A two-scale model for the periodic homogenization of the wave equation

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Abstract

We present a new mathematical object designed to analyze the oscillations occurring on both microscopic and macroscopic scales in a wave equation with oscillating coefficients and data. Through a Bloch wave homogenization method, our study addresses typical problems of two-scale convergence in the interior of the domain, and sheds some light on the behavior near the boundary. A decoupled system of (systems of) transport equations is derived in each energy band, and the total energy field is approximated. We also recover previously known results in homogenization as a restricted part of our model.

Résumé

Nous présentons un nouvel objet mathématique conçu pour analyser les oscillations aux échelles microscopique et macroscopique de la solution de l'équation des ondes à coefficients et données oscillants. Notre étude traite de problèmes de convergence à deux échelles grâce à une méthode d'homogénéisation par ondes de Bloch. Il en résulte une famille de systèmes découplés d'équations de transport associés à chaque bande d'énergie qui conduit à une approximation de l'énergie totale. Nous retrouvons des résultats connus sur l'homogénéisation de l'équation des ondes comme étant une partie de notre modèle.

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1 Introduction

We establish a homogenized model for the Cauchy problem of the wave equation

$$\rho(\frac{x}{\varepsilon})\partial_{tt}^2 u^\varepsilon(t,x) - div_x\left(a(\frac{x}{\varepsilon})\nabla_x u^\varepsilon(t,x)\right) = f^\varepsilon(t,x) \tag{1}$$
$$u^\varepsilon(t=0,x) = u_0^\varepsilon(x) \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon(x)$$

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in a domain $\Omega \subset \mathbb{R}^N$ on the boundary of which mixed Dirichlet-Neumann conditions are applied. In order to describe how the solution $u^\varepsilon(t,x)$ inherits the possible oscillations of the data $(u_0^\varepsilon, v_0^\varepsilon, f^\varepsilon)$ as $\varepsilon \to 0$, we develop an asymptotic analysis of the first-order derivatives of u^ε in the framework of periodic homogenization (ρ and a being periodic with respect to a lattice of reference cell $Y \subset \mathbb{R}^N$). This setting is a particular but typical case of a more general situation, requiring the H-convergence of the coefficients, studied by S. Brahim-Otsmane, G.A. Francfort and F. Murat in [8]. Under quite general circumstances, they showed that the solution u^ε may be decomposed as the sum of a weakly oscillating part, for which the limit of the energy equals the energy of the limit, and of a highly oscillating part, corresponding to generic solutions of weak limit 0 but of total energy 1 equidistributed between the kinetic and potential energies. To our knowledge, very little information concerning this last problematic part is available at the present time. For instance, the transport equation derived for the H-measure of the energy density and the geometrical optics antsatz developped in Sections 2 and 3 of [20] only pertain to the case of constant coefficients ρ and a. Correspondingly, there does not seem to be any fully satisfactory theory based on the microlocal analysis techniques developped in [21], that would allow the rapidly varying coefficients of (1). However, adopting a slightly different standpoint, other works certainly offer interesting results on the highly oscillating part. Such is for example the recent paper [5] which focuses on the asymptotical regime of (1) for long times.

We also have to face here the specific problems due to the boundary $\partial\Omega$, which forbids the use of the spatial Fourier transform, an essential tool in the homogenization theory by Bloch waves in the case of $\Omega = \mathbb{R}^N$, see e.g. [16] and [15], as well as in the theory of defect measures, see e.g. the Wigner side in [21]. In the same respect, it would be hopeless for ε fixed to appeal (as usual in $\mathbb{R}^N$) to the Bloch reduction of $-\Delta_\varepsilon := -div_x\left(a(\frac{x}{\varepsilon})\nabla_x\right)$ as a direct sum of Fourier multipliers $\lambda_n(\varepsilon D_x)/\varepsilon^2$ acting on generalized eigenspaces (the Floquet subspaces). In fact, any kind of homogeneous structure for the problem set in $\Omega \neq \mathbb{R}^N$ only reveals itself *after* the limiting procedure $\varepsilon \to 0$.

Consequently, the asymptotic behavior of the spectrum of $-\Delta_\varepsilon$ turns to be of great interest. Since [32], it is known that the family of eigenvalues $\lambda^\varepsilon \geq 0$ solving the corresponding spectral problem

$$-div_x\left(a(\frac{x}{\varepsilon})\nabla_x\Phi^\varepsilon(x)\right) = \lambda^\varepsilon\rho(\frac{x}{\varepsilon})\Phi^\varepsilon(x),$$

may be splitted into two subfamilies of different nature. Indeed, infinitely-many eigenvalues converge as $\varepsilon \to 0$ towards the eigenvalues of the classically homogenized eigenproblem, while many others are of order $1/\varepsilon^2$ (i.e. when multiplied by ε^2 they converge towards some limits). Between these extreme cases of low and high frequencies, there is also the difficult notion of spectrum exhibited by G. Allaire and C. Conca in [3] and [4]. This so-called boundary layer spectrum fills the gaps within the Bloch spectrum, see [2] and [3] for the specific question of completeness. Unfortunately, its description as the spectrum of a limit operator acting on the boundary seems to be very sensitive to the particular shape of Ω and Y (according to whether Ω is an exact number of εY-cells or not), and also to the possible parametrization of $\varepsilon \to 0$ along a pre-assigned sequence, see [12]. In this paper, we wilfully discarded the limiting eigenvalues corresponding to the boundary layer spectrum to avoid the related difficulties, and only kept track of the solutions to the Bloch wave eigenproblem

$$-div_y\left(a(y)\nabla_y\Phi^k(y)\right) = \lambda^k\rho(y)\Phi^k(y) \tag{2}$$

associated with k-quasiperiodic conditions for varying $k \in \mathbb{R}^N$ i.e.

$$\Phi^k(y+\ell) = \Phi^k(y)e^{2i\pi k.\ell} \text{ for all } y \in \mathbb{R}^N \text{ and all } \ell \text{ in the lattice.}$$

Incorporating the boundary layer spectrum into our study in view of some completeness theorem could be the subject of a subsequent work.

Now to the general overview of the model. We start with a reformulation of the wave equation (1) as an equivalent system (of $N+1$ partial differential equations) satisfied by the first-order derivatives $U^\varepsilon := (\sqrt{a^\varepsilon}\nabla_x u^\varepsilon, \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon)$ considered as *the* quantity of interest. Our point of view will be to treat it as a general solution to a general first-order hyperbolic system. In very few occasions will we really return to u^ε itself and to the special form due to (1). Note that consequently our method is open to greater generality.

In order to study the weak convergence of the solutions and to guarantee that Bloch waves are kept in the limit, we first apply to $U^\varepsilon(t,x)$ a custom-made two-scale transform S_k^ε acting on x, which is a k-quasiperiodic version of the usual two-scale transformation used in [6], [25], [26], [27], [11], [13], [14]. Then, a parameterized (time) two-scale transform acting on t is applied separately to each Bloch wave, its period being sized to match the corresponding wave time-period $\varepsilon\alpha_n^k$ with $\alpha_n^k := 2\pi/(\lambda_n^k)^{1/2}$ given by the eigenvalues (λ_n^k) of (2). For a given fiber k, the resulting time-space two-scale transform $W_k^\varepsilon U^\varepsilon(t,\tau,x,y)$ will be called the one-fibered wave two-scale transform of U^ε, since it has been designed to capture the space-time waves in U^ε that show some k-quasiperiodic spatial oscillations. For further details see formulas (29),(35),(36) of Section 6. Note that the idea of the wave two-scale transform originated in [28] and [23].

