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# Modelling of Thin Elastic Plates with Small Piezoelectric Inclusions and Distributed Electronic Circuits. Models for Inclusions that Are Small with Respect to the Thickness of the Plate

ÉRIC CANON and MICHEL LENCZNER
*Laboratoire de Mathématiques de Besançon UMR 6623, Équipe de Calcul Scientifique, Université de Franche-Comté, 16 Route de Gray, F-25030 Besançon Cédex, France*
*E-mail: canon@univ-st-etienne.fr*

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**Abstract.** This paper is devoted to the modelling of thin elastic plates with small, periodically distributed, piezoelectric inclusions, in view of active controlled structure design. The initial equations are those of linear elasticity coupled with the electrostatic equation. Different kinds of boundary conditions on the upper faces of inclusions are considered, corresponding to different ways of control: Dirichlet, Neumann, local or nonlocal mixed conditions. We compute effective models when the thickness $a$ of the plate, the characteristic dimension $\varepsilon$ of the inclusions, and $\varepsilon/a$ tend together to zero. Other situations will be considered in two forthcoming papers.

**Mathematics Subject Classifications (1991):** 35B27 homogenization of partial differential equations in media with periodic structures, 73B27 nonhomogeneous materials and homogenization.

**Key words:** linear elasticity, piezoelectricity, homogenization, plate theory, composite materials, prescribed electric potential, local electric circuits, nonlocal electrical circuits, transfinite networks, smart materials.

## 1. Introduction

### 1.1. GENERAL

This paper is part of systematic work devoted to the derivation of effective models for piezoelectric/elastic composite plates including elementary electronic circuits. In [4],we considered three dimensional elastic plates with a small number of piezoelectric inclusions, and we derived effective models when the thickness of the plate tends to zero. The models are static (and linear) but may be extended to dynamic via the Laplace transform.

In the present paper, we consider plates with a great number of piezoelectric transducers, periodically distributed in an elastic matrix, requiring homogenization.

Two small parameters are thus involved in our analysis: the thickness $a$ of the plate and the characteristic dimension $\varepsilon$ of inclusions. *Effective models* mean that we compute the limit models when $a$ and $\varepsilon$ simultaneously tend to zero. The fact that $a$ and $\varepsilon$ tend together to zero ensures that all the possible limit models are obtained. Three different situations actually occur according to whether $a/\varepsilon \to 0$, $\varepsilon/a \to 0$ or $\varepsilon = a$. The aim of the present paper is to obtain models in the case where the inclusions are small with respect to the thickness of the plate, that is $\varepsilon/a \to 0$. The two other situations will be treated in two forthcoming papers. We remark that (simplified) models for $a/\varepsilon \to 0$ were presented in [5].

The goal we have in mind is to control structures by electrical regulation applied to the upper and lower faces of piezoelectric transducers. More precisely, we try to conceive distributed electronic circuits which act on structures for the purpose of control. As in [4], we consider different possibilities for the boundary conditions on the upper faces of inclusions, corresponding to different kinds of control: Dirichlet conditions, if the tension is controlled, Neumann conditions, if the current is controlled, and mixed conditions, if inclusions are connected to R-L-C circuits. In this last class, we consider the case where the upper and lower faces of each inclusion are connected, and the case where, in addition, each inclusion is connected to its direct neighbours. From a mathematical point of view, this corresponds to local mixed conditions and to nonlocal mixed conditions, respectively.

For the model associated with nonlocal boundary conditions, a Laplace operator in the in-plane direction of the plate arises, acting on the transverse component $L_3^0$ of the electric field. This is a model of transfinite network type, as described, for example, in Zemanian [12, 13] with a different approach. Let us mention that one may choose a priori the form of the operator on $L_3^0$ (and the corresponding boundary conditions) by appropriately connecting the inclusions to each other (and to the outside of the domain). In fact, we get here a complete family of transfinite networks. This seems particularly interesting in the perspective of building relevant controllers.

To derive the effective models, we use a mixing of two-scale convergence [1, 10] and of classical arguments of plate theory [6, 7, 10]. Note, however, that, as in [4], the derivation is made in the space of the gradients of solutions. This seems to be unusual, but allows a more synthetic and readable presentation of the models themselves as well as of their derivation. We think that this formalism, in itself, is an interesting contribution of this work.

The obtained models do have rather a simple structure. The effective model for Dirichlet conditions has the same form as the purely elastic plate model; the influence of piezoelectrics only appears in the definition of the effective coefficients and as a source term on the right hand side. This is not the case for nonlocal mixed conditions: because of the differential operator induced by the R-L-C circuits, a coupling arises between mechanical effects and a transverse component of the electric field. For local mixed conditions, the situation is intermediate: see the comments at the end of Section 5.

For problems which include homogenization and plate theory, one must mention the work of Caillerie [3]. Caillerie treated the case of thin static elastic plates with rapidly oscillating coefficients, using the energy method of Tartar [2, 11]. It should be noted that the parameters $a$ and $\varepsilon$ tend (except for $a = \varepsilon$) successively and independently to zero in [3].

A more extensive bibliographical review on piezoelectric plate models is given in [4].

### 1.2. DETAILED CONTENTS

Section 2 is devoted to the presentation of the initial 3-dimensional equations: elasticity and piezoelectricity equations in their linear and static versions. The piezoelectric inclusions are assumed to be strictly included in the elastic matrix, which is considered to be electrically insulated. For simplicity, as it is usually the case in applications, the stiffness, piezoelectricity and permitivity tensors are assumed to be constant in the direction of thickness of the plate. In the same spirit, the upper and lower faces of the piezoelectrics are assumed to be metallized, that is, covered with a thin film of conductive metal. Concerning the equation of elasticity, standard boundary conditions are considered: Neumann conditions on the upper and lower faces of the plate, Neumann and Dirichlet conditions on the lateral boundary. For the equation of piezoelectricity, we consider Neumann conditions on the lateral boundary, Dirichlet condition on the lower faces. As mentioned in Section 1.1, various boundary conditions are considered on the upper faces, namely: Dirichlet, Neumann, local and nonlocal mixed conditions. These kinds of conditions are, to our knowledge (except Neumann conditions), unusual in plate theory. They thus constitute an interesting point of this paper.

The corresponding weak formulations are presented in Section 3. In the sequel, because of the relative formal complexity of the models, and because we want to treat the various boundary conditions together, as much as possible, we adopt synthetic tensorial notation rather than fully expanded formulae. We strongly believe that this allows a better description of our computations as well as of our limit models.

The precise assumptions on the data are presented in Section 4.1. We give, in particular, the correct scalings. From a practical point of view, this indicates how electric circuits must be chosen to obtain a significant influence on the effective behaviour of the material. Resulting a priori estimates and first convergence results are given in Sections 4.2 and 4.3.

Section 5 is devoted to the statement of the main result of the paper, Theorem 5.1: an effective 2-dimensional plate model for each type of electrical boundary condition, when the inclusions are much smaller than the thickness of the plate.

Theorem 5.1 is proved in Section 6 by letting $a$, $\varepsilon$, $\varepsilon/a$ tend simultaneously to 0 in the weak formulations of Section 3. The proof is in three steps.

The first one, which is mathematically most difficult, consists in characterizing two-scale limits of the strains and of the electric field. These results are new, even in the case of pure elasticity. Caillerie considered weak limits only; the intermediate two-scale limits were not described in [3]. These results are of general interest and may apply to various situations which concern homogenization and plate theory.

The second step consists in eliminating the local variable $y$ by computing the microscopic fields (depending on $y$) with respect to the macroscopic fields (depending only on the macroscopic variable $x$). Here, we use the classical arguments of linear homogenization.

The third step consists in eliminating (part of) the transverse components of the strains and of the electric field, which may be computed with respect to the other components of these fields. This elimination slightly departs from the classical plate theory, because of the nonstandard boundary conditions on the faces of the inclusions.

We use the same formalism as in [4], based on tensorial notation and products, and on simple algebraic operations such as projections. It allows us to deal relatively easily with complex computations. Completely explicit formulae would be lengthy and limit the readability. In our approach, steps 2 and 3 are almost formal computation and may be easily adapted to variants of our models. In this way, one could also easily extend the results to multilayered plate models, as in [4].

To conclude the paper, in Section 7, we propose, an illustration of our modelling. We consider a transversally isotropic material (a PZT ceramic, for instance), with Dirichlet conditions. This is the simplest possible example, because the effect of piezoelectricity does not occur in the isotropic material. We use the programming package Mathematica to compute, from the general formulation of Theorem 5.1, quite explicit formulae for the effective coefficients. By comparing the compact formulae of Section 5 with the expanded formulae of Section 7, one may appreciate the formalism used in [4] and in the present paper. Moreover, the use of Mathematica shows that this formalism is not only elegant; it is also practical.

## 2. Equations of 3-dimensional Piezoelectricity

This section is devoted to the presentation of the initial 3D equations. The corresponding weak formulations are given in Section 3, the corresponding effective models are calculated in Sections 4–6.

### 2.1. GEOMETRY

Let $a$ be the positive parameter measuring the thickness of the plate. The 3-dimensional plate is initially represented by $\Omega^a = \omega \times ]-a, a[$, $\omega$ being a bounded domain of $\mathbb{R}^2$. (Using the classical change of scales and variables introduced in [6], we shall in fact work on the fixed domain $\Omega = \omega \times ]-1, 1[$.)

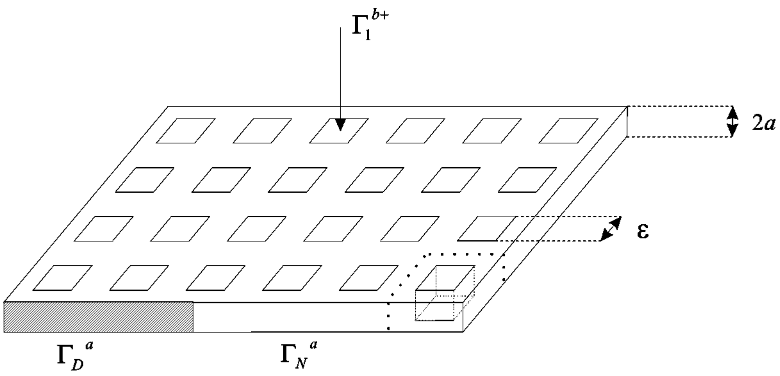


*Figure 1.* Composite plate with piezoelectric inclusions.

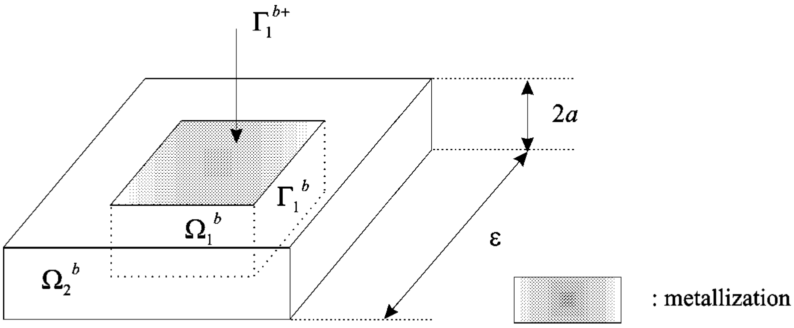


*Figure 2.* Elementary cell with piezoelectric inclusion and metallization.

Let $\varepsilon > 0$ denote the characteristic dimension of inclusions. The domain $\omega$ is divided into two subdomains $\omega_1^\varepsilon$ and $\omega_2^\varepsilon$ that are constructed as follows. Let $Y$ be a rectangle subdomain of $\mathbb{R}^2$ such that, without loss of generality, $|Y| = 1$. Let $Y_1 \subset\subset Y$ with $|Y_1| > 0$, and $Y_2 = Y \setminus Y_1$. The set $\omega_1^\varepsilon$ is a union of all the $\varepsilon Y$-periodic translations of $\varepsilon Y_1$ that are strictly contained in $\omega$, while $\omega_2^\varepsilon = \omega \setminus \overline{\omega}_1^\varepsilon$. Let $b = (a, \varepsilon)$. The elastic matrix is represented by $\Omega_2^b = \omega_2^\varepsilon \times ]-a, a[$, the set of all piezoelectric inclusions by $\Omega_1^b = \omega_1^\varepsilon \times ]-a, a[$.

The inclusions are numbered by a multi-index $\mathbf{i} = (i_1, i_2) \in \mathbb{I}^\varepsilon$. Then, $\langle\cdot\rangle_\mathbf{i}$ denotes the mean value on the upper face of the inclusion indexed by $\mathbf{i}$. For every function $\psi$ on $\Omega^a$, $\psi_\mathbf{i}$ is the restriction of $\psi$ to the inclusion $\mathbf{i}$.

The boundary of $\omega$ is assumed to be smooth and divided into two regular parts $\gamma_D$ and $\gamma_N$, with $|\gamma_D| > 0$. The boundary of $\Omega$ is divided into: $\Gamma_D^a = \gamma_D \times ]-a, a[$, $\Gamma_N^a = (\gamma_N \times ]-a, a[) \cup (\omega \times \{-a, a\})$. The boundary of $\Omega_1^b$ is divided into $\Gamma_1^{b+} = \omega_1^\varepsilon \times \{a\}$, $\Gamma_1^{b-} = \omega_1^\varepsilon \times \{-a\}$ and $\Gamma_1^b = \partial\omega_1^\varepsilon \times ]-a, a[$.

The current point in $\Omega^a$ is $x^a = (x_1, x_2, x_3^a)$, where $x_3^a \in ]-a, a[$ and $\hat{x} = (x_1, x_2) \in \omega$. The current point in $Y$ is $y = (y_1, y_2)$. The derivatives with respect to $x_\alpha$, $x_3^a$ and $y_\alpha$ are denoted by $\partial_\alpha$, $\partial_3$ and $\partial_{y_\alpha}$, respectively. The outer unit normal to the boundaries of $\Omega^a$ and $Y$ is denoted by $\mathbf{n}$ and $\mathbf{n}_Y$, respectively.

Specifically for the scaled domain $\Omega$, a constant use is made of

$$\mathcal{M}(f) = \frac{1}{2}\int_{-1}^{1} f(x_3)\,\mathrm{d}x_3 \quad \text{and} \quad \mathcal{N}(f) = f - \mathcal{M}(f) \quad \text{for } f \in L^1(-1,1). \tag{1}$$

Finally, let us mention that when referring to the fixed domain $\Omega$, the geometric notation is the same, the subscript $a$ being removed, if necessary.

### 2.2. OTHER NOTATIONS

Bold characters are used for vector and matrix valued functions and, possibly, for the corresponding functional spaces. We constantly use Einstein's convention of summation on repeated indices, with summation from one to three for Latin indices, from one to two for Greek indices.

### 2.3. EQUATIONS OF 3-DIMENSIONAL PIEZOELECTRICITY

The mechanical displacements $\mathbf{u}^b = (u_i^b)_{i=1,2,3}$ and the electrical potential $\varphi^b$ are governed by the linear equations of piezoelectricity in their static version. In this section, we recall these equations that underlie our models. The boundary conditions for the upper and lower faces of inclusions that characterize different models are specified in 2.4.

The plate is submitted to the volume mechanical forces $\mathbf{f}^b = (f_i^b)_{i=1,2,3}$ in $\Omega^a$, and to the surface mechanical forces $\mathbf{g}^b = (g_i^b)_{i=1,2,3}$ on $\Gamma_N^a$.

