of fact, this is the consequence of the additional zero voltage source that closes the loop constituted of voltage sources.

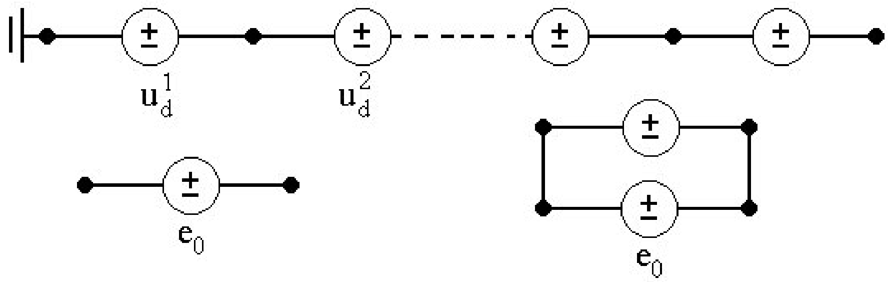


FIG. 5.1. *Example* 1: *The whole network* Θ^ε*, the cell* T*, and the cell* $\mathcal{T}$*.*

Example 2 (see Figure 5.2). This is a case where (H3) is satisfied for the ε-periodic circuit and violated for the modified circuit cell. Let us mention that this could also be the case if the subcircuit T'' was meeting the boundary of the cell. However, we already have insisted on the fact that this case is excluded from our focus in this paper.

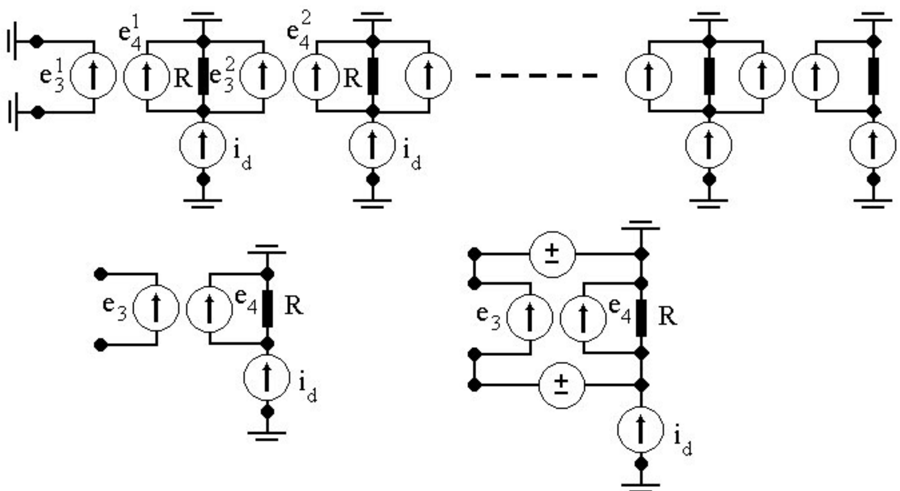


FIG. 5.2. *Example* 2: *The whole network* Θ^ε*, the cell* T*, and the cell* $\mathcal{T}$*.*

Example 3 (see Figure 5.3). This is another situation where assumption (H1) is clearly satisfied for Θ^ε, but it is not for a reason different from that invoked in Example 1. Here, the loop of voltage sources comes from the fact that a voltage source is linked to a crossing path in the original cell.

5.2. Example 4: Numerical validation. The numerical comparisons of the two solutions computed on one hand on the periodic network and on the other hand by

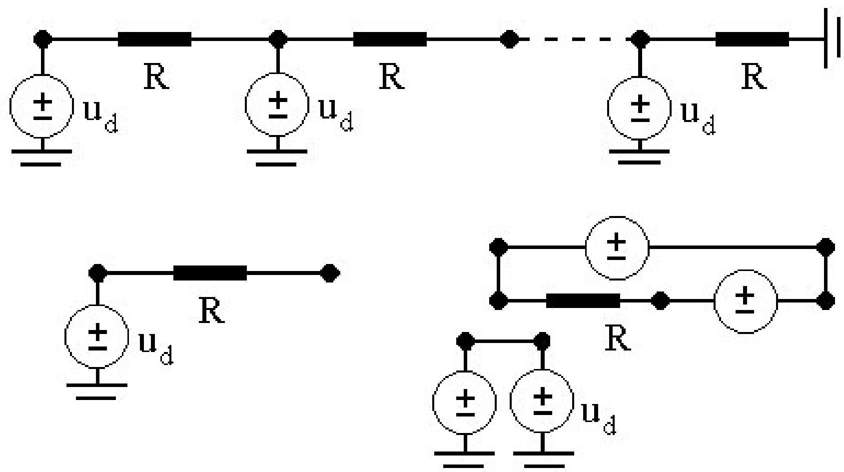


FIG. 5.3. *Example* 3: *The whole network* Θ^ε*, the cell* T*, and the cell* $\mathcal{T}$*.*