All these waves are separated from one another by spatial orthogonality (w.r.t. the microscopic variable y). To recover as many Bloch waves as possible, we proceed in the spirit of [3], [2], [4], and replace the reference cell Y by a bigger one Y_K made of K^N copies of Y, on which K^N one-fibered wave two-scale transforms are encoded into our final transformation $W^\varepsilon := \sum_k W_k^\varepsilon$. For $K \in \mathbb{N}^*$ fixed, this harmless recollection only aims at a finer model, and should not cause too much worry at first reading. As in the construction of ordinary two-scale transforms, W^ε is a (pseudo) isometry in the time-space L^2-norm, in the sense that the norm is preserved apart from ε-terms originating from erratic portions of ε-cells near the boundary. So the L^2-boundedness of the solutions U^ε guarantees the L^2-boundedness of their wave two-scale transforms $W^\varepsilon U^\varepsilon$.

Passing to the limit in the integro-differential system solved by $W^\varepsilon U^\varepsilon$ yields a set of equations satisfied by any weak limit U of $W^\varepsilon U^\varepsilon$ in L^2. Some of them involve the microscopic derivatives $(\partial_\tau, \nabla_y)$ of U and enforce the decomposition

$$U = U_H(t,x,y) + \sum_k \sum_n U_n^k(t,x)e^{2i\pi\tau}e_n^k(y) + U_{-n}^k(t,x)e^{-2i\pi\tau}e_{-n}^k(y),$$

where $e^{\pm 2i\pi\tau}e_{\pm n}^k(y)$ are two internal microscopic waves, with opposite propagation senses, amplified by the macroscopic factors $U_{\pm n}^k(t,x)$. Note therein that the family of Bloch eigenvectors (e_n^k) will be built from the eigenvectors Φ^k of (2). Some others involve the macroscopic derivatives (∂_t, ∇_x) of U and govern the homogenized evolution of the internal wave amplitudes

$$\partial_t U_{\pm n}^k \mp \sum_m \kappa_{nm}^k . \nabla_x U_{\pm m}^k = F_{\pm n}^k \text{ for all } n \text{ and } k, \tag{3}$$

where the finite sum runs over all modes m with the same eigenvalue and propagation sense as n. However, while the initial conditions for (3) are easily identified, until now appropriate boundary conditions on $\partial\Omega$ are still lacking. This drawback of the model disappears when there is no boundary i.e. for $\Omega = \mathbb{R}^N$. Accordingly, the model obtained in this case yields a unique solution U, and shows a conservation of the space L^2-norm (strictly speaking when $f^\varepsilon = 0$) i.e.

$$\|U\|^2_{L^2(\mathbb{R}^N\times Y_K)}(t,\tau) = \|U\|^2_{L^2(\mathbb{R}^N\times Y_K)}(t=0,\tau=0) \tag{4}$$
$$+2\,\mathrm{Re}\int_0^t\int_{\mathbb{R}^N} f\cdot v + 2\,\mathrm{Re}\sum_k\sum_n\int_0^t\int_{\mathbb{R}^N} F_n^k\cdot U_n^k + F_{-n}^k\cdot U_{-n}^k,$$

with $f(t,x)$ the limit of f^ε and $v(t,x)$ a component of U_H. On the contrary, in the case of $\Omega \neq \mathbb{R}^N$, our results may appear comparatively incomplete, in the sense that the solutions U to our model are not necessarily unique, because of a partial loss of boundary conditions in the homogenization process. Supplementing (3) with appropriate boundary conditions on $U_{\pm n}$ (or on certain combinations of U_{+n} and U_{-n}) would close the problem.

Note that the limit U includes a low frequency part U_H, which turns out to be purely periodic in y. This part, which shows no serious oscillations in time, naturally inherits the mixed boundary conditions of (1), and coincides with the well-posed homogenized model exhibited in [20] and [8]. At the opposite, the internal waves of U concentrate all the fast time-oscillations of U^ε, which can be revealed through the substitution $\tau = t/\varepsilon\alpha_n^k$ with $\alpha_n^k := 2\pi/(\lambda_n^k)^{1/2}$.

For a given $K \in \mathbb{N}^*$, an interpretation of our theorem of convergence (Theorem 19) expresses that the physical field U^ε can be approximated by

$$U^\varepsilon(t,x) \approx U_H(t,x,\frac{x}{\varepsilon}) + \sum_k\sum_n U_n^k(t,x)e^{2i\pi t/\varepsilon\alpha_n^k}e_n^k(\frac{x}{\varepsilon}) + U_{-n}^k(t,x)e^{-2i\pi t/\varepsilon\alpha_n^k}e_{-n}^k(\frac{x}{\varepsilon}).$$

In the special case of $\Omega = \mathbb{R}^N$, more can be said (Theorem 40) about the error made in the time-space L^2-norm. Once this model has been obtained, we can make $K \to \infty$, so that all Bloch eigenvectors e_n^k tend to take part in the decomposition.

In case of isolated bands, the system obtained in (3) reduces to a single scalar transport equation, whose constant coefficient $\kappa_{nn}^k = \nabla_k(1/\alpha_n^k)$ is the k-gradient of the corresponding n^{th} frequency of oscillations. This phenomenon is in complete agreement with the transport equations derived in [21] for the Wigner measure in an energy band.

The paper is organized as follows. In Section 2 we introduce standard notations used throughout the paper. In Section 3 we interpret the original scalar wave equation as a first-order hyperbolic system. In Section 4 we prove an existence and uniqueness property of the solutions as well as a uniform a priori estimate in the energy norm (Theorem 3). In Section 5 we detail the spectral analysis of the wave operator viewed as an action on the spatial microscopic scale only. Particularly, are gathered there all the definitions and properties of all objects of a spectral nature to be used later on. In Section 6 we define and study the space two-scale, the time two-scale and the wave two-scale transforms. A key equivalence is established in Lemmas 14 and 16 between the convergence of the two-scale transform of a sequence and the two-scale convergence in the sense of the testing method of [1]. At the beginning of Section 7, we describe the homogenized model in detail, and state our main results on the convergence of the wave two-scale transforms (Theorems 19 and 22). The remaining part of Section 7 is devoted to the proofs. Most of the work is probably contained in Proposition 30, where the decoupling of modes with non-crossing eigenvalues is exhibited in the spirit of band-Wigner-measure techniques. Note also that the special form induced by the original problem is not essentially used till Subsection 7.9, where final simplifications are taken into account. In Section 8 we conclude the paper with an approximation result in the energy norm (Theorem 39), which should convince the reader that the wave two-scale transform put forward before was indeed the right object.