For any $\mathbf{v} \in \mathbf{H}^1(\Omega^a)$, let us denote the strains by

$$s_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \quad \forall i,j \in \{1,2,3\}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega^a). \tag{2}$$

The stresses $\sigma^b = (\sigma_{ij}^b)_{i,j=1,2,3}$ and the electrical displacements $\mathbf{D}^b = (D_i^b)_{i=1,2,3}$ are then given by

$$\begin{cases} \sigma_{ij}^b = R_{ijkl}^\varepsilon s_{kl}(\mathbf{u}^b) + d_{kij}^\varepsilon \partial_k \varphi^b & \text{in } \Omega, \\ D_k^b = -d_{kij}^\varepsilon s_{ij}(\mathbf{u}^b) + c_{ki}^\varepsilon \partial_i \varphi^b & \text{in } \Omega_1^b. \end{cases} \tag{3}$$

The mechanical equilibrium equations and the mechanical boundary conditions are:

$$\begin{aligned} &-\partial_j \sigma_{ij}^b = f_i^b \quad \text{in } \Omega, \qquad \sigma_{ij}^b n_j = g_i^b \quad \text{on } \Gamma_N^a \quad \text{for } i = 1,2,3, \\ &\mathbf{u}^b = \mathbf{0} \text{ on } \Gamma_D^a. \end{aligned} \tag{4}$$

The electrostatic equation and the electrical boundary conditions on the lateral faces of the inclusions are:

$$-\partial_i D_i^b = 0 \quad \text{in } \Omega_1^b, \qquad \mathbf{D}^b \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1^b. \tag{5}$$

In (3), $\mathbf{R}^\varepsilon = (R^\varepsilon_{ijkl})_{i,j,k,l=1,2,3}$, $\mathbf{d}^\varepsilon = (d^\varepsilon_{kij})_{i,j,k=1,2,3}$ and $\mathbf{c}^\varepsilon = (c^\varepsilon_{ij})_{i,j=1,2,3}$ denote the tensors of stiffness, piezoelectricity, and permitivity. They satisfy

$$R^\varepsilon_{ijkl} = R^\varepsilon_{klij} = R^\varepsilon_{jikl}, \qquad c^\varepsilon_{ij} = c^\varepsilon_{ji}, \qquad d^\varepsilon_{ijk} = d^\varepsilon_{ikj} \quad \forall i,j,k,l \in \{1,2,3\}. \tag{6}$$

We assume that the piezoelectric inclusions are electrically insulated from the elastic matrix. The electrical influence of $\Omega_2^b$ on $\Omega_1^b$ is, therefore, neglected in our analysis. Though, it is convenient to define the tensors $\mathbf{R}^\varepsilon$, $\mathbf{d}^\varepsilon$, $\mathbf{c}^\varepsilon$ on the whole domain $\Omega^a$. We let, therefore,

$$c^\varepsilon_{ij} = 0, \qquad d^\varepsilon_{ijk} = 0 \quad \text{in } \Omega_2^b \quad \forall i,j,k \in \{1,2,3\}. \tag{7}$$

For the electrostatic equation, we go now into detail about the boundary conditions on the upper and lower faces of inclusions.

## 2.4. BOUNDARY CONDITIONS ON THE UPPER AND LOWER FACES OF THE PIEZOELECTRIC INCLUSIONS

For the sake of conciseness, we only consider situations where all the faces are metallized, as it is usually the case in applications. From a mathematical point of view, this means that the electric field is constant on each face of each inclusion. Considering nonmetallized faces would lead to unnecessary technical complications. However, let us note that nonmetallized faces were considered by the authors in [4] for models with few inclusions.

Three kinds of conditions are considered:

### 2.4.1. *Dirichlet Conditions*

$$\varphi^b = \begin{cases} \varphi^b_m + a\varphi^b_c & \text{on } \Gamma_1^{b+}, \\ \varphi^b_m - a\varphi^b_c & \text{on } \Gamma_1^{b-}, \end{cases} \tag{8}$$

where $\varphi^b_m$ and $\varphi^b_c$ are constant on each inclusion.

This condition may result from the connection of each piezoelectric to the output of a tension source providing tension $\varphi^b_c$, or to the input of a current amplifier (here $\varphi^b_c = 0$). In both cases, one of the faces is connected to the ground equal to $\varphi^b_m$.

### 2.4.2. *Neumann and Local Mixed Conditions*

$$\varphi^b = \varphi^b_m \quad \text{on } \Gamma_1^{b-}, \qquad \langle \mathbf{D}^b \cdot \mathbf{n} \rangle_{\mathbf{i}} = -\frac{G}{a}\bar{\varphi}^b + h^b \quad \text{on } \Gamma_1^{b+}\ \forall \mathbf{i} \in \mathbb{I}^\varepsilon, \tag{9}$$

where $\bar{\varphi}^b = \varphi_{|\Gamma_1^{b+}} - \varphi^b_m$. The functions $\varphi^b_m$ and $h^b$ are constant on each inclusion, $G$ is a fixed nonnegative constant.

Equation (9) covers two sorts of boundary conditions. If $G = 0$, (9) is a Neumann condition

$$\langle \mathbf{D}^b \cdot \mathbf{n} \rangle_{\mathbf{i}} = h^b \quad \text{on } \Gamma_1^{b+},$$

which arises when the inclusions are connected to the output of a current source $h^b$, or to the input of a tension amplifier ($h^b = 0$). When $G > 0$, (9) is the true mixed condition. It occurs when the upper and lower faces of each inclusion are connected by an R-L-C circuit of impedance $a/G$, $h^b$ being an additional source of current.

REMARK 2.1. The above explanations are slightly inaccurate. In fact, the current which flows out of an inclusion is the time derivative of $\langle \mathbf{D}^b \cdot \mathbf{n}\rangle_{\mathbf{i}}$. One may think of (9) as the Laplace transform of the Kirchoff law.

### 2.4.3. *Nonlocal Mixed Conditions*

They occur when dielectric inclusions are also connected together by R-L-C circuits. We consider here the case where the upper face of each inclusion is connected with each of its direct neighbours, but not to the outside of the plate. This is described as follows.

Let us introduce the shift operators

$$T^1_{+1}\colon \begin{array}{l}\mathbb{I}^\varepsilon \to \mathbb{N}^2,\\ \mathbf{i} \mapsto (i_1+1, i_2),\end{array} \qquad T^2_{+1}\colon \begin{array}{l}\mathbb{I}^\varepsilon \to \mathbb{N}^2,\\ \mathbf{i} \mapsto (i_1, i_2+1),\end{array}$$

$$T^1_{-1}\colon \begin{array}{l}\mathbb{I}^\varepsilon \to \mathbb{N}^2,\\ \mathbf{i} \mapsto (i_1-1, i_2),\end{array} \qquad T^2_{-1}\colon \begin{array}{l}\mathbb{I}^\varepsilon \to \mathbb{N}^2,\\ \mathbf{i} \mapsto (i_1, i_2-1).\end{array}$$

With the convention

$$\begin{aligned}&\bar{\varphi}^b_{T^\alpha_{-1}(\mathbf{i})} - \bar{\varphi}^b_{\mathbf{i}} = 0 \quad \text{if } T^\alpha_{-1}(\mathbf{i}) \notin \mathbb{I}^\varepsilon,\\ &\bar{\varphi}^b_{T^\alpha_{+1}(\mathbf{i})} - \bar{\varphi}^b_{\mathbf{i}} = 0 \quad \text{if } T^\alpha_{+1}(\mathbf{i}) \notin \mathbb{I}^\varepsilon, \quad \text{for } \alpha = 1, 2,\end{aligned} \tag{10}$$

the Kirchhoff law leads here to

$$\begin{cases}\langle \mathbf{D}^b \cdot \mathbf{n}\rangle_{\mathbf{i}} = \displaystyle\sum_{\alpha=1}^{2} \frac{G_1}{a\varepsilon^2}\big(\bar{\varphi}^b_{T^\alpha_{-1}(\mathbf{i})} - 2\bar{\varphi}^b_{\mathbf{i}} + \bar{\varphi}^b_{T^\alpha_{+1}(\mathbf{i})}\big) - \frac{G}{a}\bar{\varphi}^b_{\mathbf{i}} + h^b, \quad \forall \mathbf{i} \in \mathbb{I}^\varepsilon,\ \text{on } \Gamma_1^{b+},\\ \varphi^b = \varphi^b_m \quad \text{on } \Gamma_1^{b-}.\end{cases} \tag{11}$$

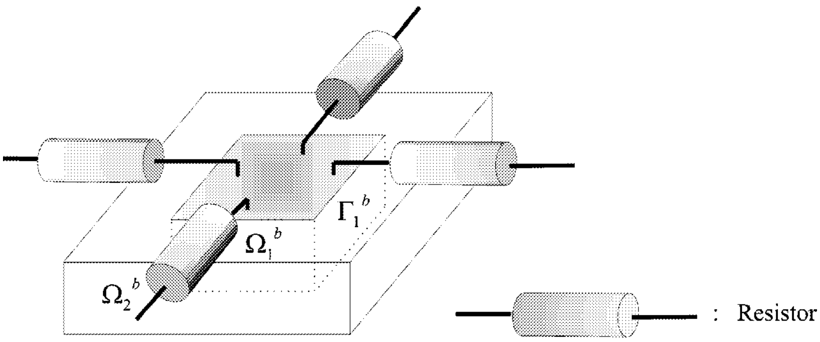


*Figure 3.* Cell with nonlocal electric circuit.

Here $a\varepsilon^2/G_1$ ($G_1 > 0$) designates the common impedance of the circuits linking two adjacent inclusions.

REMARK 2.2. Condition (11) clearly corresponds to a discrete Laplace operator in the two directions of the plate here with discrete homogeneous Neumann conditions (10). Due to the above particular scaling on the impedance, in the asymptotic process $(a, \varepsilon) \to (0, 0)$, this generates a Laplace operator on the transverse component of the electric field. It is worth emphasizing that one can choose in advance the operator (and the corresponding boundary conditions) on the transverse component of the electric field by appropriately choosing the way to connect the upper faces to each other.

### 2.4.4. *General Comments*

In the sequel, we often use common formulations for the above three boundary conditions. To do so, we need to define $h^b$, $\varphi_c^b$, $G$, and $G_1$ for all the models with the conventions

$$h^b = 0 \text{ for Dirichlet conditions}, \ \varphi_c^b = 0 \text{ for mixed conditions}, \tag{12}$$

$$\begin{cases} G \text{ and } G_1 \text{ are two given nonnegative constants}, \\ G = G_1 = 0 \quad \text{for Dirichlet conditions}, \\ G_1 = 0 \qquad \text{for local mixed conditions}, \\ GG_1 > 0 \qquad \text{for nonlocal mixed conditions}. \end{cases} \tag{13}$$

Unlike [4], we do not treat separately the case of Neumann conditions. From a mathematical point of view, it does not differ from the case of local mixed conditions. One simply has to set $G = 0$ in the local mixed condition to obtain the corresponding model.

Since all the faces are metallized, $\bar{\varphi}^b$, $\varphi_m^b$, and $\bar{\varphi}_c^b$ are constant on each face of each inclusion. Also, because the current is provided by a single wire, the same property holds for $h^b$.

The relevance of the scaling $(G/a)^{-1}$ and $(G_1/(a\varepsilon^2))^{-1}$ on the capacities will become apparent in the next sections. It will clearly indicate, according to inclusions and thickness, the type of circuit that must be chosen to obtain a significant effect on the global behaviour of the plate.

## 3. Weak Formulations

Notation, equations, and boundary conditions were stated in the previous sections. The aim of the present section is the formulation, on the fixed domain $\Omega$, of the corresponding weak formulations. The effective models are deduced from these weak formulations (18) in the next two sections.

### 3.1. SCALING OF THE EQUATIONS

Using the standard change of variables $x^a \to x = {}^t(x_1, x_2, x_3) = {}^t(x_1, x_2, x_3^a/a)$, equations of Section 2 are reformulated on $\Omega = \omega \times ]-1, 1[$. As already mentioned, the geometrical notation for the domain $\Omega$ is the same as for $\Omega^a$, the index $a$ being removed when necessary. The corresponding scaling for volume forces, surface forces, and displacements is classical [6]:

$$\begin{cases} \hat{\mathbf{u}}^b(x) = \big(u_1^b(x^a), u_2^b(x^a), a u_3^b(x^a)\big) & \text{in } \Omega, \\ \hat{\mathbf{f}}^b(x) = \big(f_1^b(x^a), f_2^b(x^a), a^{-1} f_3^b(x^a)\big) & \text{in } \Omega, \\ \hat{\mathbf{g}}^b(x) = \big(g_1^b(x^a), g_2^b(x^a), a^{-1} g_3^b(x^a)\big) & \text{on } \gamma_N \times ]-1, 1[, \\ \hat{\mathbf{g}}^b(x) = a^{-1}\big(g_1^b(x^a), g_2^b(x^a), a^{-1} g_3^b(x^a)\big) & \text{on } \omega \times \{-1, 1\}. \end{cases}$$

The current sources $h^b$, the electric potentials $\varphi^b$, $\varphi_m^b$, and $\varphi_c^b$ are unchanged. As in the sequel we only work on the reference domain $\Omega$, we use again, for simplicity, the notation $\mathbf{u}^b$, $\mathbf{f}^b$, $\mathbf{g}^b$, $h^b$, $\varphi^b$, $\varphi_m^b$, and $\varphi_c$, without hats.

For $\mathbf{V} = (\mathbf{v}, \psi) \in \mathbf{H}^1(\Omega) \times H^1(\Omega_1^\varepsilon)$, we define the scaled strain tensor and the scaled electric field $\mathbf{K}^a(\mathbf{v}) = (K_{ij}^a(\mathbf{v}))_{i,j=1,2,3}$ and $\mathbf{L}^a(\varphi) = (L_i^a(\varphi))_{i=1,2,3}$ by

$$\begin{cases} K_{\alpha\beta}^a(\mathbf{v}) = s_{\alpha\beta}(\mathbf{v}) & \text{for } \alpha, \beta = 1, 2, \\ K_{3\alpha}^a(\mathbf{v}) = K_{\alpha 3}^a(\mathbf{v}) = a^{-1} s_{\alpha 3}(\mathbf{v}) & \text{for } \alpha = 1, 2, \\ K_{33}^a(\mathbf{v}) = a^{-2} s_{33}(\mathbf{v}), & \\ L_\alpha^a(\varphi) = \partial_\alpha \varphi & \text{for } \alpha = 1, 2 \text{ and} \\ L_3^a(\varphi) = a^{-1} \partial_3 \varphi, & \end{cases} \tag{14}$$

where $\partial_3$ represents now $\partial/\partial x_3$. We also use the global notation

$$\mathbf{M}^a(\mathbf{V}) = {}^t\Big(\big(K_{\alpha\beta}^a(\mathbf{v})\big)_{\alpha,\beta=1,2}, \big(K_{\alpha 3}^a(\mathbf{v})\big)_{\alpha=1,2}, K_{33}^a(\mathbf{v}), \big(L_\alpha^a(\psi)\big)_{\alpha=1,2}, L_3^a(\psi)\Big). \tag{15}$$

### 3.2. WEAK FORMULATION

We put together the tensors $\mathbf{R}^\varepsilon$, $\mathbf{d}^\varepsilon$, and $\mathbf{c}^\varepsilon$ in a global stiffness-piezoelectricity-permitivity tensor $\mathcal{R}^\varepsilon$, which is the $10 \times 10$ symmetric matrix written in a format compatible with (15):

$$\begin{pmatrix} (R^\varepsilon_{\alpha\beta\gamma\delta})_{\alpha,\beta,\gamma,\delta=1,2} & (2R^\varepsilon_{\alpha\beta\gamma 3})_{\alpha,\beta,\gamma=1,2} & (R^\varepsilon_{\alpha\beta 33})_{\alpha,\beta=1,2} & (d^\varepsilon_{\gamma\alpha\beta})_{\alpha,\beta,\gamma=1,2} & (d^\varepsilon_{3\alpha\beta})_{\alpha\beta=1,2} \\ (2R^\varepsilon_{\alpha 3\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (4R^\varepsilon_{\alpha 3\gamma 3})_{\alpha,\gamma=1,2} & (2R^\varepsilon_{\alpha 333})_{\alpha=1,2} & (2d^\varepsilon_{\gamma\alpha 3})_{\alpha,\gamma=1,2} & (2d^\varepsilon_{3\alpha 3})_{\alpha=1,2} \\ (R^\varepsilon_{33\gamma\delta})_{\gamma,\delta=1,2} & (2R^\varepsilon_{33\gamma 3})_{\gamma=1,2} & R^\varepsilon_{3333} & (d^\varepsilon_{\gamma 33})_{\gamma=1,2} & d^\varepsilon_{333} \\ (-d^\varepsilon_{\alpha\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (-2d^\varepsilon_{\alpha\gamma 3})_{\alpha,\gamma=1,2} & (-d^\varepsilon_{\alpha 33})_{\alpha=1,2} & (c^\varepsilon_{\alpha\gamma})_{\alpha,\gamma=1,2} & (c^\varepsilon_{\alpha 3})_{\alpha=1,2} \\ (-d^\varepsilon_{3\gamma\delta})_{\gamma,\delta=1,2} & (-2d^\varepsilon_{3\gamma 3})_{3,\gamma=1,2} & -d^\varepsilon_{333} & (c^\varepsilon_{3\gamma})_{\gamma=1,2} & c^\varepsilon_{33} \end{pmatrix}. \tag{16}$$