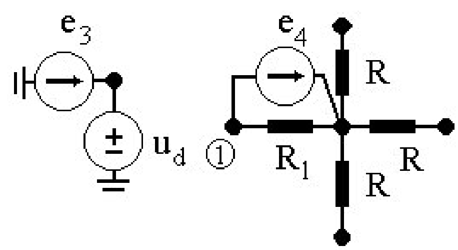


FIG. 5.4. *Example* 4: *The cell* T*.*

using the homogenized model have been carried out for the unit circuit cell represented in Figure 5.4 with the values $R = k = 1$. The four extremities of the resistors are located on the cell boundary. In spite of appearances on the figure, the block (e_3, u_d) is included inside the cell. In fact, its location in the cell does not play any role, while it does not meet the crossing subcircuits. It is easy to check that assumptions (H1)–(H4) are satisfied for the periodic circuit and that (H1)–(H3) are also satisfied for the circuit cell. The homogenized model is

$$-\frac{1}{2}\Delta\varphi^0 = \operatorname{div} H^1 \text{ in } \Omega \text{ and } \varphi^0 = 0 \text{ on } \Gamma,$$

with $H^1 \in L^2(\Omega)$. It has a unique solution $\varphi^0 \in H_0^1(\Omega)$.

The electric potential is imposed to zero on the whole boundary Γ. The homogenized distribution of the voltage source that in the homogenized model is $u_d(z) = -2\pi\cos(\pi z_1)\sin(\pi z_2)$ when its counterpart u_d^ε for the periodic circuit is taken to be equal to $u_d(x_i^\varepsilon)$, the x_i^ε being the center of the 15×15 cells. The homogenized potential $\varphi^0(z)$ is computed using a P^1 finite element method with 15 elements in both directions.

Distributed electric potentials $\varphi^\varepsilon(x_i^{1\varepsilon})$ and $\varphi^0(x_i^{1\varepsilon}, y^1)$ are compared in the first row of Figure 5.5, where $x_i^{1\varepsilon} = x_i^\varepsilon + \varepsilon y^1$ and $y^1 = (-0.5, 0)$ are the coordinates of the node 1 in Θ^ε and T, respectively. By another way, the voltages $u^\varepsilon(x_i^{2\varepsilon})$ and

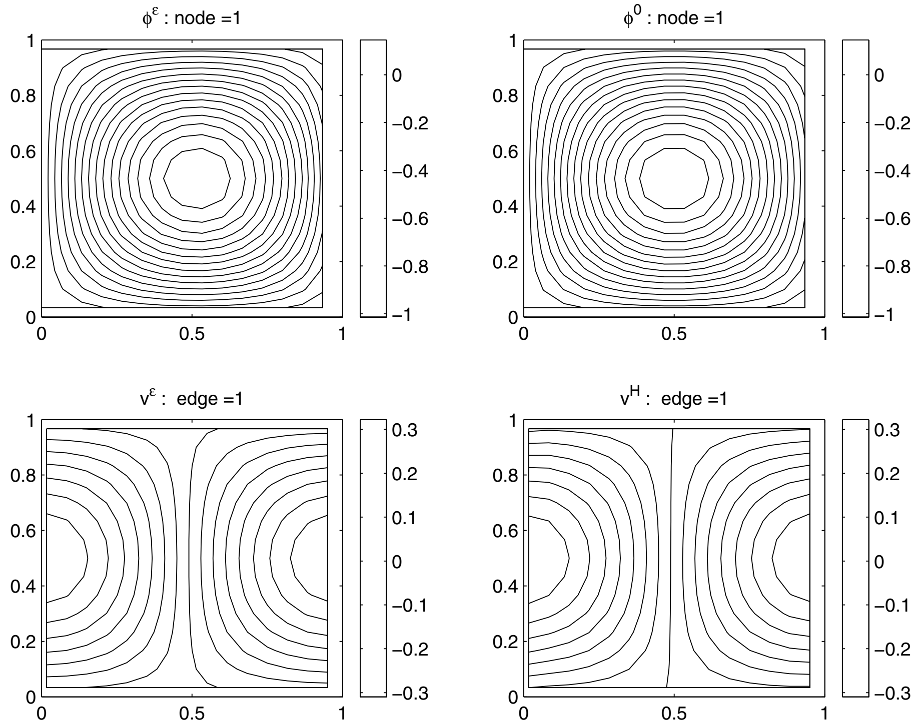


FIG. 5.5. *Example* 4: $u_d(z) = -2\pi\cos(\pi z_1)\sin(\pi z_2)$.