To finish with, we must mention that up to now we failed to generalize Theorems 19 and 22 when the coefficients $(\rho^\varepsilon, a^\varepsilon) = (\rho(x,x/\varepsilon), a(x,x/\varepsilon))$ vary on both scales. Most of the material below adapts quite well, but we found ourselves in serious trouble in the course of Proposition 30, when trying to exhibit the expected destructive interaction between internal waves with different time-frequencies $1/\alpha_n^k$ of oscillations, these frequencies depending now on x. The same obstacle occurs in other problems, for instance when a Schroedinger equation with a periodic microscopic potential is perturbed by a slowly varying potential, see [9].

2 Notations

In this section we bring together some conventions used all along the paper. The convergence symbol $\to$ always relates to the limit as $\varepsilon \to 0$. The letter C stands for possibly different constants. Every scalar quantity is complex-valued unless otherwise stated. If $U = (U_i)$ and $V = (V_i)$ are m-dimensional vectors we set $U.V := \sum_i U_i V_i$ and $U \cdot V := \sum_i U_i \overline{V_i}$ as well as $|U|^2 := U \cdot U$. In some occasions we write $U = ([U]_N, [U]_D)$ where the scalar $[U]_D$ denotes the last component of U, and where $[U]_N$ is the remaining part of U. The hilbertian space $(L^2)^m$ of square integrable m-dimensional vectors is normed by $||U||_{L^2}^2 := \int U \cdot U$ but, when the microscopic unit cell $Y \subset \mathbb{R}^N$ defined hereafter is involved, we make an exception by using the averaged norm

$$||U||_{L^2(Y)} := \left(\frac{1}{|Y|} \int_Y |U|^2 \, dy \right)^{1/2},$$

where $|Y|$ denotes the Lebesgue measure of Y. As a rule, $\sharp$ means periodic, $\mathcal{L}$ means linear continuous, $\mathbb{1}$ stands for the characteristic function of a set, and every derivative is to be understood in a distributional sense ($\mathcal{D}'$) even if actually it is almost always a function here. As usual, $\mathcal{C}^0$ is the space of all bounded continuous functions endowed with the uniform norm of L^∞. $H_\sharp^s$ refers to the periodic version of the usual Sobolev space H^s made of all L^2-functions whose $s > 0$ first generalized derivatives are in L^2. H^{div} refers to the hilbertian space of L^2-fields whose divergence is in L^2, see [18] Ch. IV and IX for a detailed description. We define $\mathcal{C}^\infty(\overline{O})$ as the space of all restrictions of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ to a given open subset $O \subset \mathbb{R}^d$, where the subscript c in $\mathcal{C}_c^\infty$ requires that the functions be compactly supported. We use N as the space dimension and ν as the smallest integer strictly greater than $N/2$. We assume the coefficients ρ and a regular in the sense that their ν first derivatives are bounded:

$$\rho \in W^{\nu,\infty}(\mathbb{R}^N) \text{ and } a \in W^{\nu,\infty}(\mathbb{R}^N)^{N\times N}. \tag{5}$$

3 The physical problem

Let $I = [0, T) \subset \mathbb{R}^+$ be a finite time interval. Let Ω be an open subset of $\mathbb{R}^N$ with a bounded Lipschitz boundary $\partial\Omega$ endowed with its natural $(N-1)$-dimensional measure $d\sigma$. We fix a possibly trivial splitting [1] of $\partial\Omega$ into two disjoint parts Γ_D and Γ_N where Dirichlet and Neumann boundary conditions are applied. We denote by $n_\Omega \in L^\infty(\partial\Omega)$ the outer unit normal of $\partial\Omega$. We consider u^ε solution to a linear scalar wave equation with time-independent oscillatory coefficients $(\rho^\varepsilon, a^\varepsilon)$ and time-dependent source term f^ε, supplemented with given initial values $(u_0^\varepsilon, v_0^\varepsilon)$ and boundary conditions $(g^\varepsilon, h^\varepsilon)$:

$$\begin{gathered} \rho^\varepsilon \partial_{tt}^2 u^\varepsilon - div(a^\varepsilon \nabla u^\varepsilon) = f^\varepsilon \text{ in } I \times \Omega, \\ u^\varepsilon(t=0) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t=0) = v_0^\varepsilon \text{ in } \Omega, \\ u^\varepsilon = g^\varepsilon \text{ on } I \times \Gamma_D \text{ and } a^\varepsilon \nabla u^\varepsilon . n_\Omega = h^\varepsilon \text{ on } I \times \Gamma_N. \end{gathered} \tag{6}$$

[1] Up to a set whose $d\sigma$ measure is null. Moreover the case $\Gamma_D = \Gamma_N = \partial\Omega = \varnothing$ is allowed ($\Omega = \mathbb{R}^N$).

Here as usual $0 < \varepsilon < 1$ denotes a small parameter intended to go to zero and indexing the data and hence the solution $u^\varepsilon = u^\varepsilon(t,x)$. Note that the setup allows general ε-dependent families apart from $(\rho^\varepsilon, a^\varepsilon)$ whose oscillations are assumed to obey a prescribed profile:

$$a^\varepsilon := a(\frac{x}{\varepsilon}), \quad \rho^\varepsilon := \rho(\frac{x}{\varepsilon}),$$

where $\rho(y)$ is real-valued and where $a(y)$ is a $N \times N$ symmetric matrix, both being Lipschitz periodic on $\mathbb{R}^N$ with the same periodicity in y. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions:

$$\rho_0 \le \rho(y) \le \rho_1 \text{ and } a_0|\xi|^2 \le a(y)\xi.\xi \le a_1|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N,$$

for some given positive ρ_0, ρ_1, a_0 and a_1. By setting

$$U^\varepsilon := (\sqrt{a^\varepsilon}\nabla u^\varepsilon, \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon), \quad F^\varepsilon := (0, f^\varepsilon/\sqrt{\rho^\varepsilon}), \tag{7}$$

$$U_0^\varepsilon := (\sqrt{a^\varepsilon}\nabla u_0^\varepsilon, \sqrt{\rho^\varepsilon}v_0^\varepsilon), \quad G^\varepsilon := \begin{pmatrix} \mathbb{1}_{\Gamma_D}\partial_t g^\varepsilon \sqrt{a^\varepsilon} n_\Omega \\ \mathbb{1}_{\Gamma_N} h^\varepsilon/\sqrt{\rho^\varepsilon} \end{pmatrix}, \tag{8}$$

$$A^\varepsilon := \begin{pmatrix} 0 & \sqrt{a^\varepsilon}\nabla(\frac{1}{\sqrt{\rho^\varepsilon}}.) \\ \frac{1}{\sqrt{\rho^\varepsilon}} div(\sqrt{a^\varepsilon}.) & 0 \end{pmatrix}, \; n_A^\varepsilon := \frac{1}{\sqrt{\rho^\varepsilon}} \begin{pmatrix} 0 & \sqrt{a^\varepsilon} n_\Omega \\ \sqrt{a^\varepsilon} n_\Omega & 0 \end{pmatrix}, \tag{9}$$

we recast the scalar wave equation (6) as a first-order system of size $N+1$,

$$\begin{gathered} (\partial_t - A^\varepsilon)U^\varepsilon = F^\varepsilon \text{ in } I \times \Omega, \\ U^\varepsilon(t=0) = U_0^\varepsilon \text{ in } \Omega, \\ [n_A^\varepsilon U^\varepsilon]_D = [G^\varepsilon]_D \text{ on } I \times \Gamma_D \text{ and } [n_A^\varepsilon U^\varepsilon]_N = [G^\varepsilon]_N \text{ on } I \times \Gamma_N. \end{gathered} \tag{10}$$