The linear forms associated with the mechanical and electric loads are

$$l_u^b(\mathbf{v}) = \int_\Omega f_i^b v_i \,\mathrm{d}x + \int_{\Gamma_N} g_i^b v_i \,\mathrm{d}s, \quad \text{and} \quad l_\varphi^b\big(\widetilde{L}_3\big) = \int_{\Omega_1^\varepsilon} h^b \widetilde{L}_3 \,\mathrm{d}x.$$

Given the assumption on metallization, the set of admissible electric potentials is chosen by

$$H_c^1(\Omega_1^\varepsilon) = \{\psi \in H^1(\Omega_1^\varepsilon);\ \psi \text{ is constant in each connected part of } \Gamma_1^{\varepsilon+} \cup \Gamma_1^{\varepsilon-}\}. \tag{17}$$

The Hilbert spaces $\mathbf{W}^\varepsilon$ and $\mathbf{W}_D^b$ are defined by

(i) *Dirichlet conditions*:

$$\begin{cases} \mathbf{W}_D^b = \mathbf{W}^\varepsilon(\varphi_m^b, \varphi_c^b) \\ \quad := \{(\mathbf{v}, \varphi) \in \mathbf{H}^1(\Omega) \times H_c^1(\Omega_1^\varepsilon);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \\ \qquad \varphi = \varphi_m^b \pm a\varphi_c^b \text{ on } \Gamma_1^{\varepsilon\pm}\}, \\ \mathbf{W}^\varepsilon = \mathbf{W}^\varepsilon(0, 0). \end{cases}$$

(ii) *Mixed conditions*:

$$\mathbf{W}_D^b = \mathbf{W}^\varepsilon(\varphi_m^b) := \{(\mathbf{v}, \varphi) \in \mathbf{H}^1(\Omega) \times H_c^1(\Omega_1^\varepsilon);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D,$$
$$\varphi = \varphi_m^b \text{ on } \Gamma_1^{\varepsilon-}\},$$
$$\mathbf{W}^\varepsilon = \mathbf{W}^\varepsilon(0).$$

The backward difference operator $\nabla_{\hat{x}}^\varepsilon$ is defined inclusion by inclusion by

$$(\nabla_{\hat{x}}^\varepsilon \psi)_{\mathbf{i}} = \varepsilon^{-1}\, {}^t(\psi_{\mathbf{i}} - \psi_{T_{-1}^1(\mathbf{i})}, \psi_{\mathbf{i}} - \psi_{T_{-1}^2(\mathbf{i})}) \quad \forall \mathbf{i} \in \mathbb{I}^\varepsilon\ \forall \psi \in H_c^1(\Omega_1^\varepsilon).$$

The weak formulations on the scaled domain $\Omega$ for the coupled problems (3)–(5) and (8)–(11), with the conventions (7), (10), (12), (13), are then summarized by:

$$\begin{cases} \int_\Omega {}^t\mathbf{M}^a(\mathbf{V})\mathcal{R}^\varepsilon\mathbf{M}^a(\mathbf{U}^b)\mathrm{d}x + 2\int_{\Omega_1^\varepsilon} G\mathcal{M}(L_3^a(\varphi^b))\mathcal{M}(L_3^a(\psi))\mathrm{d}x \\ \quad + 2\int_{\Omega_1^\varepsilon} G_1\nabla_{\hat{x}}^\varepsilon\mathcal{M}(L_3^a(\varphi^b)) \cdot \nabla_{\hat{x}}^\varepsilon\mathcal{M}(L_3^a(\psi))\mathrm{d}x = l_u^b(\mathbf{v}) + l_\varphi^b(L_3^a(\psi)) \\ \forall \mathbf{V} = (\mathbf{v}, \psi) \in \mathbf{W}^\varepsilon, \quad \text{with } \mathbf{U}^b = (\mathbf{u}^b, \varphi^b) \in \mathbf{W}_D^b. \end{cases} \tag{18}$$

The mean operator $\mathcal{M}$ is defined in (1). We used (6) to reorganize the first term and the relations

$$\bar{\varphi}^b_{|\Gamma_1^{\varepsilon+}} = \varphi^b_{|\Gamma_1^{\varepsilon+}} - \varphi^b_{|\Gamma_1^{\varepsilon-}} = 2\mathcal{M}(\partial_3\varphi^b) \quad \text{and}$$
$$\psi_{|\Gamma_1^{\varepsilon+}} = 2\mathcal{M}(\partial_3\psi) \quad \text{for the other terms.}$$

REMARK 3.1. It is worth pointing out that, for mixed conditions, the connection of the two faces of each inclusion introduces the mean value of the transverse component of the electric field in the equations.

In (18) and throughout the paper, the products between matrices have to be understood as bloc matrices products, where in each bloc Einstein's convention of summation is used. For example, $\mathcal{R}^\varepsilon \mathbf{M}^a(\mathbf{U}^b)$ is equal to

$$\begin{pmatrix} \left(R^\varepsilon_{\alpha\beta\gamma\delta}K^a_{\gamma\delta}(\mathbf{u}^b) + 2R^\varepsilon_{\alpha\beta\gamma 3}K^a_{\gamma 3}(\mathbf{u}^b) + R^\varepsilon_{\alpha\beta 33}K^a_{33}(\mathbf{u}^b) + d^\varepsilon_{\gamma\alpha\beta}L^a_\gamma(\varphi^b) + d^\varepsilon_{3\alpha\beta}L^a_3(\varphi^b)\right)_{\alpha,\beta=1,2} \\ \left(2R^\varepsilon_{\alpha 3\gamma\delta}K^a_{\gamma\delta}(\mathbf{u}^b) + 4R^\varepsilon_{\alpha 3\gamma 3}K^a_{\gamma 3}(\mathbf{u}^b) + 2R^\varepsilon_{\alpha 333}K^a_{33}(\mathbf{u}^b) + 2d^\varepsilon_{\gamma\alpha 3}L^a_\gamma(\varphi^b) + 2d^\varepsilon_{3\alpha 3}L^a_3(\varphi^b)\right)_{\alpha,\beta=1,2} \\ R^\varepsilon_{33\gamma\delta}K^a_{\gamma\delta}(\mathbf{u}^b) + 2R^\varepsilon_{33\gamma 3}K^a_{\gamma 3}(\mathbf{u}^b) + R^\varepsilon_{3333}K^a_{33}(\mathbf{u}^b) + d^\varepsilon_{\gamma 33}L^a_\gamma(\varphi^b) + d^\varepsilon_{333}L^a_3(\varphi^b) \\ \left(-d^\varepsilon_{\alpha\gamma\delta}K^a_{\gamma\delta}(\mathbf{u}^b) - 2d^\varepsilon_{\alpha\gamma 3}K^a_{\gamma 3}(\mathbf{u}^b) - d^\varepsilon_{\alpha 33}K^a_{33}(\mathbf{u}^b) + c^\varepsilon_{\alpha\gamma}L^a_\gamma(\varphi^b) + c^\varepsilon_{\alpha 3}L^a_3(\varphi^b)\right)_{\alpha=1,2} \\ -d^\varepsilon_{3\gamma\delta}K^a_{\gamma\delta}(\mathbf{u}^b) - 2d^\varepsilon_{3\gamma 3}K^a_{\gamma 3}(\mathbf{u}^b) - d^\varepsilon_{333}K^a_{33}(\mathbf{u}^b) + c^\varepsilon_{3\gamma}L^a_\gamma(\varphi^b) + c^\varepsilon_{33}L^a_3(\varphi^b) \end{pmatrix}.$$

## 4. Assumptions on the Data. A Priori Estimates. Convergences

The aim of this section is twofold. The detailed assumptions on the data are stated in 4.1. The resulting a priori estimates and first convergence results are given in 4.2.

### 4.1. ASSUMPTIONS ON THE DATA

We use in this paper the notion of two-scale convergence of Allaire [1] and Nguentseng [9]. Since we also need two-scale convergence for functions defined on $\Omega_1^\varepsilon$, we use the following practical definition.

DEFINITION 4.1. A sequence $(\psi^b)$ of $L^2(\Omega_1^\varepsilon)$ is said to two-scale converge to a limit $\psi$ in $L^2(\Omega \times Y_1)$ if $\psi \in L^2(\Omega \times Y_1)$ and if $(P^\varepsilon \psi^b)$ two-scale converges to $P\psi$ in $L^2(\Omega \times Y)$, where $P^\varepsilon$ and $P$ denote the extension by 0 from $\Omega_1^\varepsilon$ to $\Omega$ and from $\Omega \times Y_1$ to $\Omega \times Y$, respectively.

In addition to the standard symmetry assumptions (6), the tensors $\mathbf{R}^\varepsilon$, $\mathbf{d}^\varepsilon$, and $\mathbf{c}^\varepsilon$ constituting the stiffness-piezoelectricity-permitivity tensor $\mathcal{R}^\varepsilon$ are assumed to satisfy

$$\begin{cases} \left(\mathcal{R}^\varepsilon\right) \text{ two-scale converges in } \mathbf{L}^2(\Omega \times Y) \text{ to some limit } \mathcal{R} \in \mathbf{L}^\infty(\Omega \times Y), \\ \|\mathcal{R}^\varepsilon\|_{\mathbf{L}^\infty(\Omega)} \leqslant C, \quad \mathcal{R}^\varepsilon \text{ does not depend on } x_3, \\ \lim_{\varepsilon\to 0}\left\|\mathcal{R}^\varepsilon\right\|_{\mathbf{L}^2(\Omega)} = \|\mathcal{R}\|_{\mathbf{L}^2(\Omega\times Y)}, \\ {}^t\mathbf{K}\mathbf{R}^\varepsilon\mathbf{K} \geqslant c\|\mathbf{K}\|^2 \quad \forall \mathbf{K} \in \mathbb{R}^9 \text{ with } K_{ij} = K_{ji}, \text{ a.e. in } \omega, \\ {}^t\mathbf{L}\mathbf{c}^\varepsilon\mathbf{L} \geqslant c\|\mathbf{L}\|^2 \quad \forall \mathbf{L} \in \mathbb{R}^3, \text{ a.e. in } \omega_1^\varepsilon. \end{cases} \tag{19}$$

Here and throughout the paper, $c$ and $C$ designate generic positive constants, not depending on $a$ and $\varepsilon$.

REMARK 4.1. In view of the symmetry relations $d^\varepsilon_{ijk} = d^\varepsilon_{ikj}$, coercivity for $\mathbf{c}^\varepsilon$ and $\mathbf{R}^\varepsilon$ implies coercivity for $\mathcal{R}^\varepsilon$. Conversely, two-scale convergence for $\mathcal{R}^\varepsilon$ implies two-scale convergence for $\mathbf{R}^\varepsilon$, $\mathbf{c}^\varepsilon$, and $\mathbf{d}^\varepsilon$. The corresponding limits are naturally denoted by $\mathbf{R}$, $\mathbf{c}$, and $\mathbf{d}$.

The mechanical forces are assumed to satisfy

$$\begin{cases} \mathbf{f}^b \in \mathbf{L}^2(\Omega), \mathbf{g}^b \in \mathbf{H}^{1/2}(\Gamma_N), \\ (\mathbf{f}^b) \text{ converges weakly in } \mathbf{L}^2(\Omega) \text{ to some limit } \mathbf{f}, \\ (\mathbf{g}^b) \text{ converges weakly in } \mathbf{L}^2(\Gamma_N) \text{ to some limit } \mathbf{g}. \end{cases} \tag{20}$$

The assumptions relative to the electrical boundary conditions are:

$$\begin{cases} h^b, \varphi_m^b, \text{ and } \varphi_c^b \text{ are constant on each inclusion}, \\ (h^b) \text{ two-scale converges in } L^2(\omega \times Y_1) \text{ to some limit } h \in L^2(\omega), \\ (\varphi_m^b) \text{ two-scale converges in } L^2(\omega \times Y_1) \text{ to some limit } \varphi_m \in H^1(\omega), \\ (\varphi_c^b) \text{ two-scale converges in } L^2(\omega \times Y_1) \text{ to some limit } \varphi_c \in L^2(\omega). \end{cases} \tag{21}$$

REMARK 4.2. Since $\varphi_c^b$, $\varphi_m^b$, and $h^b$ are constant on each inclusion, their two-scale limits do not depend on $\mathbf{y}$ in $Y_1$.

Let us recall that the convention (12)–(13) have been chosen to define $h^b$, $\varphi_c^b$, $G$, and $G_1$ for all the models.

## 4.2. A PRIORI ESTIMATES. CONVERGENCES

Let us introduce the space of Kirchhoff–Love's displacement fields

$$\mathbf{V}_{KL} = \left\{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D,\ (s_{i3}(\mathbf{v}))_{i=1,2,3} = \mathbf{0}\right\},$$

or equivalently,

$$\begin{aligned} \mathbf{V}_{KL} = &\left\{{}^t(\bar{v}_1 - x_3\partial_1 v_3, \bar{v}_2 - x_3\partial_2 v_3, v_3); \bar{v}_1, \bar{v}_2 \in H^1(\omega), v_3 \in H^2(\omega),\right. \\ &\left.\bar{v}_1 = \bar{v}_2 = v_3 = 0 \text{ on } \Gamma_D\right\}. \end{aligned}$$

In the sequel, for $\mathbf{v} \in \mathbf{V}_{KL}$, we frequently use the practical notation $\bar{\mathbf{v}} = {}^t(\bar{v}_1, \bar{v}_2)$.

A priori estimates and the resulting convergence results for the sequence $(\mathbf{u}^b, \varphi^b)$ are summarized in the following lemma. The convergence statements hold a priori for a subsequence. However, since we see with hindsight (from uniqueness of the solution to the limit problem) that the complete sequences converge, we omit to mention the extractions of subsequences.

LEMMA 4.1. *If assumptions* (6), (19)–(21) *and conventions* (12), (13) *hold, then for sufficiently small* $b$:

(i) *for each fixed* $b$ *there is a unique solution to problem* (18);

(ii) $\|\mathbf{K}^a(\mathbf{u}^b)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{L}^a(\varphi^b)\|_{\mathbf{L}^2(\Omega_1^\varepsilon)} + G_1\|\nabla_{\hat{x}}^\varepsilon \mathcal{M}(L_3^a(\varphi^b))\|_{\mathbf{L}^2(\Omega_1^\varepsilon)} \leqslant C$;

(iii) *there exists* $\mathbf{M} = (\mathbf{K}, \mathbf{L}) \in (L^2(\Omega\times Y))^7 \times (L^2(\Omega\times Y_1))^3$ *such that* $(\mathbf{M}^a(\mathbf{u}^b))$ *two-scale converges to* $\mathbf{M}$ *in* $\mathbf{L}^2(\Omega \times Y) \times \mathbf{L}^2(\Omega \times Y_1)$;

(iv) *there exists* $\mathbf{u} \in \mathbf{V}_{KL}$ *and* $\mathbf{u}^1 = {}^t(u_1^1, u_2^1, 0)$ *with* $u_1^1, u_2^1 \in L^2(\Omega; H_\sharp^1(Y)/\mathbb{R})$, *such that* $(\mathbf{u}^b)$ *converges weakly to* $\mathbf{u}$ *in* $\mathbf{H}^1(\Omega)$, $(\nabla_{\hat{x}}\mathbf{u}^b)$ *and* $(\partial_3\mathbf{u}^b)$ *two-scale converge to* $\nabla_{\hat{x}}\mathbf{u} + \nabla_y\mathbf{u}^1$ *and* $\partial_3\mathbf{u}$, *respectively, in* $\mathbf{L}^2(\Omega \times Y)$;

(v) $(\varphi^b)$ *two-scale converges to* $\varphi_m$ *in* $L^2(\Omega \times Y_1)$;

(vi) *there exists* $\varphi^1 \in L^2(\Omega; H^1(Y_1))$ *such that* ${}^t(L_1, L_2) = \nabla_y \varphi^1$;

(vii) $\mathcal{M}(L_3)$ *is independent of* $y$, *and for Dirichlet conditions* $\mathcal{M}(L_3) = \varphi_c$;

(viii) *In the case of nonlocal mixed conditions,* $\mathcal{M}(L_3) \in H^1(\omega)$ *and* $(\nabla^\varepsilon_{\hat{x}} \mathcal{M}(L^a_3(\varphi^b)))$ *two-scale converge to* $\nabla_{\hat{x}} \mathcal{M}(L_3)$ *in* $\mathbf{L}^2(\Omega \times Y_1)$.