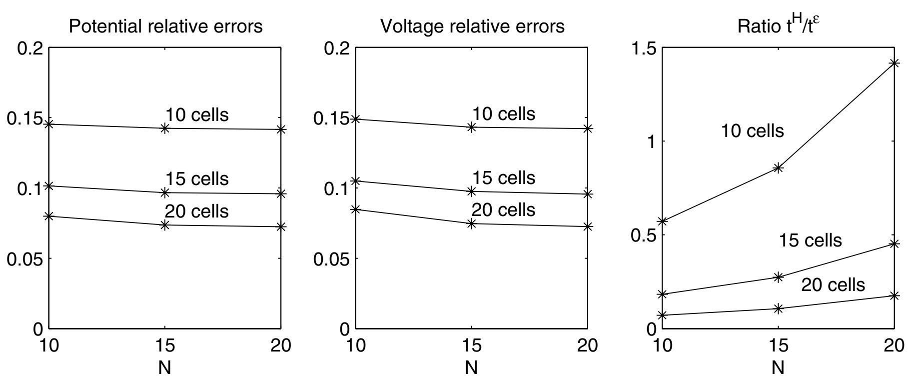


FIG. 5.6. *Example* 4: *Errors and simulation times.*

$u^H = L^\varepsilon D(\varphi^0,\varphi^1)(x_i^{2\varepsilon}, y^2)$ are represented in the second row, where $x_i^{2\varepsilon} = x_i^\varepsilon + \varepsilon y^2$ and $y^2 = (-0.25, 0)$ are the middle of the resistors R_1 in Θ^ε and T, respectively. The results show a good qualitative agreement between the two models.

Quantitative comparisons are detailed in Figure 5.6. Global relative errors, in the L^2 norm, for potentials and voltages are compared when Θ^ε has 10, 15, or 20 cells in each direction and when the finite element method is with $N = 10$, 15, or 20 elements in each direction. It shows that, in this case, the errors diminish with the increase of the number of cells but are not influenced much by the number of finite elements.

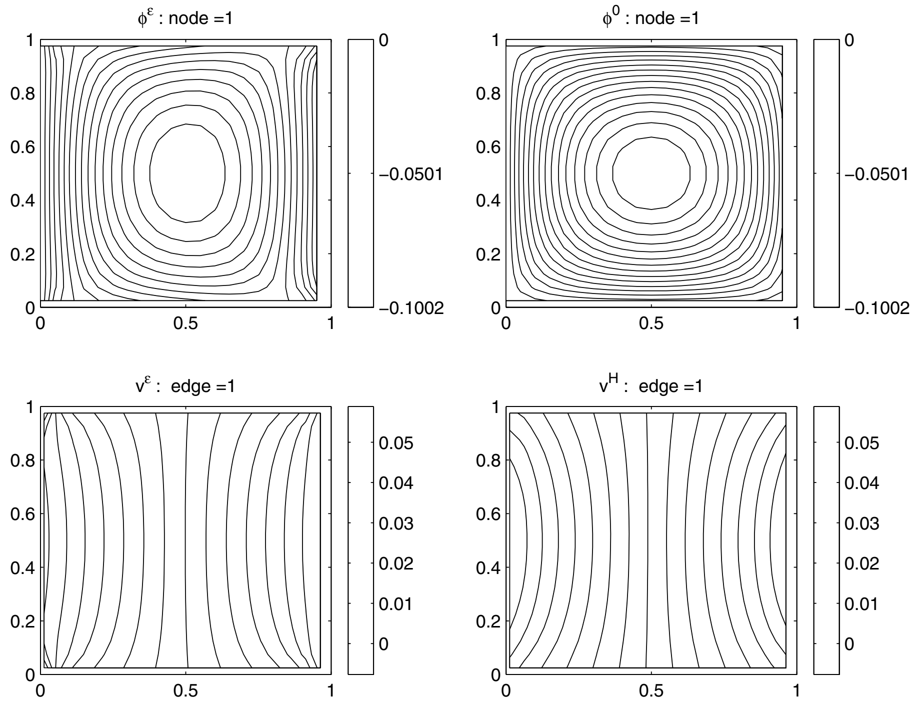


FIG. 5.7. *Example* 4: $u_d(z) = 1 + z_1$.

The observation of the ratio $\frac{t^H}{t^\varepsilon}$ of the simulation times of the two models yields the conclusion that the homogenized model presents great interest in this point of view. This is particularly true for large numbers of cells. Moreover, we have observed that the more the complexity of the circuit increases, the more the ratio is favorable to the homogenized model.

It has already been emphasized that conditions (H1)–(H3) imposed on the cell circuit are not sufficient to ensure the uniform estimate (H6). This can be observed in this example when $u_d(z) = 1 + z_1$. An analytical calculation shows that the convergence does not hold because a phenomenon of boundary layer occurs in the vicinity of the boundary Γ. This may be observed in our simulation results, with 20×20 circuit cells and 20×20 finite elements, that are reported in Figure 5.7.

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