From now on, this system will be referred to as the physical problem, and will be understood in a distributional sense including boundary conditions, namely:

$$\begin{aligned} \int_{I\times\Omega} F^\varepsilon \cdot \psi \; dxdt + \int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi \; dxdt + \int_\Omega U_0^\varepsilon \cdot \psi(t=0) \; dx \\ + \int_{I\times\partial\Omega} G^\varepsilon \cdot \psi \; d\sigma dt = 0 \end{aligned} \tag{11}$$

for all admissible test functions ψ in

$$\mathcal{V}^\varepsilon := \{\psi \in H^1(I\times\Omega)^{N+1} \mid \psi(t,.) \in D(A^\varepsilon) \text{ a.e. in } t \in I \text{ and } \psi(T,.) = 0\},$$

where the dense domain $D(A^\varepsilon) \subset L^2(\Omega)^{N+1}$ is defined by

$$\begin{gathered} D(A^\varepsilon) := \{(\varphi, \phi) \in L^2(\Omega)^N \times L^2(\Omega) \mid \sqrt{a^\varepsilon}\varphi \in H^{div}(\Omega), \phi/\sqrt{\rho^\varepsilon} \in H^1(\Omega), \\ \gamma_n(\sqrt{a^\varepsilon}\varphi) = 0 \text{ on } \Gamma_N \text{ and } \gamma(\phi/\sqrt{\rho^\varepsilon}) = 0 \text{ on } \Gamma_D\}. \end{gathered}$$

Here $\gamma \in \mathcal{L}(H^1(\Omega); H^{1/2}(\partial\Omega))$ and $\gamma_n \in \mathcal{L}(H^{div}(\Omega); H^{-1/2}(\partial\Omega))$ stand for the usual trace operator and the usual normal-trace operator.

4 Uniform a priori estimates

This section is mainly concerned with the properties of the ε-parameterized wave equation (6). We establish an existence and uniqueness result based on the self-adjointness of iA^ε, together with an L^2-bound of the solution uniformly in ε.

Theorem 1 *The operator iA^ε with domain $D(A^\varepsilon)$ is self-adjoint on $L^2(\Omega)^{N+1}$. Moreover,*

$$D := \{(\varphi,\phi) \in \mathcal{C}^\infty(\overline{\Omega})^N \times \mathcal{C}^\infty(\overline{\Omega}) \mid \varphi = 0 \text{ on } \Gamma_N \text{ and } \phi = 0 \text{ on } \Gamma_D\} \tag{12}$$

is a core for iA^ε, in other words iA^ε with domain D is essentially self-adjoint on $L^2(\Omega)^{N+1}$.

We will not report the proof of the self-adjointness of iA^ε because it would be much the same as that of iA_k proved in Theorem 9 below. We just recall here that $D \subset D(A^\varepsilon)$ is said to be a core for A^ε when $D \subset D(A^\varepsilon)$ is dense in the graph norm sense $\psi \mapsto ||\psi||_{L^2(\Omega)} + ||A^\varepsilon\psi||_{L^2(\Omega)}$. If need be, see [24] Problems p. 269, for a short account of general properties of essentially self-adjoint operators.

Remark 2 *The use of D instead of $D(A^\varepsilon)$ in $\mathcal{V}^\varepsilon$ gives rise to a new subspace*

$$\mathcal{V} := \{\psi \in \mathcal{C}^\infty(\overline{I\times\Omega})^{N+1} \mid \psi \text{ has compact support in } I \text{ and } \psi(t,.) \in D \text{ for every } t \in I\}$$

of admissible test functions, but with the help of Theorem 1 it is an easy matter to check that both physical problems (11) with test functions in $\mathcal{V}^\varepsilon$ or $\mathcal{V}$ are in fact equivalent, so $\mathcal{V}^\varepsilon$ and $\mathcal{V}$ can be used indifferently. To motivate the introduction of D and $\mathcal{V}$, we refer the reader to Propositions 26, 34, 36 below, to see how our asymptotic analysis of (6) will ultimately rely on essential self-adjointness rather than on self-adjointness itself.

Theorem 3 *For any fixed ε, the physical problem (11) has a unique solution $U^\varepsilon \in L^2(I\times\Omega)^{N+1}$ for any $U_0^\varepsilon \in L^2(\Omega)^{N+1}$, $F^\varepsilon \in L^2(I\times\Omega)^{N+1}$, $\partial_t g^\varepsilon \in H^1(I;H^{1/2}(\Gamma_D))$, $h^\varepsilon \in H^1(I;H^{-1/2}(\Gamma_N))$. Moreover, U^ε satisfies the estimate*

$$||U^\varepsilon||_{L^2(I\times\Omega)} \le C(||F^\varepsilon||_{L^2(I\times\Omega)} + ||U_0^\varepsilon||_{L^2(\Omega)} + ||\partial_t g^\varepsilon||_{H^1(I;H^{1/2}(\Gamma_D))} + ||h^\varepsilon||_{H^1(I;H^{-1/2}(\Gamma_N))})$$

uniformly in ε.

Throughout the sequel, we will assume that the data are bounded in the sense [2]

$$||f^\varepsilon||_{L^2(I\times\Omega)} + ||\nabla u_0^\varepsilon||_{L^2(\Omega)} + ||v_0^\varepsilon||_{L^2(\Omega)} + ||\partial_t g^\varepsilon||_{H^1(I;H^{1/2}(\Gamma_D))} + ||h^\varepsilon||_{H^1(I;L^2(\Gamma_N))} \le C, \tag{13}$$

so that the solution U^ε be also bounded in $L^2(I\times\Omega)^{N+1}$ uniformly in ε.

Perhaps some comment is needed at this stage to explain how the norms of $\partial_t g^\varepsilon$ and h^ε originated. The point at stake was to provide handy conditions on G^ε for the physical problem (11) to admit uniformly bounded solutions U^ε. The question would have been fairly classical if the coefficients $(\rho^\varepsilon, a^\varepsilon)$ were not so heavily dependent on ε. As far as we know, any standard trace estimate on $(\partial_t - A^\varepsilon)U^\varepsilon$ would involve some derivatives of $(\rho^\varepsilon, a^\varepsilon)$, and as such would create an $1/\varepsilon$ explosion. A way to circumvent the problem could have been to assume that G^ε derives from such a bounded sequence U^ε. Considering this point of view as unsatisfactory, we prefered to taylor-make sufficient conditions on G^ε ensuring that G^ε does admit a bounded extension:

Proposition 4 *For any boundary conditions*

$$\partial_t g^\varepsilon \in H^1(I;H^{1/2}(\Gamma_D)) \text{ and } h^\varepsilon \in H^1(I;H^{-1/2}(\Gamma_N)),$$

there exists a solution V^ε to the physical problem (11) for some initial value and source term, satisfying the uniform estimate

$$\begin{aligned}||V^\varepsilon||_{L^2(I\times\Omega)} + ||\partial_t V^\varepsilon||_{L^2(I\times\Omega)} + ||A^\varepsilon V^\varepsilon||_{L^2(I\times\Omega)} + ||V^\varepsilon(t=0)||_{L^2(\Omega)}&\\ \le C(||\partial_t g^\varepsilon||_{H^1(I;H^{1/2}(\Gamma_D))} + ||h^\varepsilon||_{H^1(I;H^{-1/2}(\Gamma_N))}).&\end{aligned}$$

[2]The need to strengthen $H^{-1/2}(\Gamma_N)$ into $L^2(\Gamma_N)$ will be justified by (i) of Proposition 31 but is unessential in most cases in view of Remark 27.