*Proof.* Point (i) is a direct application of Lax–Milgram's lemma. Point (ii) is obtained with standard arguments by choosing $(\mathbf{v}, \psi) = (\mathbf{u}^b, \varphi^b - (\varphi^b_m + a x_3 \varphi^b_c))$ in the case of Dirichlet conditions, $(\mathbf{v}, \psi) = (\mathbf{u}^b, \varphi^b - \varphi^b_m)$, otherwise. Point (iii) is a direct consequence of (ii).

Let us prove (iv). First, as $(\mathbf{K}^a(\mathbf{u}^b))$ is bounded in $\mathbf{L}^2(\Omega)$, Korn's inequality implies that $(\mathbf{u}^b)$ is bounded in $\mathbf{H}^1(\Omega)$. Then, from [1, Proposition 1.14], there exists $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\mathbf{u}^1 = {}^t(u^1_1, u^1_2, u^1_3) \in \mathbf{L}^2(\Omega; \mathbf{H}^1_\sharp(Y)/\mathbb{R})$, such that (after extraction of a subsequence, if necessary) $(\mathbf{u}^b)$ converges weakly to $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$, $(\nabla_{\hat{x}} \mathbf{u}^b)$ two-scale converges to $\nabla_{\hat{x}} \mathbf{u} + \nabla_y \mathbf{u}^1$ in $\mathbf{L}^2(\Omega \times Y)$, $(\partial_3 \mathbf{u}^b)$ two-scale converges to $\partial_3 \mathbf{u}$ in $\mathbf{L}^2(\Omega \times Y)$. Now, the sequences $(K^a_{i3}(\mathbf{u}^b))$ being bounded, in view of (14), the sequences $(s_{i3}(\mathbf{u}^b))$ strongly tend to 0. As they also converge to $(s_{i3}(\mathbf{u}))$, this proves $\mathbf{u} \in \mathbf{V}_{KL}$.

To complete the proof of (iv), it remains for us to show that $u^1_3$ may be taken as equal to zero. We re-use the fact that the sequences $(s_{\alpha 3}(\mathbf{u}^b))$ strongly converge to 0. Strong convergence implies two-scale convergence, and thus, $s_{\alpha 3}(\mathbf{u}) + \partial_{y_\alpha} u^1_3 = \partial_{y_\alpha} u^1_3 = 0$. Then, because $u^1_3$ is defined up to the function of $x$, we are free to choose $u^1_3 = 0$.

To prove (v) we note that $(\varphi^b)$ is bounded in $L^2(\Omega^\varepsilon_1)$ (see (ii)). Thus, $(\varphi^b)$ two-scale converges to some limit $\varphi$ in $L^2(\Omega \times Y_1)$. As $(L^a_3(\varphi^b)) = (a^{-1} \partial_3 \varphi^b)$ is bounded, $\varphi$ does not depend on $x_3$. Similarly, as the quantities $(L^a_\alpha(\varphi^b)) = (\partial_\alpha \varphi^b)$ are bounded, $\varphi$ does not depend on $y$. Hence, $\varphi \in L^2(\omega)$. Now, to compute $\varphi$, we only need to pass to the limit in

$$\int_{\Omega^\varepsilon_1} \partial_3 \varphi^b \psi (x_3 - 1)\, \mathrm{d}x = -\int_{\Omega^\varepsilon_1} \varphi^b \psi \, \mathrm{d}x + 2 \int_{\omega^\varepsilon_1} \left(\varphi^b_m - a\varphi^b_c\right) \psi \, \mathrm{d}\hat{x} \quad \forall \psi \in H^1(\omega).$$

The left-hand side tends to 0 because $(L^a_3(\varphi^b))$ is bounded and therefore $\varphi = \varphi_m$.

To prove (vi), we use the identity

$$\int_{\Omega^\varepsilon_1} \nabla_{\hat{x}} \varphi^b \cdot \psi^\varepsilon \mathrm{d}x = -\int_{\Omega^\varepsilon_1} \varphi^b (\mathrm{div}_{\hat{x}} \psi)^\varepsilon \mathrm{d}x - \int_{\Omega^\varepsilon_1} \frac{\varphi^b}{\varepsilon} (\mathrm{div}_y \psi)^\varepsilon \mathrm{d}x + \int_{\Gamma^\varepsilon_1} \varphi^b \psi^\varepsilon \cdot \mathbf{n} \, \mathrm{d}\sigma^\varepsilon \quad \forall \psi \in D(\Omega \times Y),$$

where $\psi^\varepsilon$ denotes the function $x \mapsto \psi(x, \hat{x}/\varepsilon)$. We choose $\psi$ such that $\mathrm{div}_y \psi = 0$ in $\Omega \times Y_1$ and $\psi \cdot \mathbf{n}_Y = 0$ in $\Omega \times \partial Y_1$, and pass to the limit as $b \to \mathbf{0}$. With (v), we get

$$\int_{\Omega \times Y_1} \left({}^t(L_1, L_2) - \nabla_{\hat{\mathbf{x}}} \varphi_m\right) \cdot \psi \, \mathrm{d}x \mathrm{d}y = 0.$$

This proves that ${}^t(L_1, L_2) - \nabla_{\hat{x}}\varphi_m$ is a gradient with respect to $y$. Remarking that $\nabla_{\hat{x}}\varphi_m = \nabla_y(\mathbf{y} \cdot \nabla_{\hat{x}}\varphi_m)$, this proves that ${}^t(L_1, L_2)$ is a gradient with respect to $y$.

The first part of (vii) is obtained by remarking that $\mathcal{M}(L_3^a(\varphi^b)) = a^{-1}\mathcal{M}(\partial_3\varphi^b)$ $= a^{-1}(\varphi^b_{|\Gamma^{\varepsilon+}} - \varphi^b_{|\Gamma^{\varepsilon-}})$ is constant on each inclusion (because $\varphi^b \in H^1_c(\Omega_1^\varepsilon)$). Hence, its two-scale limit does not depend on $y$.

For Dirichlet conditions, (8) also implies that $\mathcal{M}(L_3^a(\varphi^b)) = \varphi^b_c$. Hence, passing to the limit: $\mathcal{M}(L_3) = \varphi_c$.

Let us prove (viii). Let $\xi$ designate the two-scale limit of $(\nabla^\varepsilon_{\hat{x}}\mathcal{M}(L_3^a(\varphi^b)))$. Let $\psi \in \mathcal{D}(\Omega \times Y_1)$. For $\varepsilon$ small enough, $\psi^\varepsilon(x) = \psi(x, \hat{x}/\varepsilon)$ vanishes in all noninternal inclusions. Then, the following integration by parts formula holds:

$$\varepsilon^{-1}\int_{\Omega_1^\varepsilon}\sum_{\mathbf{i}\in\mathbb{I}^\varepsilon}\mathcal{M}\big(L_3^a\big(\varphi^b_{\mathbf{i}} - \varphi^b_{T^\alpha_{-1}(\mathbf{i})}\big)\big)\psi^\varepsilon_{\mathbf{i}}\,\mathrm{d}x$$
$$= \varepsilon^{-1}\int_{\Omega_1^\varepsilon}\sum_{\mathbf{i}\in\mathbb{I}^\varepsilon}\mathcal{M}\big(L_3^a\big(\varphi^b_{\mathbf{i}}\big)\big)\big(\psi^\varepsilon_{\mathbf{i}} - \psi^\varepsilon_{T^\alpha_{+1}(\mathbf{i})}\big)\mathrm{d}x$$

for $\alpha = 1, 2$. Passing to the limit, as $\psi$ is regular, this yields

$$\int_{\Omega\times Y_1}\xi_\alpha\psi\,\mathrm{d}x\mathrm{d}y = -\int_{\Omega\times Y_1}\mathcal{M}(L_3)\partial_\alpha\psi\,\mathrm{d}x, \quad \alpha = 1, 2.$$

This proves that $\mathcal{M}(L_3) \in H_1(\omega)$ and $\xi_\alpha = \partial_\alpha\mathcal{M}(L_3)$. □

## 5. Main Result: Limit Models

This section is devoted to the presentation of the effective 2-dimensional plate models. They are obtained by letting $a$, $\varepsilon$, $\varepsilon/a$ tend simultaneously to 0 in (18). As a unique asymptotic situation arises here, we subsequently know that the same models would be obtained by deriving the first 3-dimensional homogenized equations by letting $\varepsilon$ tend to 0 ($a$ fixed), and then applying the asymptotic method in the plate theory as $a \to 0$.

The derivation of limit models is made up of three steps. The first one, the most difficult mathematically, consists in characterizing the limits $(\mathbf{K}, \mathbf{L})$ defined in Lemma 4.1. The second step consists, as usual, in linear homogenization, in eliminating the local variable $y$. This is realized by computing the microscopic fields (depending on $y$) with respect to the macroscopic fields (depending only on $x$). The third step consists in eliminating (part of) the transverse components of the fields (homogenized strains and the electric field) that we compute with respect to the other components. This elimination differs from the classical plate theory because of the nonstandard boundary conditions on the faces of inclusions.

The notation related to step 2 is presented in Section 5.1. The notation related to step 3 is presented in Section 5.2. The effective models are then summarized by Theorem 5.1, in Section 5.3. The proof of Theorem 5.1 is postponed to Section 6.

Let us stress again that our approach allows a synthetic and readable presentation of our results. Compare, for example, the notation of Section 5.2 below, with the expanded expression presented in Section 7 for a transversally isotropic material. Our notation is also practical: the calculations of Section 7 (based on the general notation of Section 5) have been worked out using Mathematica.

### 5.1. NOTATION RELATED TO HOMOGENIZATION

Similarly to (2), let

$$S_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(\partial_{y_\alpha} v_\beta + \partial_{y_\beta} v_\alpha), \qquad S_{\alpha 3}(\mathbf{v}) = \frac{1}{2}\partial_{y_\alpha} v_3, \quad \alpha, \beta = 1, 2. \tag{22}$$

Let us define

$$Z = \{(1,1), (1,2), (2,1), (2,2), (1,3), (2,3), (3,3), 3\}.$$

The local variables $(\mathbf{u}^i, \varphi^i) \in (H^1_\sharp(Y))^3 \times H^1(Y_1)$, needed to compute the homogenized elasticity-piezoelectricity-permitivity tensor $\mathcal{R}^H$, are defined, for each $i \in \mathbb{Z}$, as the solutions of the local problems:

$$\begin{aligned}
&\int_Y (S_{\alpha\beta}(\mathbf{v}), S_{\alpha 3}(v_3), \partial_{y_\alpha}\psi) \begin{pmatrix} R_{\alpha\beta\gamma\delta} & 2R_{\alpha\beta\gamma 3} & d_{\gamma\alpha\beta} \\ 2R_{\alpha 3\gamma\delta} & 4R_{\alpha 3\gamma 3} & 2d_{\gamma\alpha 3} \\ -d_{\alpha\gamma\delta} & -2d_{\alpha\gamma 3} & c_{\alpha\gamma} \end{pmatrix} \begin{pmatrix} S_{\gamma\delta}(\mathbf{u}^i) \\ S_{\gamma 3}(u_3^i) \\ \partial_{y_\gamma}\varphi^i \end{pmatrix} \mathrm{d}y \\
&\quad = \int_Y (S_{\alpha\beta}(\mathbf{v}), S_{\alpha 3}(\mathbf{v}), \partial_{y_\alpha}\psi) \begin{pmatrix} R_{\alpha\beta\gamma\delta} & 2R_{\alpha\beta\gamma 3} & R_{\alpha\beta 33} & d_{\gamma\alpha\beta} & d_{3\alpha\beta} \\ 2R_{\alpha 3\gamma\delta} & 4R_{\alpha 3\gamma 3} & 2R_{\alpha 333} & 2d_{\gamma\alpha 3} & 2d_{3\alpha 3} \\ -d_{\alpha\gamma\delta} & -2d_{\alpha\gamma 3} & -d_{\alpha 33} & c_{\alpha\gamma} & c_{\alpha 3} \end{pmatrix} \\
&\quad\quad \times \begin{pmatrix} \delta_{i,\gamma\delta} \\ 0_\gamma \\ \delta_{i,33} \\ \delta_{i,\gamma} \\ \delta_{i,3} \end{pmatrix} \mathrm{d}y \quad \forall (\mathbf{v}, \psi) \in \left(H^1_\sharp(Y)\right)^3 \times H^1(Y_1),
\end{aligned} \tag{23}$$

where $\delta_{i,j}$ is the Kronecker symbol for $i, j \in \mathbb{Z}$.

The tensor $\mathcal{L}$, stored in a format compatible with $\mathcal{R}$, is defined as

$$\mathcal{L} = \begin{pmatrix} (S_{\alpha\beta}(\mathbf{u}^{\mu\rho}))_{\alpha,\beta,\mu,\rho=1,2} & (S_{\alpha\beta}(\mathbf{u}^{\mu 3}))_{\alpha,\beta,\mu=1,2} & (S_{\alpha\beta}(\mathbf{u}^{33}))_{\alpha,\beta=1,2} & \mathbf{0}_{2\times 2\times 2} & (S_{\alpha\beta}(\mathbf{u}^{3}))_{\alpha,\beta=1,2} \\ (S_{\alpha 3}(\mathbf{u}^{\mu\rho}))_{\alpha,\mu,\rho=1,2} & (S_{\alpha 3}(\mathbf{u}^{\mu 3}))_{\alpha,\mu=1,2} & (S_{\alpha 3}(\mathbf{u}^{33}))_{\alpha=1,2} & \mathbf{0}_{2\times 2} & (S_{\alpha 3}(\mathbf{u}^{3}))_{\alpha=1,2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 0 \\ (\partial_{y_\alpha}\varphi^{\mu\rho})_{\alpha,\mu,\rho=1,2} & (\partial_{y_\alpha}\varphi^{\mu 3})_{\alpha,\mu=1,2} & (\partial_{y_\alpha}\varphi^{33})_{\alpha=1,2} & \mathbf{0}_{2\times 2} & (\partial_{y_\alpha}\varphi^{3})_{\alpha=1,2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 0 \end{pmatrix}.$$

The homogenized stiffness-piezoelectricity-permitivity coefficients are then given by

$$\mathcal{R}^H = \int_Y \left(\mathbf{Id} + {}^t\mathcal{L}\right)\mathcal{R}(\mathbf{Id} + \mathcal{L})\,\mathrm{d}y. \tag{24}$$

### 5.2. NOTATION RELATED TO PLATE THEORY

The notation is:

$$\left\{\begin{array}{l}\Pi \text{ and } \Pi_1 \text{ are the projections from } (L^2(\Omega))^{10} \text{ onto its subspaces of the form} \\ {}^t\big(\mathbf{0}_4, (K_{i3})_{i=1,2,3} \text{ and } \mathbf{0}_2, L_3\big), \quad {}^t(\mathbf{0}_9, L_3), \text{ respectively}, \quad \Pi_2 = \Pi - \Pi_1, \\ \mathbf{T}_{\mathcal{N}} = -\big(\Pi \mathcal{R}^H \Pi\big)^{-1} \Pi \mathcal{R}^H, \\ \mathbf{T}_{\mathcal{M}} = -\big(\Pi_2 \mathcal{R}^H \Pi_2\big)^{-1} \Pi_2 \mathcal{R}^H \quad \text{for Dirichlet and nonlocal conditions}, \\ \mathbf{T}_{\mathcal{M}} = -(\Pi \mathcal{R}^H \Pi + 2G\Pi_1)^{-1} \Pi \mathcal{R}^H \quad \text{for local mixed conditions}, \\ \mathcal{R}_{\mathcal{N}} = \big(Id + {}^t\mathbf{T}_{\mathcal{N}}\big) \mathcal{R}^H (Id + \mathbf{T}_{\mathcal{N}}), \\ \mathcal{R}_{\mathcal{M}} = \big(Id + {}^t\mathbf{T}_{\mathcal{M}}\big)\big(\mathcal{R}^H + 2G\Pi_1\big)(Id + \mathbf{T}_{\mathcal{M}}), \\ \mathcal{R}^{\text{Mix}}_{\mathcal{M}} = |Y_1|\big({}^t\mathbf{T}_{\mathcal{M}} - \big(Id + {}^t\mathbf{T}_{\mathcal{M}}\big)\big(\mathcal{R}^H + 2|Y_1|G\Pi_1\big) \\ \qquad\qquad \times \big(\Pi \mathcal{R}^H \Pi + 2|Y_1|G\Pi_1\big)^{-1}\big). \end{array}\right. \tag{25}$$

The notation in (25) is not completely correct. The inverted matrices are not in fact invertible as applications from $(L^2(\Omega))^{10}$ to $(L^2(\Omega))^{10}$, but on the relevant subspaces. For example, the inversion of $\Pi \mathcal{R}^H \Pi$ is meant for the restricted application $\Pi(L^2(\Omega))^{10} \mapsto \Pi(L^2(\Omega))^{10}$. In practice, $(\Pi \mathcal{R}^H \Pi)^{-1}$ is obtained by deleting the zero lines and columns of $\Pi \mathcal{R}^H \Pi$, inverting the resulting matrix and incorporating the results in the right place in a $10 \times 10$ matrix of format (16). A detailed example is given in Section 7.