Proof. Let us fix two bounded sequences $\partial_t g^\varepsilon \in H^1(I; H^{1/2}(\Gamma_D))$ and $h^\varepsilon \in H^1(I; H^{-1/2}(\Gamma_N))$ of boundary conditions. Since the usual restricted trace $\gamma \in \mathcal{L}(H^1(I\times\Omega); H^1(I; H^{1/2}(\Gamma_D)))$ is onto, Lemma 5 to come provides a bounded sequence q^ε in $H^1(I\times\Omega)$ such that $\partial_t g^\varepsilon = \gamma(q^\varepsilon)$. By the same argument, the restricted normal trace $\gamma_n \in \mathcal{L}(H^1(I; H^{div}(\Omega)); H^1(I; H^{-1/2}(\Gamma_N)))$ being onto, there exists a bounded sequence $p^\varepsilon \in H^1(I; H^{div}(\Omega))$ such that $h^\varepsilon = \gamma_n(p^\varepsilon)$.

By construction, $V^\varepsilon := ((a^\varepsilon)^{-1/2} p^\varepsilon, \sqrt{\rho^\varepsilon} q^\varepsilon)$ is a bounded sequence in $L^2(I\times\Omega)^{N+1}$ such that $\partial_t V^\varepsilon$ and $A^\varepsilon V^\varepsilon$ are bounded in $L^2(I\times\Omega)^{N+1}$. The same is true of $V^\varepsilon(t=0) = V^\varepsilon(t) - \int_0^t \partial_t V^\varepsilon(s)\, ds$ for I finite. Now, we remark that V^ε is sufficiently regular for us to apply a Green-like formula

$$\begin{aligned}\int_{I\times\Omega} (\partial_t - A^\varepsilon)V^\varepsilon\cdot\psi + V^\varepsilon\cdot(\partial_t - A^\varepsilon)\psi\; dxdt + \int_\Omega V^\varepsilon(t=0)\cdot\psi(t=0)\; dx \\ + \int_{I\times\partial\Omega} G^\varepsilon\cdot\psi\; d\sigma dt = \int_{I\times\partial\Omega} (G^\varepsilon - n_A^\varepsilon V^\varepsilon)\cdot\psi\; d\sigma dt \end{aligned} \tag{14}$$

for all $\psi \in \mathcal{C}^\infty(\overline{I\times\Omega})^{N+1}$ with compact support in I, where

$$n_A^\varepsilon V^\varepsilon = (\gamma(q^\varepsilon)\sqrt{a^\varepsilon} n_\Omega, \gamma_n(p^\varepsilon)/\sqrt{\rho^\varepsilon})$$

coincides with G^ε on $I\times\partial\Omega$ in the sense that

$$\gamma(q^\varepsilon)\sqrt{a^\varepsilon} n_\Omega = \partial_t g^\varepsilon \sqrt{a^\varepsilon} n_\Omega \text{ on } I\times\Gamma_D \text{ and } \gamma_n(p^\varepsilon)/\sqrt{\rho^\varepsilon} = h^\varepsilon/\sqrt{\rho^\varepsilon} \text{ on } I\times\Gamma_N.$$

As a consequence, (14) is null whenever $\psi(t,.) \in D$. According to Remark 2, the physical problem (11) has a solution V^ε associated with some $U_0^\varepsilon := V^\varepsilon(t=0)$ and $F^\varepsilon := (\partial_t - A^\varepsilon)V^\varepsilon$ as claimed. ■

Lemma 5 *Let E, F, G be three Hilbert spaces with a continuous embedding $G \subset F$ and let $\Phi \in \mathcal{L}(E,F)$. If G is a subset of the range of Φ then there exists $\Psi \in \mathcal{L}(G,E)$ such that $\Phi\circ\Psi = 1$ on G.*

Proof. Our assumptions imply that $||v||_H := \sqrt{||v||_E^2 + ||\Phi(v)||_G^2}$ is a hilbertian norm on $H := \Phi^{-1}(G)$ and that $\Phi \in \mathcal{L}(H;G)$ is onto. Since the kernel of Φ in H has a topological complement subspace (typically its orthogonal subspace), Φ turns out to be invertible on the right, see [10] Ch. II Th. II.10, i.e. there exists $\Psi \in \mathcal{L}(G;H)$ such that $\Phi\circ\Psi = 1$. Obviously $\Psi \in \mathcal{L}(G;E)$. ■

Once Proposition 4 has been fully established, we are in a position to prove Theorem 3:

Proof. By the theory of unitary groups generated by self-adjoint operators, we already know that the physical problem (11) has a unique solution (a so-called mild solution in [31] built as a strong limit of classical solutions) for any $U_0^\varepsilon \in L^2(\Omega)^{N+1}$ and $F^\varepsilon \in L^1(I; L^2(\Omega))^{N+1}$ whenever $G^\varepsilon = 0$. Of course, the uniqueness property for the general non-homogeneous problem (with $G^\varepsilon \neq 0$) follows at once by linearity. As for the existence, let V^ε be as built in the proof of Proposition 4, and let W^ε be the mild solution to the physical problem (11) with initial value $U_0^\varepsilon - V^\varepsilon(t=0) \in L^2(\Omega)^{N+1}$, source term $F^\varepsilon - (\partial_t - A^\varepsilon)V^\varepsilon \in L^2(I\times\Omega)^{N+1}$ and null boundary conditions. For I finite, it satisfies the classical estimate of continuity with respect to data

$$||W^\varepsilon||_{L^\infty(I;L^2(\Omega))} \leq C(||U_0^\varepsilon - V^\varepsilon(t=0)||_{L^2(\Omega)} + ||F^\varepsilon - (\partial_t - A^\varepsilon)V^\varepsilon||_{L^1(I;L^2(\Omega))}).$$

Then $U^\varepsilon := V^\varepsilon + W^\varepsilon$ is a solution to (11) with initial value U_0^ε, source term F^ε, and boundary condition G^ε thanks to (14). Moreover,

$$\begin{aligned}||U^\varepsilon||_{L^2(I\times\Omega)} \leq ||V^\varepsilon||_{L^2(I\times\Omega)} + ||W^\varepsilon||_{L^2(I\times\Omega)} \\ \leq C(||V^\varepsilon||_{L^2(I\times\Omega)} + ||U_0^\varepsilon - V^\varepsilon(t=0)||_{L^2(\Omega)} + ||F^\varepsilon - (\partial_t - A^\varepsilon)V^\varepsilon||_{L^2(I\times\Omega)}),\end{aligned}$$

from which the announced estimate follows using Proposition 4. ■

5 Multi-fibered spectral analysis

We recall parts of the classical discrete Bloch-wave machinery for second order elliptic operators with periodic coefficients.