REMARK 5.1. Matrices $\mathcal{R}_{\mathcal{M}}$, $\mathcal{R}_{\mathcal{N}}$, and $\mathcal{R}^{\text{Mix}}_{\mathcal{M}}$ have the same format (16) as $\mathcal{R}$. The corresponding submatrices are naturally denoted by

$$\mathbf{R}_{\mathcal{M}}, \mathbf{d}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}}, \mathbf{R}_{\mathcal{N}}, \mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}}, \mathbf{R}^{\text{Mix}}_{\mathcal{M}}, \mathbf{d}^{\text{Mix}}_{\mathcal{M}}, \text{ and } \mathbf{c}^{\text{Mix}}_{\mathcal{M}}.$$

Because of the projections, these matrices are sparse matrices. Only the coefficients $R_{\mathcal{M}\alpha\beta\gamma\delta}$, $R_{\mathcal{N}\alpha\beta\gamma\delta}$, $d_{\mathcal{M}3\alpha\beta}$, $d_{\mathcal{M}3\alpha\beta}$, $c_{\mathcal{M}33}$, and $d^{\text{Mix}}_{\mathcal{M}3\alpha\beta}$ are needed in the following models.

### 5.3. MODELS

Let

$$l(\mathbf{v}) = \left\{\begin{array}{l} \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma^N} g_i v_i \,\mathrm{d}s - 2\displaystyle\int_\omega s_{\alpha\beta}(\bar{\mathbf{v}}) d_{\mathcal{M}3\alpha\beta} \varphi_c \,\mathrm{d}\hat{x} \\ \quad \text{for Dirichlet conditions}, \\ \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma^N} g_i v_i \,\mathrm{d}s + 2\displaystyle\int_\omega s_{\alpha\beta}(\bar{\mathbf{v}}) d^{\text{Mix}}_{\mathcal{M}3\alpha\beta} h \,\mathrm{d}\hat{x} \\ \quad \text{for local mixed conditions}, \\ \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma^N} g_i v_i \,\mathrm{d}s \\ \quad \text{for nonlocal mixed conditions}. \end{array}\right.$$

THEOREM 5.1. *Assume that the hypothesis of Lemma* 4.1 *holds. Assume that* $a$, $\varepsilon$, *and* $\varepsilon/a$ *tend to zero. Then*:

(i) *in the case of Dirichlet or local mixed electrical boundary conditions, the sequence* $(\mathbf{u}^b)$ *converges to* $\mathbf{u} = {}^t(\bar{u}_1 - x_3\partial_1 u_3, \bar{u}_2 - x_3\partial_2 u_3, u_3) \in \mathbf{V}_{KL}$ *which is the unique solution of*:

$$\int_\omega \left(2s_{\alpha\beta}(\bar{\mathbf{v}})R_{\mathcal{M}\alpha\beta\gamma\delta}s_{\gamma\delta}(\bar{\mathbf{u}}) + \frac{2}{3}\partial^2_{\alpha\beta}v_3 R_{\mathcal{N}\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\right)\mathrm{d}\hat{x} = l(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_{KL};$$

(ii) *in the case of nonlocal mixed electrical boundary conditions, the sequence* $(\mathbf{u}^b, \mathcal{M}(L_3^a(\varphi^b)))$ *converges to* $(\mathbf{u}, L_3^0) \in \mathbf{V}_{KL} \times H^1(\omega)$ *which is the unique solution to*:

$$\begin{aligned}\int_\omega \Bigg(2\big(s_{\alpha\beta}(\bar{\mathbf{v}}), \widetilde{L}_3\big) &\begin{pmatrix} R_{\mathcal{M}\alpha\beta\gamma\delta} & d_{\mathcal{M}3\alpha\beta} \\ e_{\mathcal{M}3\gamma\delta} & c_{\mathcal{M}33} + 2|Y_1|G \end{pmatrix} \begin{pmatrix} s_{\gamma\delta}(\bar{\mathbf{u}}) \\ L_3^0 \end{pmatrix} \\ &+ \frac{2}{3}\partial^2_{\alpha\beta}v_3 R_{\mathcal{N}\alpha\beta\gamma\delta}\partial^2_{\gamma\delta}u_3\Bigg)\mathrm{d}\hat{x} + 4|Y_1| \int_\omega G_1 \partial_\alpha \widetilde{L}_3 \partial_\alpha L_3^0 \,\mathrm{d}\hat{x} \\ &= l(\mathbf{v}) + 2\int_\omega \widetilde{L}_3 h \,\mathrm{d}\hat{x} \quad \forall (\mathbf{v}, \widetilde{L}_3) \in \mathbf{V}_{KL} \times H^1(\omega).\end{aligned}$$

5.3.1. *Comments*

- All the models are independent of $\varphi_m$. Only the difference of potential between the upper and lower faces does influence the effective behaviour of the plate.
- In both cases, equations for $u^3$ and $\bar{\mathbf{u}}$ are uncoupled. This would, however, no longer be the case for multilayered plates. See [4].
- For Dirichlet and local mixed conditions, the limit model has the standard form of a two-dimensional elastic plate. The influence of inclusions only appears in the definition of the effective coefficients, and as a source term on the right-hand side.
- For nonlocal conditions, the situation is more interesting. The coupling arises between the mechanical effects and the transverse electric field induced by the inclusions. The form of the differential operator (here, a Laplace operator) acting on $L_3^0$ depends only of the choice of connections between inclusions. However, given that in (ii) the equations for $u_3$ on the one hand, for $(\bar{u}_1, \bar{u}_2, L_0^3)$ on the other hand, are uncoupled, the transverse displacement control would require the consideration of multilayered plates, as in [4].
- Formulation (ii) is more general than formulation (i). First, for time dependent problems, even for local mixed conditions, this formulation could be applied because $G$ could be a combination of time derivatives which cannot be simply inverted. Second, also for local conditions, when thinking in terms of control, one may prefer to consider model (ii) (with $G_1 = 0$) rather than model (i). The role of $G$ is more apparent in (ii): the local mixed conditions actually correspond to the operator on $L_3^0$ without derivatives.

## 6. Proof of Theorem 5.1

### 6.1. STEP 1: CHARACTERIZATION OF THE LIMIT $\mathbf{M}$

#### 6.1.1. *Some Notation*

We give here some notation that allows a more elegant presentation of the results.

Let $\mathcal{C}^\infty_\sharp(Y)$ denote the subspace of $Y$-periodic functions of $\mathcal{C}^\infty(\mathbb{R}^2)$. For any function $v \in \mathcal{D}(\overline{\Omega}, \mathcal{C}^\infty_\sharp(Y))$, we systematically denote by $v^\varepsilon \in \mathcal{D}(\overline{\Omega})$ the function $x \mapsto v(x, \hat{x}/\varepsilon)$. A similar convention is used for functions of $\mathcal{D}(\Omega, \mathcal{C}^\infty_\sharp(Y_1))$.

In what follows, we consider in (18) two-scale admissible test functions in

$$\mathbf{W}^1_{ad} = \Big\{\mathbf{V}^1 = (\mathbf{v}^1, \psi^1) \in \big(\mathcal{D}\big(\overline{\Omega}, \mathcal{C}^\infty_\sharp(Y)\big)^3 \times \mathcal{D}\big(\Omega, \mathcal{C}^\infty_\sharp(Y_1)\big)\big);\\ \mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_D \times Y\Big\},$$

and test functions in

$$\mathbf{W}_{ad} = \big\{(\mathbf{v}, \psi) \in \mathbf{H}^1(\Omega) \times \Psi_{ad}(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\big\},$$

where

$$\begin{array}{ll} \Psi_{ad}(\Omega) = \mathcal{D}(]-1, 1] \times \omega) & \text{for mixed conditions},\\ \Psi_{ad}(\Omega) = \mathcal{D}(]-1, 1[\times \omega) & \text{for Dirichlet conditions}. \end{array}$$

For $\mathbf{V} \in \mathbf{W}^1_{ad}$, we introduce the key decomposition of $\mathbf{M}^a(\mathbf{V}^\varepsilon)$:

$$\mathbf{M}^a\big(\mathbf{V}^\varepsilon\big) = \big(\mathbf{M}^{00}(\mathbf{V})\big)^\varepsilon + \frac{1}{\varepsilon}\big(\mathbf{M}^{10}(\mathbf{V})\big)^\varepsilon + \frac{1}{a}\big(\mathbf{M}^{01}(\mathbf{V})\big)^\varepsilon \\ + \frac{1}{a\varepsilon}\big(\mathbf{M}^{11}(\mathbf{V})\big)^\varepsilon + \frac{1}{a^2}\big(\mathbf{M}^{02}(\mathbf{V})\big)^\varepsilon, \tag{26}$$

and we recall that for $\mathbf{V} \in \mathbf{W}_{ad}$:

$$\mathbf{M}^a(\mathbf{V}) = \mathbf{M}^{00}(\mathbf{V}) + \frac{1}{a}\mathbf{M}^{01}(\mathbf{V}) + \frac{1}{a^2}\mathbf{M}^{02}(\mathbf{V}), \tag{27}$$

where, with definition (2) for $S_{\alpha i}$, as $\mathbf{V} = (\mathbf{v}, \psi)$:

$$\begin{cases} \mathbf{M}^{00}(\mathbf{V}) = {}^t\big(\big(s_{\alpha\beta}(\mathbf{v})\big)_{\alpha,\beta=1,2}, \mathbf{0}_3, (\partial_\alpha \psi)_{\alpha=1,2}, 0\big),\\ \mathbf{M}^{10}(\mathbf{V}) = {}^t\big(\big(S_{\alpha\beta}(\mathbf{v})\big)_{\alpha,\beta=1,2}, \mathbf{0}_3, (\partial_{y_\alpha} \psi)_{\alpha=1,2}, 0\big),\\ \mathbf{M}^{01}(\mathbf{V}) = {}^t\big(\mathbf{0}_{2\times2}, \big(s_{\alpha3}(\mathbf{v})\big)_{\alpha=1,2}, \mathbf{0}_4, \partial_3\psi\big),\\ \mathbf{M}^{11}(\mathbf{V}) = {}^t\big(\mathbf{0}_{2\times2}, \big(S_{\alpha3}(\mathbf{v})\big)_{\alpha=1,2}, \mathbf{0}_4\big),\\ \mathbf{M}^{02}(\mathbf{V}) = {}^t\big(\mathbf{0}_{2\times2}, \mathbf{0}_2, s_{33}(\mathbf{v}), \mathbf{0}_3\big). \end{cases} \tag{28}$$

Associated subspaces $\mathbb{M}$, $\mathbb{M}^{-2}$, $\mathbb{M}^{-1}$ and $\mathbb{M}^0$ of $(L^2(\Omega \times Y))^7 \times (L^2(\Omega \times Y_1))^3$ are defined by

$$\mathbb{M}^{-2} = \left\{ {}^t(\mathbf{0}_{2\times 2}, \mathbf{0}_2, K_{33}, \mathbf{0}_3);\ K_{33} \in L^2(\Omega) \right\},$$
$$\mathbb{M}^{-1} = \Big\{ {}^t\big(\mathbf{0}_{2\times 2}, (k_{\alpha 3}/2)_{\alpha=1,2}, \mathbf{0}_3, L_3\big) + \mathbf{M}^{11}\big(\mathbf{V}^2\big);\ k_{\alpha 3}, L_3 \in L^2(\Omega),$$
$$v_3^2 \in L^2\big(\Omega; H^1_\sharp(Y)\big),\ \text{with } \mathcal{M}(L_3) = 0 \text{ for Dirichlet conditions} \Big\},$$
$$\mathbb{M}^0 = \Big\{ \mathbf{M}^{00}(\mathbf{v}, 0) + \mathbf{M}^{10}\big(\mathbf{V}^1\big);$$
$$\mathbf{v} \in \mathbf{V}_{KL}, \mathbf{V}^1 \in \mathbf{L}^2\big(\Omega; \mathbf{H}^1_\sharp(Y)\big)^2 \times \{0\} \times \in L^2\big(\Omega; H^1(Y_1)\big) \Big\},$$
$$\mathbb{M} = \mathbb{M}^{-2} \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^0.$$

REMARK 6.1. Each $\mathbf{M} \in \mathbb{M}$ is associated with $(\mathbf{v}, \tilde{\mathbf{v}}^1, \psi) \in \mathbf{V}_{KL} \times \mathbf{L}^2(\Omega; \mathbf{H}^1_\sharp(Y)) \times L^2(\Omega; H^1(Y_1))$, where $\tilde{\mathbf{v}}^1 = {}^t(v_1^1, v_2^1, v_3^2)$.

### 6.1.2. *Three Preliminary Lemmas*

The first two lemmas are density results that allow us to pass from admissible test functions to test functions in $\mathbb{M}$, $\mathbb{M}^{-2}$, $\mathbb{M}^{-1}$, and $\mathbb{M}^0$. Lemma 6.1 deals with mixed conditions. For Dirichlet conditions, each function of $\Psi_{ad}(\Omega)$ is trivially identified to a function of $H^1_c(\Omega_1^\varepsilon)$. This is no longer the case for mixed conditions (see definition (17) of $H^1_c(\Omega_1^\varepsilon)$). It is the aim of Lemma 6.1 to overcome this difficulty.

LEMMA 6.1. *For mixed conditions, for each $\psi \in \Psi_{ad}(\Omega)$, there exists a sequence $(\tilde{\psi}^\varepsilon)$ with $\tilde{\psi}^\varepsilon_{|\Omega_1^\varepsilon} \in H^1_c(\Omega_1^\varepsilon)$ such that $(\partial_3 \tilde{\psi}^\varepsilon)$ strongly converges to $\partial_3 \psi$ in $L^2(\Omega)$.*

*Proof.* Let $\omega^\varepsilon_{\mathbf{i}}$ denote the mean section of the inclusion number $\mathbf{i}$. To obtain Lemma 6.1, we simply need to choose $\tilde{\psi}^\varepsilon$ defined by

$$\tilde{\psi}^\varepsilon = \frac{1}{|\omega^\varepsilon_{\mathbf{i}}|} \int_{\omega^\varepsilon_{\mathbf{i}}} \psi(x)\, \mathrm{d}\hat{x} \quad \text{in } \omega^\varepsilon_{\mathbf{i}}\ \forall \mathbf{i} \in \mathbb{I}^\varepsilon.$$ □

LEMMA 6.2.

(i) *The set $\{\mathbf{M}^{02}(\mathbf{V}); \mathbf{V} \in \mathbf{W}_{ad}\}$ is dense in $\mathbb{M}^{-2}$,*

(ii) *The set $\{\mathbf{M}^{01}(\mathbf{V}) + \mathbf{M}^{11}(\mathbf{V}^1); \mathbf{V} \in \mathbf{W}_{ad}, \mathbf{V}^1 \in \mathbf{W}^1_{ad}, v_3 = 0\}$ is dense in $\mathbb{M}^{-1}$,*

(iii) *The set $\{\mathbf{M}^{00}(\mathbf{V}) + \mathbf{M}^{10}(\mathbf{V}^1); \mathbf{V} \in \mathbf{V}_{KL} \times \{0\}, \mathbf{V}^1 \in \mathbf{W}^1_{ad}\}$ is dense in $\mathbb{M}^0$.*

*Proof.* Point (i) follows, for instance, from the density of $\{\partial_3 v_3; v_3 \in \mathcal{D}(\bar{\omega} \times ]-1, 1])\}$ in $L^2(\Omega)$. Point (ii) is similar. For Dirichlet conditions, we remark that the density of $\{\partial_3 \psi; \psi \in \mathcal{D}(\bar{\omega} \times ]-1, 1[)\}$ is in $\{L_3 \in L^2(\Omega); \mathcal{M}(L_3) = 0\}$ only. Point (iii) is straightforward. □

LEMMA 6.3. *Let $(u^\varepsilon)$ be a bounded sequence in $H^1(\Omega)$. Let $u \in H^1(\Omega)$ and $u^1 \in L^2(\Omega; H^1_\sharp(Y))$ be functions such that $(u^\varepsilon)$ weakly converges to $u$ in $H^1(\Omega)$, $(\nabla u^\varepsilon)$ two-scale converges to $\nabla u + \nabla_y u^1$ in $L^2(\Omega \times Y)$. Then*

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{u^\varepsilon}{\varepsilon} \partial_{y_\alpha} v^\varepsilon \,\mathrm{d}x = \int_{\Omega \times Y} u^1 \partial_{y_\alpha} v \,\mathrm{d}x\mathrm{d}y \quad \forall v \in \mathcal{D}\big(\Omega; C^\infty_\sharp(Y)\big).$$

*Proof.* We simply need to pass to the limit in

$$\int_\Omega \partial_\alpha u^\varepsilon v^\varepsilon \,\mathrm{d}x = -\int_\Omega u^\varepsilon (\partial_\alpha v)^\varepsilon \,\mathrm{d}x - \int_\Omega \frac{u^\varepsilon}{\varepsilon}\big(\partial_{y_\alpha} v\big)^\varepsilon \mathrm{d}x.$$

The integration by parts of the first term on the right-hand side then yields the result. □

### 6.1.3. *Characterization of* **M**

Define $\phi_c$ by $\phi_c = {}^t(\mathbf{0}_9, \varphi_c)$ for Dirichlet conditions, $\phi_c = \mathbf{0}_{10}$ for mixed conditions.

LEMMA 6.4. *Assume that assumptions of Lemma* 4.1 *hold. Assume that* $a$, $\varepsilon$, *and* $\varepsilon/a$ *tend to zero. Then* $(\mathbf{M}^a(\mathbf{U}^b))$ *two-scale converges to*

$$\mathbf{M} = {}^t\bigg(\big(s_{\alpha\beta}(\mathbf{u}) + S_{\alpha\beta}\big(\mathbf{u}^1\big)\big)_{\alpha,\beta=1,2}, \frac{1}{2}\big(k_{\alpha 3} + \partial_{y_\alpha} u_3^2\big)_{\alpha=1,2}, K_{33}, \big(\partial_{y_\alpha}\varphi^1\big)_{\alpha=1,2}, L_3\bigg) \in \phi_c + \mathbb{M}$$

*which is the unique solution of*

$$\int_{\Omega\times Y} {}^t\widetilde{\mathbf{M}}\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y + 2G\int_{\Omega\times Y_1}\mathcal{M}(L_3)\mathcal{M}\big(\widetilde{L}_3\big)\mathrm{d}x\mathrm{d}y + 2G_1\int_{\Omega\times Y_1}\partial_\alpha\mathcal{M}(L_3)\partial_\alpha\mathcal{M}\big(\widetilde{L}_3\big)\mathrm{d}x\mathrm{d}y = l_u(\mathbf{v}) + l_\varphi\big(\widetilde{L}_3\big) \quad \forall\widetilde{\mathbf{M}}\in\mathbb{M}, \tag{29}$$

*where*

$$\begin{aligned} l_u(\mathbf{v}) &= \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma^N} g_i v_i \,\mathrm{d}x,\\ l_\varphi(L_3) &= \int_{\Omega\times Y_1} hL_3 \,\mathrm{d}x\mathrm{d}y = |Y_1|\int_\Omega hL_3\,\mathrm{d}x. \end{aligned} \tag{30}$$

*Proof.* The proof is in two steps. We first establish that **M** satisfies the weak formulation (29). We then show that $\mathbf{M} \in \phi_c + \mathcal{M}$. Uniqueness of the solution of (29) is a simple consequence of Lax–Milgram's lemma.

In the case of Dirichlet conditions, we choose $\mathbf{V} \in \mathbf{W}_{ad}$ as a test function in the weak formulation (18). We then multiply by $a^2$, $a$, and 1 successively and pass to the limit in each case. With definitions (26), (27), and (28) we, thus, get

$$\begin{cases} \int_{\Omega\times Y} {}^t\mathbf{M}^{02}(\mathbf{V})\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y = 0 \quad \forall\mathbf{V}\in\mathbf{W}_{ad},\\ \int_{\Omega\times Y} {}^t\mathbf{M}^{01}(\mathbf{V})\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y + 2\int_{\Omega\times Y_1} G\mathcal{M}(L_3)\mathcal{M}(\partial_3\psi)\,\mathrm{d}x\mathrm{d}y\\ \quad +2\int_{\Omega\times Y_1} G_1\partial_\alpha\mathcal{M}(L_3)\partial_\alpha\mathcal{M}(\psi)\,\mathrm{d}x\mathrm{d}y = l_\varphi(\partial_3\psi)\\ \quad \forall\mathbf{V}\in\mathbf{W}_{ad} \quad \text{with } \mathbf{M}^{02}(\mathbf{V}) = \mathbf{0},\\ \int_{\Omega\times Y} {}^t\mathbf{M}^{00}(\mathbf{V})\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y = l_u(\mathbf{v}) \quad \forall\mathbf{V}\in\mathbf{W}_{ad}\\ \quad \text{with } \mathbf{M}^{02}(\mathbf{V}) = \mathbf{M}^{01}(\mathbf{V}) = \mathbf{0}. \end{cases} \tag{31}$$

For mixed conditions, (31) also holds, but we need to start in (18) with $\widetilde{\psi}^{\varepsilon}$ as in Lemma 6.1 instead of $\psi$.

Choose now $\mathbf{V}^{1\varepsilon}$: $x \mapsto \mathbf{V}^1(x, x/\varepsilon)$, where $\mathbf{V^1} \in \mathbf{W}^1_{ad}$, as a test function in (18). With definition (26), multiplication by $a\varepsilon$ and $\varepsilon$ yields

$$\begin{cases} \int_{\Omega\times Y} {}^t\mathbf{M}^{11}(\mathbf{V}^1)\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y = 0 \quad \forall \mathbf{V^1} \in \mathbf{W}^1_{ad}, \\ \int_{\Omega\times Y} {}^t\mathbf{M}^{10}(\mathbf{V}^1)\mathcal{R}\mathbf{M}\,\mathrm{d}x\mathrm{d}y = 0 \quad \forall \mathbf{V^1} \in \mathbf{W}^1_{ad}, \\ \quad \text{with } \mathbf{M}^{11}(\mathbf{V}) = \mathbf{M}^{02}(\mathbf{V}) = \mathbf{0}. \end{cases} \tag{32}$$

Now, using Lemma 6.2, point (i), the first equation in (31) is equivalent to the weak formulation (29) with $\mathbb{M}^{-2}$ instead of $\mathbb{M}$. Also, using Lemma 6.2 (ii), the second equation in (31) (with $v_3 = 0$) and the first equation in (32) are equivalent to (29) with $\mathbb{M}^{-1}$ instead of $\mathbb{M}$. Last, using Lemma 6.2 (iii), the third equation in (31) (with $v_3 \in \mathbf{V}_{KL}$, that ensures $\mathbf{M}^{02}(\mathbf{V}) = \mathbf{M}^{01}(\mathbf{V}) = \mathbf{0}$) and the second equation in (32) (with $v_3^1 = 0$) are equivalent to (29) with $\mathbb{M}^0$ instead of $\mathbb{M}$. As (29) holds for any $\widetilde{M}$ in $\mathbb{M}^{-2}$, $\mathbb{M}^{-1}$, and $\mathbb{M}^0$, it holds in $\mathbb{M} = \mathbb{M}^{-2} \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^0$. This ends the proof of the first part of Lemma 6.4.

Now we prove that $\mathbf{M} \in \phi_c + \mathcal{M}$, or in other words, that $\mathbf{M}$ has the form as announced in the lemma.

The form $K_{\alpha\beta} = s_{\alpha\beta}(\mathbf{u}) + S_{\alpha\beta}(\mathbf{u}^1)$ for $\alpha, \beta = 1, 2$ is a direct consequence of Lemma 4.1, point (iv) and of definition (14): $K^a_{\alpha\beta}(\mathbf{v}^b) = s_{\alpha\beta}(\mathbf{v}^b)$. The form $L_\alpha = \partial_\alpha \varphi^1$ is proved in Lemma 4.1, point (vi).

Concerning $K_{33}$, we simply need to show that it is independent of $y$. To do this, we pass to the limit in the identity

$$\begin{aligned} &\frac{1}{a^2}\int_\Omega \partial_3 u_3^b\big(\varepsilon(\partial_\beta\partial_{y_\alpha} v)^\varepsilon + (\partial^2_{y_\alpha y_\beta} v)^\varepsilon\big)\mathrm{d}x \\ &\quad = 2\frac{\varepsilon}{a}\int_\Omega K^a_{\beta 3}(\mathbf{u}^\varepsilon)(\partial_{y_\alpha}\partial_3 v)^\varepsilon \mathrm{d}x + \frac{\varepsilon^2}{a^2}\int_\Omega \frac{1}{\varepsilon} u^b_\beta \partial_{y_\alpha}\partial^2_{33} v^\varepsilon \mathrm{d}x \end{aligned}$$

which holds for $\alpha, \beta \in \{1, 2\}$ and $v \in \mathcal{D}(\Omega; \mathcal{C}^\infty_\sharp(Y))$ (recall that this implies that $\partial_{y_\alpha} v$ is Y-periodic). The first term on the right-hand side tends to zero because $(K^a_{\beta 3}(\mathbf{u}^\varepsilon))$ is bounded. Using Lemma 6.3, the second term on the right-hand side also tends to zero. Hence,

$$\int_{\Omega\times Y} K_{33}\partial^2_{y_\beta y_\alpha} v\,\mathrm{d}x\mathrm{d}y = 0 \quad \forall v \in \mathcal{D}\big(\Omega; \mathcal{C}^\infty_\sharp(Y)\big).$$

This proves that $K_{33}$ does not depend on $\mathbf{y}$.

For $(K_{13}, K_{23})$, first note, using a few integrations by parts, that

$$\begin{aligned} &\int_\Omega \big(\partial_\alpha u^b_\beta - \partial_\beta u^b_\alpha\big)\partial_3 v^\varepsilon \mathrm{d}x \\ &\quad = 2a\int_\Omega K^a_{\beta 3}(\mathbf{u}^b)\left(\partial_\alpha v^\varepsilon + \frac{1}{\varepsilon}\partial_{y_\alpha} v^\varepsilon\right)\mathrm{d}x - 2a\int_\Omega K^a_{\alpha 3}(\mathbf{u}^b)\left(\partial_\beta v^\varepsilon + \frac{1}{\varepsilon}\partial_{y_\beta} v^\varepsilon\right)\mathrm{d}x \\ &\qquad \forall v \in \mathcal{D}\big(\Omega; \mathcal{C}^\infty_\sharp(Y)\big). \end{aligned} \tag{33}$$

By multiplying (33) by $\varepsilon/a$ and passing to the limit, one gets

$$\int_{\Omega\times Y}(K_{\beta 3}\partial_{y_\alpha}v - K_{\alpha 3}\partial_{y_\beta}v)\,\mathrm{d}x\mathrm{d}y = 0 \quad \forall v\in\mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(Y)\big).$$

Hence, ${}^t(K_{13}, K_{23})$ is curl-free with respect to $y$. Thus, there exist $(k_{13}, k_{23}) \in \mathbf{L}^2(\Omega)$ and $u_3^1 \in L^2(\Omega; H^1_\sharp(Y))$ such that $2K_{\alpha 3} = k_{\alpha 3} + \partial_{y_\alpha}u_3^1$ (see [8, Section 3], if necessary for this well-known orthogonality result in the context of periodic functions).

To complete the proof, it remains for us to examine $L_3$. Passing to the limit in

$$\varepsilon\int_{\Omega_1^\varepsilon} L_3^a\big(\varphi^b\big)\bigg((\partial_\alpha\psi)^\varepsilon + \frac{1}{\varepsilon}(\partial_{y_\alpha}\psi)^\varepsilon\bigg)\mathrm{d}x = \frac{\varepsilon}{a}\int_{\Omega_1^\varepsilon} L_\alpha^a\big(\varphi^b\big)\partial_3\psi^\varepsilon\mathrm{d}x, \quad \alpha = 1,2,$$

as the quantities $L_i^a(\varphi^b)$ are bounded, one gets

$$\int_{\Omega_1^\varepsilon} L_3\partial_{y_\alpha}\psi\,\mathrm{d}x\mathrm{d}y = 0, \quad \alpha = 1,2.$$

This proves that $L_3$ does not depend on $y$ and thus completes the proof of Lemma 6.1 if the mixed conditions case. To conclude for Dirichlet conditions, it suffices to remember that $\mathcal{M}(L_3) = \varphi^c$ (see Lemma 4.1 (vii)). Hence, $\mathcal{M}(L_3 - \varphi^c) = 0$. □

### 6.2. STEP 2: HOMOGENIZATION

The weak formulation (29) being established, the next step consists in eliminating the local variable $y$. This requires the auxiliary functions $(\mathbf{u}^{\gamma\delta}, \mathbf{u}^{\gamma 3}, \mathbf{u}^{33}, \mathbf{u}^3)$ defined in (23). We use the decomposition

$$\mathbf{M} = \mathbf{M}_x + \mathbf{M}^{11}\big(\mathbf{U}^1\big) + \mathbf{M}^{10}\big(\mathbf{U}^1\big)$$

in (29), where

$$\mathbf{M}_x = {}^t\Big(\big(s_{\alpha\beta}(\mathbf{u})\big)_{\alpha,\beta=1,2},\ (k_{\alpha 3}/2)_{\alpha=1,2}, K_{33}, \mathbf{0}_2, L_3\Big),$$
$$\mathbf{U}^1 = {}^t\big(u_1^1, u_2^1, u_3^2, \varphi^1\big).$$

LEMMA 6.5. *Let* $\mathbf{M}$ *be the solution of* (29). *Then,*

$$\mathbf{M}^{11}\big(\mathbf{U}^1\big) + \mathbf{M}^{10}\big(\mathbf{U}^1\big) = \mathcal{L}\mathbf{M}_x,$$

*and* $(\mathbf{u}, (k_{\alpha 3})_{\alpha=1,2}, K_{33}, L_3) \in \mathbf{V}_{KL}\times(L^2(\Omega))^4$ *is the unique solution of*

$$\int_\Omega\big({}^t\widetilde{\mathbf{M}}\mathcal{R}^H\mathbf{M}_x + 2|Y_1|G\widetilde{L}_3\mathcal{M}(L_3) + 2|Y_1|G_1\partial_\alpha\mathcal{M}(L_3)\partial_\alpha\widetilde{L}_3\big)\mathrm{d}x$$
$$= l_u(\mathbf{v}) + l_\varphi(\widetilde{L}_3) \quad \forall\mathbf{v}\in\mathbf{V}_{KL}, \quad \forall\big((\tilde{k}_{\alpha 3})_{\alpha=1,2}, \widetilde{K}_{33}, \widetilde{L}_3\big)\in\mathbf{L}^2(\Omega). \qquad (34)$$

The linear forms $l_u$ and $l_\varphi$ are defined in (30).