5.1 Bloch decompositions

Let $L \subset \mathbb{R}^N$ denote the N-dimensional lattice of $\mathbb{R}^N$ associated with the periodicity of (ρ, a), and let $Y \subset \mathbb{R}^N$ be a unit parallelepiped open cell such that $\mathbb{R}^N = \overline{Y} + L$. Through a choice of basis, the dual lattice $L^* \subset \mathbb{R}^N$ of L can be described as $L^* = \mathbb{Z}b_1^* + ... + \mathbb{Z}b_N^*$, where $(b_1^*, ..., b_N^*)$ is the dual basis of an arbitrary $\mathbb{Z}$-basis $(b_1, ..., b_N)$ taken in L. As a consequence, $\ell.\ell^* \in \mathbb{Z}$ for all $\ell \in L$ and $\ell^* \in L^*$, and L^* is in fact the largest such lattice-solution. The corresponding torus $\mathbb{R}^N/L^*$ can then be identified with an arbitrarily chosen cell $Y^* \subset \mathbb{R}^N$ of L^*.

Now, given $K \in \mathbb{N}^*$, we observe that the dual lattices KL and L^*/K satisfy $L = L_K + KL$ and $L^* + L_K^* = L^*/K$ for some fundamental subsets $L_K \subset L$ and $L_K^* \subset L^*/K$ of common cardinal K^N, such that $L_K \cap (KL) = \{0\}$ and $L_K^* \cap L^* = \{0\}$. Also, we introduce a set Y_K made of K^N cells indexed by L_K and translated from Y, such that Y_K tends to cover $\mathbb{R}^N$ when K increases.

Example 6 *If $L = \mathbb{Z}^N$ then $L^* = \mathbb{Z}^N$. We can choose the canonical basis of $\mathbb{R}^N$ as direct and dual basis, $Y = Y^* = (0,1)^N$ as unit cell, and $L_K^* = L_K/K$ with*

$$\begin{cases} L_K = \{-\frac{K}{2}, ..., \frac{K}{2} - 1\}^N, Y_K = (-\frac{K}{2}, \frac{K}{2})^N \text{ if } K \text{ is even,} \\ \\ L_K = \{-\frac{K-1}{2}, ..., \frac{K-1}{2}\}^N, Y_K = (-\frac{K-1}{2}, \frac{K+1}{2})^N \text{ if } K \text{ is odd.} \end{cases}$$

For any $k \in Y^*$, we define the k-quasiperiodic L^2-space by

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}^N) \mid u(x+\ell) = u(x)e^{2i\pi k.\ell} \text{ a.e. for all } \ell \in L\},$$

or equivalently

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}^N) \mid \exists v \in L_\sharp^2 \text{ such that } u(x) = v(x)e^{2i\pi k.x} \text{ a.e.}\},$$

where $L_\sharp^2$ is the traditional notation for the periodic case L_k^2 with $k = 0$. Likewise, we set $H_k^s := L_k^2 \cap H_{loc}^s(\mathbb{R}^N)$ and $H_k^{div} := (L_k^2)^N \cap H_{loc}^{div}(\mathbb{R}^N)$, bearing in mind that the subscript $\sharp$ would be more appropriate in the periodic case $k = 0$. If ϖ_k denotes the k-quasiperiodic extension operator

$$\varpi_k : L^2(Y) \to L_k^2 \tag{15}$$

which maps any $u \in L^2(Y)$ on the unique $v \in L_k^2$ such that $u = v$ in Y, then the following characterizations may be checked:

$$\begin{aligned} \forall u \in H^1(Y),\ \varpi_k u \in H_{loc}^1(\mathbb{R}^N) \iff u \in H_k^1(Y), \\ \forall u \in H^{div}(Y),\ \varpi_k u \in H_{loc}^{div}(\mathbb{R}^N) \iff u \in H_k^{div}(Y), \end{aligned} \tag{16}$$

with

$$\begin{aligned} H_k^1(Y) &:= \{u \in H^1(Y) \mid u_{|E_+} = e^{2i\pi k.\ell_E} u_{|E_-} \text{ for any opposite edges } E = (E_+, E_-)\}, \\ H_k^{div}(Y) &:= \{u \in H^{div}(Y) \mid (u.n_Y)_{|E_+} = e^{2i\pi k.\ell_E} (u.n_Y)_{|E_-} \text{ for any } E = (E_+, E_-)\}. \end{aligned}$$

Here n_Y represents the outward unit normal of ∂Y, and $\ell_E \in L$ stands everywhere for the unique L-translation mapping $E_- \subset \partial Y$ onto its opposite edge $E_+ \subset \partial Y$. As a matter of fact, these properties identify H^1_k with $H^1_k(Y)$ and H^{div}_k with $H^{div}_k(Y)$, in the same way as L^2_k is naturally identified with the space $L^2(Y)$ of its restrictions to Y.

Theorem 7 *If $L^2(Y_K)$ is viewed as the space $L^2_\sharp(Y_K)$ of all KL-periodic $L^2_{loc}(\mathbb{R}^N)$-functions, endowed with the usual norm $u \mapsto (\frac{1}{|Y_K|}\int_{Y_K} |u|^2\, dy)^{1/2}$, then $L^2(Y_K)$ is the hilbertian sum of K^N subspaces*

$$L^2(Y_K) = \overset{\perp}{\bigoplus_{k \in L^*_K}} L^2_k.$$

Moreover, the hilbertian structures induced by $L^2(Y)$ and $L^2(Y_K)$ coincide on L^2_k. In particular, any family of hilbertian bases $(e^k_n)_n$ of $L^2(Y)$ for varying $k \in L^*_K$ provides a hilbertian basis $(\varpi_k e^k_n)_{n,k}$ of $L^2(Y_K)$ by union and k-quasiperiodization.

Proof. On the one hand, for any $u \in L^2_{k_u}$ with $k_u \in L^*_K$ and $v \in L^2_{k_v}$ with $k_v \in L^*_K$,

$$\frac{1}{|Y_K|}\int_{Y_K} u \cdot v\; dy = \mathbb{1}_{L^*}(k_u - k_v)\frac{1}{|Y|}\int_Y u \cdot v\; dy,$$

because of the orthogonality identity $\displaystyle\frac{1}{K^N}\sum_{\ell \in L_K} e^{2i\pi k.\ell} = \mathbb{1}_{L^*}(k)$ for all $k \in L^*/K$.

On the other hand, any $u \in L^2_\sharp(Y_K)$ is a sum $u = \sum u_k$ indexed by $k \in L^*_K$ of

$$u_k := \frac{1}{K^N}\sum_{\ell \in L_K} u(.+\ell)e^{-2i\pi k.\ell} \in L^2_k,$$

because of the dual orthogonality identity $\displaystyle\frac{1}{K^N}\sum_{k \in L^*_K} e^{-2i\pi k.\ell} = \mathbb{1}_{KL}(\ell)$ for all $\ell \in L$.