*Proof.* In (29), we choose test functions $\widetilde{\mathbf{M}} \in \mathbb{M}$ of the form $\widetilde{\mathbf{M}} = \mathbf{M}^{11}(\mathbf{V}^1) + \mathbf{M}^{10}(\mathbf{V}^1)$, where

$$\mathbf{V}^1 = {}^t\big(v_1^1, v_2^1, v_3^2, \psi^1\big) \in \mathbf{L}^2\big(\Omega; \mathbf{H}^1_\sharp(Y)\big) \times L^2\big(\Omega; H^1(Y_1)\big).$$

We obtain

$$\begin{aligned}&\int_Y {}^t\big(\mathbf{M}^{11}\big(\mathbf{V}^1\big) + \mathbf{M}^{10}\big(\mathbf{V}^1\big)\big)\mathcal{R}\big(\mathbf{M}^{11}\big(\mathbf{U}^1\big) + \mathbf{M}^{10}\big(\mathbf{U}^1\big)\big)\mathrm{d}y\\ &\quad= -\int_Y {}^t\big(\mathbf{M}^{11}\big(\mathbf{V}^1\big) + \mathbf{M}^{10}\big(\mathbf{V}^1\big)\big)\mathcal{R}\,\mathrm{d}y\mathbf{M}_x\\ &\quad\forall \mathbf{V}^1 = {}^t\big(v_1^1, v_2^1, v_3^2, \psi^1\big) \in \mathbf{H}^1_\sharp(Y) \times H^1(Y_1), \quad \text{almost everywhere in } \Omega.\end{aligned}$$

Hence, $\mathbf{U}^1$ is the unique solution (up to a function of $x$) to the above variational problem. But as $\mathbf{M}_x$ is independent of $y$, one may choose $\mathbf{U}^1 = \mathbf{u}^{\gamma\delta}s_{\gamma\delta}(\mathbf{u}) + \frac{1}{2}\mathbf{u}^{\gamma 3}k_{\gamma 3} + \mathbf{u}^{33}K_{33} + \mathbf{u}^3 L_3$. Using the definition of $\mathcal{L}$ (see Section 5.1), it follows that $\mathbf{M}^{11}(\mathbf{U}^1) + \mathbf{M}^{10}(\mathbf{U}^1) = \mathcal{L}\mathbf{M}_x$ and, therefore, $\mathbf{M} = (\mathbf{Id}+\mathcal{L})\mathbf{M}_x$. Choosing now in (29) test functions of the form $(\mathbf{Id} + \mathcal{L})\widetilde{\mathbf{M}}$, where

$$\begin{aligned}&\widetilde{\mathbf{M}} = {}^t\big(\big(s_{\alpha\beta}(\mathbf{v})\big)_{\alpha,\beta=1,2}, \big(\tilde{k}_{\alpha 3}/2\big)_{\alpha=1,2}, \widetilde{K}_{33}, \mathbf{0}_2, \widetilde{L}_3\big),\\ &\big(\mathbf{v}, \big(\tilde{k}_{\alpha 3}\big)_{\alpha=1,2}, \widetilde{K}_{33}, \widetilde{L}_3\big) \in \mathbf{V}_{KL} \times \big(L^2(\Omega)\big)^4,\end{aligned}$$

keeping definition (24): $\mathcal{R}^H = \int_Y (\mathbf{Id} + {}^t\mathcal{L})\mathcal{R}(\mathbf{Id} + \mathcal{L})\,\mathrm{d}y$ in mind, we are led to the weak formulation (34). This ends the proof. □

### 6.3. STEP 3: PLATE THEORY

We complete the proof of Theorem 5.1 by eliminating $k_{\gamma 3}$, $K_{33}$, $\mathcal{N}(L_3)$, and possibly, $\mathcal{M}(L_3)$. The proof is based on the decomposition $\mathbb{M} = \mathbb{M}^0 \oplus \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$ where all $y$-terms are killed because everything in (34) depends only on $x$. Given that there is no confusion possible, we keep the same notation as before for the corresponding functional spaces. We also use again $\mathbf{M}$ instead of $\mathbf{M}_x$. In simpler terms, in the sequel $\mathbb{M}^0$, $\mathbb{M}^{-1}$, $\mathbb{M}^{-2}$, $\mathbf{M}$ designate

$$\begin{cases}\mathbb{M}^{-2} = \big\{{}^t(\mathbf{0}_{2\times 2}, \mathbf{0}_2, K_{33}, \mathbf{0}_3);\ K_{33} \in L^2(\Omega)\big\},\\ \mathbb{M}^{-1} = \big\{{}^t(\mathbf{0}_{2\times 2}, (k_{\alpha 3}/2)_{\alpha=1,2}, \mathbf{0}_3, L_3) \text{ with } \mathcal{M}(L_3) = 0\\ \qquad\qquad \text{for Dirichlet conditions}\big\},\\ \mathbb{M}^0 = \big\{\mathbf{M}^{00}(\mathbf{v}, 0);\ \mathbf{v} \in \mathbf{V}_{KL}\big\},\\ \mathbf{M} = {}^t\big(s_{\alpha\beta}(\mathbf{u})_{\alpha,\beta=1,2}, (k_{\alpha 3}/2)_{\alpha=1,2}, K_{33}, \mathbf{0}_2, L_3\big).\end{cases} \tag{35}$$

6.3.1. *Case of Dirichlet Conditions*

Here, we eliminate the components $k_{\gamma 3}$, $K_{33}$, $L_3$, that is, $\Pi\mathbf{M}$, where $\Pi$ is defined in (25). They are computed with respect to $\varphi_c$ (recall that $\mathcal{M}(L_3) = \varphi_c$ for Dirichlet conditions) and $\mathbf{M}_0 := \mathbf{M} - \Pi\mathbf{M}$. As usual in the plate theory, we are led to distinguish $\mathcal{M}(\mathbf{M}^0) = {}^t((s_{\gamma\delta}(\overline{\mathbf{u}}))_{\gamma,\delta=1,2}, \mathbf{0}_6)$ which contains the terms in $\overline{\mathbf{u}}$, and $\mathcal{N}(\mathbf{M}^0) = {}^t((-x_3\partial^2_{\gamma\delta}u_3)_{\gamma,\delta=1,2}, \mathbf{0}_6)$ which contains the terms in $\mathbf{u}_3$. Let us also recall the definition $\phi_c = {}^t(\mathbf{0}_9, \varphi_c)$.

As for Dirichlet conditions $G = G_1 = h = 0$, problem (34) simply becomes

$$\int_\Omega {}^t\widetilde{\mathbf{M}}\mathcal{R}^H\mathbf{M}\,\mathrm{d}x = l_u(\mathbf{v}) \quad \forall\widetilde{\mathbf{M}} \in \mathbb{M}, \tag{36}$$

where $\mathbf{v}$ is the vector of $\mathbf{V}_{KL}$ associated with $\widetilde{\mathbf{M}}^0$. Choose $\widetilde{\mathbf{M}} \in \mathcal{M}(\mathbb{M}^{-1} \oplus \mathbb{M}^{-2})$ in (36). Using that $\mathcal{R}^H$ like $\mathcal{R}$ does not depend on $x_3$, then

$$\int_\Omega {}^t\widetilde{\mathbf{M}}\mathcal{R}^H\mathbf{M}\,\mathrm{d}x = \int_\Omega {}^t\widetilde{\mathbf{M}}\mathcal{R}^H\mathcal{M}(\mathbf{M})\,\mathrm{d}x = 0 \quad \forall\widetilde{\mathbf{M}} \in \mathcal{M}\big(\mathbb{M}^{-1} \oplus \mathbb{M}^{-2}\big).$$

Hence, $\mathcal{R}^H\mathcal{M}(\mathbf{M}) \in (\mathcal{M}(\mathbb{M}^{-1} \oplus \mathbb{M}^{-2}))^\perp$. However, as $\mathcal{R}^H\mathcal{M}(\mathbf{M}) = \mathcal{M}(\mathcal{R}^H\mathbf{M})$ evidently belongs to $(\mathcal{N}(\mathbb{M}^{-1}\oplus\mathbb{M}^{-2}))^\perp$, this is equivalent to $\mathcal{R}^H\mathcal{M}(\mathbf{M}) \in (\mathbb{M}^{-1}\oplus \mathbb{M}^{-2})^\perp$. This implies $\Pi_2\mathcal{R}\mathcal{M}(\mathbf{M}) = \mathbf{0}$. Also, (36) implies that $\Pi\mathcal{R}\mathcal{N}(\mathbf{M}) = \mathbf{0}$. Using the decomposition $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0$, where $\mathbf{M}^0 \in \mathbb{M}^0$, we get

$$\begin{aligned}
\mathcal{M}(\mathbf{M}) &= \Pi\mathcal{M}(\mathbf{M}) + \mathcal{M}(\mathbf{M}_0) = \Pi_2\mathcal{M}(\mathbf{M}) + \mathcal{M}(\mathbf{M}_0) + \phi_c,\\
\mathcal{N}(\mathbf{M}) &= \Pi\mathcal{N}(\mathbf{M}) + \mathcal{N}(\mathbf{M}_0).
\end{aligned}$$

Multiplying by $\Pi_2\mathcal{R}^H$ and $\Pi\mathcal{R}^H$, respectively, we thus obtain

$$\begin{aligned}
\Pi_2\mathcal{R}^H\mathcal{M}(\mathbf{M}) &= \mathbf{0} = \big(\Pi_2\mathcal{R}^H\Pi_2\big)\mathcal{M}(\mathbf{M}) + \Pi_2\mathcal{R}^H\big(\mathcal{M}(\mathbf{M}_0) + \phi_c\big),\\
\Pi\mathcal{R}^H\mathcal{N}(\mathbf{M}) &= \mathbf{0} = \big(\Pi\mathcal{R}^H\Pi\big)\mathcal{N}(\mathbf{M}) + \Pi\mathcal{R}^H\mathcal{N}(\mathbf{M}_0),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\Pi_2\mathcal{M}(\mathbf{M}) &= -\big(\Pi_2\mathcal{R}^H\Pi_2\big)^{-1}\Pi_2\mathcal{R}^H\big(\mathcal{M}(\mathbf{M}^0) + \phi_c\big),\\
\Pi\mathcal{N}(\mathbf{M}) &= -\big(\Pi\mathcal{R}^H\Pi\big)^{-1}\Pi\mathcal{R}^H\mathcal{N}\big(\mathbf{M}^0\big).
\end{aligned}$$

Finally, using definition (25) of $\mathbf{T}_\mathcal{N}$ and $\mathbf{T}_\mathcal{M}$ and $\mathcal{M}(L_3) = \varphi_c$, we obtain

$$\begin{aligned}
\mathcal{M}(\mathbf{M}) &= (\mathbf{Id} + \mathbf{T}_\mathcal{M})\big(\mathcal{M}(\mathbf{M}^0) + \phi_c\big),\\
\mathcal{N}(\mathbf{M}) &= (\mathbf{Id} + \mathbf{T}_\mathcal{N})\mathcal{N}\big(\mathbf{M}^0\big).
\end{aligned}$$

Now, we choose in (36) test functions $\widetilde{\mathbf{M}} \in \mathbb{M}$ of the form $\widetilde{\mathbf{M}} = (\mathbf{Id}+\mathbf{T}_\mathcal{N})\mathcal{N}(\widetilde{\mathbf{M}}^0)+ (\mathbf{Id}+ \mathbf{T}_\mathcal{M})\mathcal{M}(\widetilde{\mathbf{M}}^0)$, where $\widetilde{\mathbf{M}}^0 \in \mathbb{M}^0$. We get

$$\begin{aligned}
&\int_\Omega \big({}^t\mathcal{M}\big(\widetilde{\mathbf{M}}^0\big)\mathcal{R}_\mathcal{M}\mathcal{M}\big(\mathbf{M}^0\big) + {}^t\mathcal{N}\big(\widetilde{\mathbf{M}}^0\big)\mathcal{R}_\mathcal{N}\mathcal{N}\big(\mathbf{M}^0\big)\big)\,\mathrm{d}x\\
&\quad = l_u(\mathbf{v}) - \int_{\Omega_1} {}^t\mathcal{M}\big(\widetilde{\mathbf{M}}^0\big)\mathcal{R}_\mathcal{M}\phi_c\,\mathrm{d}x \quad \forall\mathbf{M}^0 \in \mathbb{M}^0.
\end{aligned}$$

Noting that

$$\begin{cases} \mathcal{M}(\widetilde{\mathbf{M}}^0) = \big((s_{\alpha\beta}(\overline{\mathbf{v}}))_{\alpha,\beta=1,2}, \mathbf{0}_6\big), \ \mathcal{N}(\widetilde{\mathbf{M}}^0) = \big((-x_3\partial^2_{\alpha\beta}v_3)_{\alpha,\beta=1,2}, \mathbf{0}_6\big), \\ \mathcal{M}(\mathbf{M}^0) = \big((s_{\gamma\delta}(\overline{\mathbf{u}}))_{\gamma,\delta=1,2}, \mathbf{0}_6\big), \ \mathcal{N}(\mathbf{M}^0) = \big((-x_3\partial^2_{\gamma\delta}u_3)_{\gamma,\delta=1,2}, \mathbf{0}_6\big), \end{cases} \tag{37}$$

this completes the proof of Theorem 5.1 for Dirichlet electrical boundary conditions.

### 6.3.2. *Case of Local Mixed Conditions*

Here, as in the Dirichlet conditions case, we eliminate $k_{\gamma 3}$, $K_{33}$, and $L_3$. We also use notation (35).

The variational formulation (34) is here reduced to

$$\int_\Omega \big({}^t\widetilde{\mathbf{M}}\mathcal{R}^H\mathbf{M} + 2|Y_1|G\widetilde{L}_3\mathcal{M}(L_3)\big)\mathrm{d}x = l_u(\mathbf{v}) + l_\varphi\big(\mathcal{M}(\widetilde{L}_3)\big) \quad \forall\widetilde{\mathbf{M}} \in \mathbb{M}. \tag{38}$$

For test functions in $\mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$ one gets

$$\int_\Omega \big({}^t\mathcal{N}(\widetilde{\mathbf{M}})\mathcal{R}^H\mathcal{N}(\mathbf{M}) + {}^t\mathcal{M}(\widetilde{\mathbf{M}})\big(\mathcal{R}^H + 2|Y_1|G\Pi_1\big)\mathcal{M}(\mathbf{M})\big)\mathrm{d}x \\ = |Y_1|\int_\Omega \mathcal{H}\mathcal{M}(\Pi\mathbf{M})\,\mathrm{d}x, \tag{39}$$

where

$$\mathcal{H} := {}^t(\mathbf{0}, h).$$

Arguing as in the case of Dirichlet conditions, (39) implies $\Pi\mathcal{R}_G\mathcal{M}(\mathbf{M}) = |Y_1|\mathcal{H}$ and $\Pi\mathcal{R}^H\mathcal{N}(\mathbf{M}) = 0$, where $\mathcal{R}_G = \mathcal{R}^H + 2|Y_1|G\Pi_1$. Writing $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0$ with $\mathbf{M}^0 \in \mathbb{M}^0$, one gets then $\mathcal{N}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}^H\Pi)^{-1}\Pi\mathcal{R}^H\mathcal{N}(\mathbf{M}^0)$ and $\mathcal{M}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}_G\Pi)^{-1}\,(\Pi\mathcal{R}^H(\mathcal{M}(\mathbf{M}^0) - |Y_1|\mathcal{H})$. With definition (25) of $\mathbf{T}_\mathcal{N}$ and $\mathbf{T}_\mathcal{N}$ this is

$$\begin{cases} \mathcal{N}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{N})\mathcal{N}\big(\mathbf{M}^0\big), \\ \mathcal{M}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{M})\mathcal{M}\big(\mathbf{M}^0\big) + |Y_1|(\Pi\mathcal{R}_G\Pi)^{-1}\mathcal{H}. \end{cases} \tag{40}$$

For $\widetilde{\mathbf{M}}^0 \in \mathbb{M}^0$, let us define $\widetilde{\mathbf{M}} \in \mathbb{M}$ by: $\mathcal{N}(\widetilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_\mathcal{N})\mathcal{N}(\widetilde{\mathbf{M}}^0)$, $\mathcal{M}(\widetilde{\mathbf{M}}) = (\mathbf{Id} + \mathbf{T}_\mathcal{M})\mathcal{M}(\widetilde{\mathbf{M}}^0)$. Then from (38) and (40):

$$\int_\Omega \big({}^t\mathcal{M}(\widetilde{\mathbf{M}}^0)\mathcal{R}_\mathcal{M}\mathcal{M}(\mathbf{M}^0) + {}^t\mathcal{N}(\widetilde{\mathbf{M}}^0)\mathcal{R}_\mathcal{N}\mathcal{N}(\mathbf{M}^0)\big)\mathrm{d}x = l_u(\mathbf{v}), \\ + |Y_1|\int_\Omega {}^t\mathcal{M}(\widetilde{\mathbf{M}}^0)\big({}^t\mathbf{T}_\mathcal{M} - \big(\mathbf{Id} + {}^t\mathbf{T}_\mathcal{M}\big)\mathcal{R}_G(\Pi\mathcal{R}_G\Pi)^{-1}\big)\mathcal{H}\,\mathrm{d}x \quad \forall\widetilde{\mathbf{M}}^0 \in \mathbb{M}^0,$$

where $\mathbf{v}$ is the vector of $\mathbf{V}_{KL}$ associated with $\widetilde{\mathbf{M}}$. With (37), this proves Theorem 5.1 for local mixed conditions.