Obviously, each space L^2_k is a closed subspace of $L^2(Y_K)$ after restriction to Y_K. Altogether, this proves that the orthogonal sum $\bigoplus_{k \in L^*_K} L^2_k$ equals $L^2(Y_K)$ as claimed. Moreover, the equality

$$\frac{1}{|Y_K|}\int_{Y_K} u \cdot v\; dy = \frac{1}{|Y|}\int_Y u \cdot v\; dy \quad \text{for all } u, v \in L^2_k \tag{17}$$

shows that the scalar products of $L^2(Y)$ and $L^2(Y_K)$ are identical when restricted to L^2_k. ∎

Remark 8 *The above theorem has been stated in the scalar case. However, it applies equally to the decomposition of the space $L^2(Y_K)^{N+1}$ used throughout the paper. It will mainly be used to build a hilbertian basis of $L^2(Y_K)^{N+1}$ out of quasiperiodic eigenvectors of realizations of a standard elliptic operator. The procedure will roughly be as follows: first, build a hilbertian basis of $L^2(Y)$ made of $H^1_k(Y)$-eigenvectors for any fixed k, and second, recollect the global basis as the fiber k varies in L^*_K. Taken as a whole, this will constitute a multi-fibered spectral analysis of $L^2(Y_K)^{N+1}$.*

5.2 Spectral decompositions

We introduce the classical elliptic operators that govern the spectral analysis of the wave equation (6). Viewing y as the current variable, we set

$$\Delta_k := \frac{1}{\sqrt{\rho}} div_y(a\nabla_y \frac{1}{\sqrt{\rho}}.)$$

on the dense domain $D(\Delta_k) := \{\phi \in L^2(Y) \mid \phi/\sqrt{\rho} \in H^2_k(Y)\} \subset L^2(Y)$ for any $k \in Y^*$, and we classically check that $-\Delta_k$ is a non-negative self-adjoint operator with compact resolvent, see [32] Ch. IV Sect. 5 among others. As such, $-\Delta_k$ is reduced by a spectral hilbertian basis (ϕ^k_n) of $L^2(Y)$ such that

$$\phi^k_n \in D(\Delta_k) \text{ and } -\Delta_k \phi^k_n = \lambda^k_n \phi^k_n,$$

where λ^k_n is the non-negative increasing sequence of repeated eigenvalues of $-\Delta_k$. Note that the kernel of $-\Delta_k$ is null for $k \notin L^*$ and one-dimensional (generated by ϕ^0_1) otherwise, a reason for us to enumerate the spectral family (ϕ^k_n) by $n \in \mathbb{M}^k_+$ with $\mathbb{M}^k_+ := \mathbb{N}^*$ for $k \notin L^*$ and $\mathbb{M}^k_+ := \mathbb{N}^* - \{1\}$ otherwise, so that in either case $\phi^k_n \notin Ker(A_k)$ if $n \in \mathbb{M}^k_+$. We also agree to extend these sets by symmetry

$$\mathbb{M}^k := \mathbb{Z}^* \text{ for } k \notin L^* \text{ and } \mathbb{M}^k := \mathbb{Z}^* - \{-1, 1\} \text{ otherwise.} \tag{18}$$

Likewise we set

$$A_k := \begin{pmatrix} 0 & \sqrt{a}\nabla_y(\frac{1}{\sqrt{\rho}}.) \\ \frac{1}{\sqrt{\rho}} div_y(\sqrt{a}.) & 0 \end{pmatrix} \tag{19}$$

on the dense domain

$$D(A_k) := \{(\varphi, \phi) \in L^2(Y)^N \times L^2(Y) \mid \sqrt{a}\varphi \in H^{div}_k(Y),\ \phi/\sqrt{\rho} \in H^1_k(Y)\} \subset L^2(Y)^{N+1}.$$

Theorem 9 *For each $k \in Y^*$, the self-adjoint operator iA_k on $L^2(Y)^{N+1}$ is reduced by a spectral orthonormal family $(e^k_n)_{n\in\mathbb{M}^k}$ of $L^2(Y)^{N+1}$ in the sense that*

$$A_k = i \sum_{n\in\mathbb{M}^k} s_n \sqrt{\lambda^k_{|n|}} \Pi^k_n = 2i\pi \sum_{n\in\mathbb{M}^k} \frac{s_n}{\alpha^k_n} \Pi^k_n \quad \text{with } \alpha^k_n := \frac{2\pi}{\sqrt{\lambda^k_{|n|}}},$$

where s_n denotes the sign of n and Π^k_n the one-dimensional orthogonal projector onto

$$e^k_n := \frac{1}{\sqrt{2}} \begin{pmatrix} -i\frac{s_n}{\sqrt{\lambda^k_{|n|}}} \sqrt{a}\nabla_y(\phi^k_{|n|}/\sqrt{\rho}) \\ \phi^k_{|n|} \end{pmatrix}.$$

Moreover, the global sum $\Pi^k := \sum_{n\in\mathbb{M}^k} \Pi^k_n \in \mathcal{L}(L^2(Y)^{N+1})$ defines the orthogonal projector $1 - \Pi^k$ onto the kernel $Ker(A_k)$ of A_k.

Remark 10 *(i) In agreement with (16) and Remark 8, the eigenvector $e^k_n \in H^1_k(Y)^{N+1}$ in Theorem 9 will always be identified with its k-quasiperiodic extension $\varpi_k e^k_n \in H^1_{loc}(\mathbb{R}^N)^{N+1}$.*

(ii) It so happens that the spectrum $\sigma(iA_k) \subset \mathbb{R}$ of iA_k is symmetric with respect to the origin and purely punctual. The eigenvectors $\{e^k_{+n}, e^k_{-n}\}$ associated with opposite non-zero eigenvalues of iA_k are very similar since $\sqrt{2}e^k_n = -is_n v^k_{|n|} + w^k_{|n|}$ for all $n \in \mathbb{M}^k$ with

$$v^k_n := \frac{1}{\sqrt{\lambda^k_n}} \begin{pmatrix} \sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho}) \\ 0 \end{pmatrix} \text{ and } w^k_n := \begin{pmatrix} 0 \\ \phi^k_n \end{pmatrix} \text{ for all } n \in \mathbb{M}^k_+. \tag{20}$$

Of course $\{e^k_{+n}, e^k_{-n}\}$ and $\{v^k_n, w^k_n\}$ are two equivalent orthogonal bases in the range of $\Pi_{+n} + \Pi_{-n}$, which more or less play the same role in what follows. The first will lead to simplified microscopic equations, see (58), while the second will prove relevant to handle the boundary conditions on $\partial\Omega$ inherited from the physical problem (11), see Proposition 26.

(iii) It is worth noticing that iA_k is an operator with non-compact resolvent when $N > 1$, because the kernels $Ker(\frac{1}{z} - (z - A_k)^{-1}) \supset Ker(A_k)$ are then infinite-dimensional for any $z \in \mathbb{C}^* - \sigma(A_k)$. Typically, when $N = 2$ or $N = 3$, the kernel contains infinitely-many curl-functions: $Ker(A_k) \supset \frac{1}{\sqrt{a}} curl\ H^1_k(Y) \times \{0\}$. Nevertheless, the spectral resolution of iA_k can be carried out thanks to that of $-\Delta_k$, see (ii) of the following proof.