### 6.3.3. *Case of Nonlocal Mixed Conditions*

Here, only $k_{\alpha 3}$, $K_{33}$, and $\mathcal{N}(L_3)$ are eliminated. Because of the nonlocal term in $G_1$, $\mathcal{M}(L_3)$ cannot be eliminated. We also use notation (35). The variational formulation of Lemma 6.5 is here

$$\int_\Omega \big(\widetilde{\mathbf{M}}\mathcal{R}^H\mathbf{M} + 2|Y_1|G\widetilde{L}_3\mathcal{M}(L_3)\big)\mathrm{d}x + 2|Y_1|G_1\int_\Omega \partial_\alpha\mathcal{M}(L_3)\partial_\alpha\widetilde{L}_3\,\mathrm{d}x$$
$$= l_u(\mathbf{v}) + l_\varphi\big(\mathcal{M}\big(\widetilde{L}_3\big)\big) \quad \forall\widetilde{\mathbf{M}} \in \mathbb{M}. \tag{41}$$

For $\widetilde{\mathbf{M}} \in \mathbb{M}^{-1} \oplus \mathbb{M}^{-2}$, we thus obtain

$$\int_\Omega \big({}^t\mathcal{N}\big(\widetilde{\mathbf{M}}\big)\mathcal{R}\mathcal{N}(\mathbf{M}) + {}^t\mathcal{M}\big(\widetilde{\mathbf{M}}\big)\big(\mathcal{R} + 2|Y_1|G\Pi_1\big)\mathcal{M}(\mathbf{M})\big)\mathrm{d}x$$
$$+ 2|Y_1|G_1\int_\Omega \partial_\alpha\mathcal{M}(L_3)\partial_\alpha\widetilde{L}_3\mathrm{d}x = 0. \tag{42}$$

With the decomposition $\mathbf{M} = \Pi\mathbf{M} + \mathbf{M}^0$, we deduce as before $\mathcal{N}(\Pi\mathbf{M}) = -(\Pi\mathcal{R}^H\Pi)^{-1}\mathcal{N}(\mathbf{M}^0)$. Concerning $\mathcal{M}(\mathbf{M})$, we use the decomposition $\mathbf{M} = \Pi_1\mathbf{M} + \Pi_2\mathbf{M} + \mathbf{M}^0$. With test functions $\widetilde{\mathbf{M}} \in \mathcal{M}(\Pi_2\mathbb{M})$ in (42), we therefore get $\mathcal{M}(\Pi_2\mathbf{M}) = -(\Pi_2\mathcal{R}^H\Pi_2)^{-1}(\mathcal{M}(\mathbf{M}^0) + \mathcal{M}(\Pi_2\mathbf{M})$. This leads to

$$\begin{cases} \mathcal{N}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{N})\mathcal{N}\big(\mathbf{M}^0\big), \\ \mathcal{M}(\mathbf{M}) = (\mathbf{Id} + \mathbf{T}_\mathcal{M})\mathcal{M}\big(\mathbf{M}^0 + \Lambda_3\big). \end{cases} \tag{43}$$

With the appropriate choice of test functions in (41) the weak formulation of Theorem 5.1 (ii) follows as in Sections 6.3.1 and 6.3.2.

## 7. Example

In this section, we propose quite explicit formulae for operators and effective coefficients described in Section 5. We consider the particular case of a plate made up of transversally isotropic material with Dirichlet electrical boundary conditions.

### 7.1. PRELIMINARIES

The projections $\Pi$, $\Pi_1$ and $\Pi_2$ are:

$$\Pi = \begin{pmatrix} \mathbf{0}_{2\times2\times2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2\times2} & (\delta_{\alpha,\mu})_{\alpha,\mu=1,2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 1 & \mathbf{0}_2 & 0 \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 1 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} \mathbf{0}_{2\times2\times2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 0 \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 1 \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} \mathbf{0}_{2\times2\times2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2\times2} & (\delta_{\alpha,\mu})_{\alpha,\mu=1,2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 1 & \mathbf{0}_2 & 0 \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & \mathbf{0}_2 & 0 & \mathbf{0}_2 & 0 \end{pmatrix}.$$

Then,

$$\Pi\mathcal{R} = \begin{pmatrix} \mathbf{0}_{2\times2\times2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} \\ (R_{\alpha3\gamma\delta})_{\alpha,\gamma,\delta=1,2} & (4R_{\alpha3\gamma3})_{\alpha,\gamma=1,2} & (2R_{\alpha333})_{\alpha=1,2} & (2d_{\gamma\alpha3})_{\alpha,\gamma=1,2} & (2d_{3\alpha3})_{\alpha=1,2} \\ (R_{33\gamma\delta})_{\gamma,\delta=1,2} & (2R_{33\gamma3})_{\gamma=1,2} & R_{3333} & (d_{\gamma33})_{\gamma=1,2} & d_{333} \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ (-d_{3\gamma\delta})_{\gamma,\delta=1,2} & (-2d_{3\gamma3})_{3,\gamma=1,2} & -d_{333} & (c_{3\gamma})_{\gamma=1,2} & c_{33} \end{pmatrix}$$

and

$$\Pi\mathcal{R}\Pi = \begin{pmatrix} \mathbf{0}_{2\times2\times2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2\times2} & (4R_{\alpha3\gamma3})_{\alpha,\gamma=1,2} & (2R_{\alpha333})_{\alpha=1,2} & \mathbf{0}_{2\times2} & (2d_{3\alpha3})_{\alpha=1,2} \\ \mathbf{0}_{2\times2} & (2R_{33\gamma3})_{\gamma=1,2} & R_{3333} & \mathbf{0}_2 & d_{333} \\ \mathbf{0}_{2\times2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_2 \\ \mathbf{0}_{2\times2} & (-2d_{3\gamma3})_{3,\gamma=1,2} & -d_{333} & \mathbf{0}_2 & c_{33} \end{pmatrix}.$$

The matrix $(\Pi\mathcal{R}\Pi)^{-1}$ is computed by inverting the above matrix, omitting the rows and columns of zeros, that is:

$$\begin{pmatrix} (4R_{\alpha3\gamma3})_{\alpha,\gamma=1,2} & (2R_{\alpha333})_{\alpha=1,2} & (2d_{3\alpha3})_{\alpha=1,2} \\ (2R_{33\gamma3})_{\gamma=1,2} & R_{3333} & d_{333} \\ (-2d_{3\gamma3})_{3,\gamma=1,2} & -d_{333} & c_{33} \end{pmatrix}.$$

The result of inversion is then replaced in a $10 \times 10$ matrix.

With this, the computation of $\mathbf{T}_{\mathcal{N}} = -(\Pi\mathcal{R}^H\Pi)^{-1}\Pi\mathcal{R}^H$ is clear. The computation of $\mathbf{T}_{\mathcal{M}}$ is similar, and the effective coefficients follow. We go into detail about these last computations in the next subsection.

### 7.2. EXPLICIT FORMULAE FOR A TRANSVERSALLY ISOTROPIC MATERIAL WITH DIRICHLET CONDITIONS

The first step consists in computing the effective homogenized coefficients $\mathcal{R}^H$. This cannot be done analytically, but can be obtained by standard numerical computation. Here, we assume that $\mathcal{R}^H$ is known and has the same form of isotropy as $\mathcal{R}$. We compute the stiffness tensor of the two-dimensional plate model. For simplicity, the index $^H$ on the coefficient of $\mathcal{R}^H$ has been removed.

For a transversally isotropic material, the global stiffness-piezoelectricity-permitivity tensor

$$\mathcal{R} = \left( \begin{array}{cccccc} R_{1111} & R_{1112} & R_{1121} & R_{1122} & 2R_{1113} & 2R_{1123} \\ R_{1211} & R_{1212} & R_{1221} & R_{1222} & 2R_{1213} & 2R_{1223} \\ R_{2111} & R_{2112} & R_{2121} & R_{2122} & 2R_{2113} & 2R_{2123} \\ R_{2211} & R_{2212} & R_{2221} & R_{2222} & 2R_{2213} & 2R_{2223} \\ 2R_{1311} & 2R_{1312} & 2R_{1321} & 2R_{1322} & 4R_{1313} & 4R_{1323} \\ 2R_{2311} & 2R_{2312} & 2R_{2321} & 2R_{2322} & 4R_{2313} & 4R_{2323} \\ R_{3311} & R_{3312} & R_{3321} & R_{3322} & R_{3313} & R_{3323} \\ -d_{111} & -d_{112} & -d_{121} & -d_{122} & -2d_{113} & -2d_{123} \\ -d_{211} & -d_{212} & -d_{221} & -d_{222} & -2d_{213} & -2d_{223} \\ -d_{311} & -d_{312} & -d_{321} & -d_{322} & -2d_{313} & -2d_{323} \end{array} \right.$$

$$\left. \begin{array}{cccc} R_{1133} & d_{111} & d_{211} & d_{311} \\ R_{1233} & d_{112} & d_{212} & d_{312} \\ R_{2133} & d_{121} & d_{221} & d_{221} \\ R_{2233} & d_{122} & d_{222} & d_{322} \\ R_{1333} & 2d_{113} & 2d_{213} & 2d_{313} \\ R_{2333} & 2d_{123} & 2d_{223} & 2d_{323} \\ R_{3333} & d_{133} & d_{233} & d_{333} \\ -d_{133} & c_{11} & c_{12} & c_{13} \\ -d_{233} & c_{21} & c_{22} & c_{23} \\ -d_{333} & c_{31} & c_{31} & c_{33} \end{array} \right)$$

reduces to

$$\left( \begin{array}{ccccc} C_{11} & 0 & 0 & C_{12} & 0 \\ 0 & (C_{11}-C_{12})/2 & 0 & 0 & 0 \\ 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ C_{12} & 0 & 0 & C_{11} & 0 \\ 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \\ 0 & 0 & 0 & 0 & 0 \\ C_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2E_{15} \\ 0 & 0 & 0 & 0 & 0 \\ -E_{31} & 0 & 0 & -E_{31} & 0 \end{array} \right.$$

$$\left. \begin{array}{ccccc} 0 & C_{12} & 0 & 0 & E_{31} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & C_{12} & 0 & 0 & E_{31} \\ 0 & 0 & 2E_{15} & 0 & 0 \\ 2(C_{11}-C_{12}) & 0 & 0 & 2E_{24} & 0 \\ 0 & C_{11} & 0 & 0 & E_{33} \\ 0 & 0 & \varepsilon_{11} & 0 & 0 \\ -2E_{24} & 0 & 0 & \varepsilon_{11} & 0 \\ 0 & -E_{33} & 0 & 0 & \varepsilon_{22} \end{array} \right)$$

where $C_{11}$, $C_{12}$, $E_{31}$, $E_{24}$, $E_{15}$, $\varepsilon_{11}$, and $\varepsilon_{22}$ are fixed positive constants. Application of the formulas to Dirichlet conditions: $\mathcal{R}_{\mathcal{N}} = (Id + {}^t\mathbf{T}_{\mathcal{N}})\mathcal{R}(Id+\mathbf{T}_{\mathcal{N}})$ and $\mathcal{R}_{\mathcal{M}} = (Id + {}^t\mathbf{T}_{\mathcal{M}})\mathcal{R}(Id + \mathbf{T}_{\mathcal{M}})$ gives the expression of the stiffness coefficients of the two-dimensional model:

$$\begin{pmatrix} R_{\mathcal{M}1111} & R_{\mathcal{M}1112} & R_{\mathcal{M}1121} & R_{\mathcal{M}1122} \\ R_{\mathcal{M}1211} & R_{\mathcal{M}1212} & R_{\mathcal{M}1221} & R_{\mathcal{M}1222} \\ R_{\mathcal{M}2111} & R_{\mathcal{M}2112} & R_{\mathcal{M}2121} & R_{\mathcal{M}2122} \\ R_{\mathcal{M}2211} & R_{\mathcal{M}2212} & R_{\mathcal{M}2221} & R_{\mathcal{M}2222} \end{pmatrix} = \begin{pmatrix} C_{11} - C_{12}^2/C_{11} & 0 & 0 & C_{12} \\ 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ C_{12} - C_{12}^2/C_{11} & 0 & 0 & C_{11} \end{pmatrix},$$

and

$$\begin{pmatrix} R_{\mathcal{N}1111} & R_{\mathcal{N}1112} & R_{\mathcal{N}1121} & R_{\mathcal{N}1122} \\ R_{\mathcal{N}1211} & R_{\mathcal{N}1212} & R_{\mathcal{N}1221} & R_{\mathcal{N}1222} \\ R_{\mathcal{N}2111} & R_{\mathcal{N}2112} & R_{\mathcal{N}2121} & R_{\mathcal{N}2122} \\ R_{\mathcal{N}2211} & R_{\mathcal{N}2212} & R_{\mathcal{N}2221} & R_{\mathcal{N}2222} \end{pmatrix} = \begin{pmatrix} R_{\mathcal{N}1111} & 0 & 0 & R_{\mathcal{N}1122} \\ 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ R_{\mathcal{N}2211} & 0 & 0 & R_{\mathcal{N}2222} \end{pmatrix},$$

where

$$\begin{aligned}
R_{\mathcal{N}1111} &= \big(\big(C_{12}^2E_{33}^2\varepsilon_{22} + E_{31}^2\varepsilon_{22}^3 + C_{11}E_{33}\big(2C_{12}^2E_{33} + E_{31}^2E_{33} + E_{33}^3 \\
&\quad - 2C_{12}E_{31}\varepsilon_{22}\big) - 2C_{11}^2\big(-C_{12}E_{31}E_{33} + C_{12}^2\varepsilon_{22} + E_{33}^2\varepsilon_{22}\big) \\
&\quad + 2C_{12}E_{31}E_{33}\big(E_{33}^2 - \varepsilon_{22}^2\big) + C_{11}^3\big(C_{12}^2 + \varepsilon_{22}^2\big)\big)\big/\big(E_{33}^2 - C_{11}\varepsilon_{22}\big)^2, \\
R_{\mathcal{N}1122} &= \big(E_{31}^2\varepsilon_{22}^3 + C_{11}E_{33}\big(E_{31}^2E_{33} - C_{12}E_{31}\varepsilon_{22} - 2C_{12}E_{33}\varepsilon_{22}\big) \\
&\quad + C_{11}^2C_{12}\big(E_{31}E_{33} + \varepsilon_{22}^2\big) + C_{12}\big(-E_{31}E_{33}^3 + E_{33}^4 - E_{31}E_{33}\varepsilon_{22}^2\big)\big) \\
&\quad \big/\big(E_{33}^2 - C_{11}\varepsilon_{22}\big)^2, \\
R_{\mathcal{N}2211} &= \big(E_{31}^2\varepsilon_{22}^3 + C_{11}E_{33}\big(C_{12}^2E_{33} + E_{31}^2E_{33} - 2C_{12}(E_{31} + E_{33})\varepsilon_{22}\big) \\
&\quad + C_{12}\big(4E_{31}E_{33}^3 + E_{33}^4 - E_{31}E_{33}\varepsilon_{22}^2\big) + C_{11}^2C_{12}(E_{31}E_{33} \\
&\quad + \varepsilon_{22}(-C_{12} + \varepsilon_{22}))\big)\big/\big(E_{33}^2 - C_{11}\varepsilon_{22}\big)^2, \\
R_{\mathcal{N}2222} &= \big(-2C_{11}^2E_{33}^2\varepsilon_{22} + C_{11}^3\varepsilon_{22}^2 + C_{11}E_{33}\big(E_{31}^2E_{33} + E_{33}^3 - C_{12}E_{31}\varepsilon_{22}\big) \\
&\quad + E_{31}\big(C_{12}E_{33}^3 + E_{31}\varepsilon_{22}^3\big)\big)\big/\big(E_{33}^2 - C_{11}\varepsilon_{22}\big)^2\big).
\end{aligned}$$

The piezoelectric coefficients for the 2-dimensional plate model are given by

$$\begin{pmatrix} d_{\mathcal{M}311} \\ d_{\mathcal{M}312} \\ d_{\mathcal{M}321} \\ d_{\mathcal{M}322} \end{pmatrix} = \begin{pmatrix} E_{31} - (C_{12}E_{33})/C_{11} \\ 0 \\ 0 \\ E_{31} - (C_{12}E_{33})/C_{11} \end{pmatrix}.$$

REMARK 7.1.

(i) As compared with the complexity of these formulae, the formulation of Section 5 is very synthetic.
(ii) The above computations have been carried out with Mathematica.
(iii) For multilayered 2-dimensional plate models the results are much more complicated. Using our approach, the complexity is the same.

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