Proof. (i) SELF-ADJOINTNESS. The arguments given below to establish the self-adjointness of iA_k are based on elementary facts taken from the theory of H^1 and H^{div}-type spaces, as set out in [18] Ch. IV and IX. As often, everything will center on the ability for us to write fully general Green-like formulae. We start with the symmetry of iA_k. Taking $(p, q) \in D(A_k)$ and $(\varphi, \phi) \in D(A_k)$, we can express

$$\int_Y div_y(\sqrt{a}p) \cdot \phi/\sqrt{\rho} + \sqrt{a}p \cdot \nabla_y(\phi/\sqrt{\rho})\ dy = \langle \gamma_n(\sqrt{a}p) | \gamma(\overline{\phi}/\sqrt{\rho}) \rangle_{H^{-1/2}(\partial Y) \times H^{1/2}(\partial Y)} \tag{21}$$

$$\int_Y \nabla_y(q/\sqrt{\rho}) \cdot \sqrt{a}\varphi + \frac{q}{\sqrt{\rho}} \cdot div_y(\sqrt{a}\varphi)\ dy = \langle \gamma_n(\sqrt{a}\overline{\varphi}) | \gamma(q/\sqrt{\rho}) \rangle_{H^{-1/2}(\partial Y) \times H^{1/2}(\partial Y)} \tag{22}$$

by means of a duality bracket involving the usual trace γ and normal trace γ_n built on ∂Y, because $(\sqrt{a}p, \phi/\sqrt{\rho}) \in H^{div}(Y) \times H^1(Y)$ and $(\sqrt{a}\varphi, q/\sqrt{\rho}) \in H^{div}(Y) \times H^1(Y)$. As a consequence,

$$\int_Y A_k(p, q) \cdot (\varphi, \phi) + (p, q) \cdot A_k(\varphi, \phi)\ dy = \langle \gamma_n(\sqrt{a}p) | \gamma(\overline{\phi}/\sqrt{\rho}) \rangle + \langle \gamma_n(\sqrt{a}\overline{\varphi}) | \gamma(q/\sqrt{\rho}) \rangle$$

is seen to be zero by the balance of k-quasiperiodic conditions between (p, q) and (φ, ϕ). Stated another way, iA_k is a symmetric operator. Therefrom, the self-adjointness property only consists in identifying $D(A_k)$ with the adjoint domain $D(A^*_k)$ made of all $(\varphi, \phi) \in L^2(Y)^N \times L^2(Y)$ for which the linear form

$$(p, q) \mapsto \int_Y A_k(p, q) \cdot (\varphi, \phi)\ dy = \int_Y div_y(\sqrt{a}p) \cdot \phi/\sqrt{\rho} + \nabla_y(q/\sqrt{\rho}) \cdot \sqrt{a}\varphi\ dy \tag{23}$$

is continuous on $D(A_k)$ in the norm of $L^2(Y)$. But, particularizing (23) to $p = 0$ and $q/\sqrt{\rho} \in C^\infty_c(Y)$, respectively to $\sqrt{a}p \in C^\infty_c(Y)^N$ and $q = 0$, we see that $div_y(\sqrt{a}\varphi) \in L^2(Y)$ and $\phi/\sqrt{\rho} \in H^1(Y)$, so (21) and (22) make sense not only for $(p, q) \in D(A_k)$ but also for more general $\sqrt{a}p \in H^{div}(Y)$ and $q/\sqrt{\rho} \in H^1(Y)$.

It turns out that any non-zero linear form built on the boundary ∂Y is necessarily discontinuous in the sense of $L^2(Y)$. Therefore, the boundary forms

$$\begin{cases} \sqrt{a}p \in H^{div}_k(Y) \mapsto \langle \gamma_n(\sqrt{a}p) | \gamma(\overline{\phi}/\sqrt{\rho}) \rangle_{H^{-1/2}(\partial Y) \times H^{1/2}(\partial Y)} \in \mathbb{C} \\ q/\sqrt{\rho} \in H^1_k(Y) \mapsto \langle \gamma_n(\sqrt{a}\overline{\varphi}) | \gamma(q/\sqrt{\rho}) \rangle_{H^{-1/2}(\partial Y) \times H^{1/2}(\partial Y)} \in \mathbb{C} \end{cases}$$

given in (21) (22), and $L^2(Y)$-continuous by (23), must vanish identically. In other words, $\gamma_n(\sqrt{a}\varphi) \in H^{-1/2}(\partial Y)$ is orthogonal to $H^{1/2}_k(\partial Y) = \gamma(H^1_k(Y))$, and in the same way $\gamma(\phi/\sqrt{\rho}) \in H^{1/2}(\partial Y)$ is orthogonal to $H^{-1/2}_k(\partial Y) = \gamma_n(H^{div}_k(Y))$ with

$$H^{1/2}_k(\partial Y) := \{u \in H^{1/2}(\partial Y) \mid u \text{ is } k\text{-quasiperiodic on } \partial Y\},$$
$$H^{-1/2}_k(\partial Y) := \{v \in H^{-1/2}(\partial Y) \mid v \text{ is null on } H^{1/2}_k(\partial Y)\}.$$

Thanks to the description (16) of $H^1_k(Y)$ and $H^{div}_k(Y)$ from the inside, this well and truly means that $(\sqrt{a}\varphi, \phi/\sqrt{\rho}) \in H^{div}_k(Y) \times H^1_k(Y)$. Thus, we have checked that any element of $D(A_k^*)$ belongs to $D(A_k)$, which finally proves the expected equality of domains $D(A_k) = D(A_k^*)$.

(ii) Spectral decomposition. Using the orthogonality relations satisfied by $(\phi^k_n)_{n\in\mathbb{M}^k_+}$, namely:

$$\begin{cases} 0 = \displaystyle\int_Y \phi^k_n \cdot \phi^k_m \, dy = \int_Y a\nabla_y(\phi^k_n/\sqrt{\rho}) \cdot \nabla_y(\phi^k_m/\sqrt{\rho}) \, dy \quad \text{for any } m \neq n, \\ 1 = ||\phi^k_n||^2_{L^2(Y)} = ||\sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho})||^2_{L^2(Y)}/\lambda^k_n \quad \text{for any } n, \end{cases}$$

we see that $(e^k_n)_{n\in\mathbb{M}^k}$ is a hilbertian basis of the closed subspace $\mathcal{F} \subset L^2(Y)^{N+1}$ generated by the set of all e^k_n for varying $n \in \mathbb{M}^k$, these eigenvectors of A_k being associated with the corresponding eigenvalues $2i\pi s_n/\alpha^k_n$. We now prove the equality $\mathcal{F} = (Ker(A_k))^\perp$, or equivalently $\mathcal{F}^\perp = Ker(A_k)$, by checking a double inclusion. If $\psi \in Ker(A_k)$ then the relation

$$2i\pi s_n \int_Y e^k_n \cdot \psi \, dy = \alpha^k_n \int_Y A_k e^k_n \cdot \psi \, dy = \alpha^k_n \int_Y e^k_n \cdot A^{(*)}_k \psi \, dy = 0 \text{ for all } n$$

shows that $\psi \in \mathcal{F}^\perp$. Conversely, if $\psi = (\varphi, \phi) \in \mathcal{F}^\perp$ then $\int_Y e^k_n \cdot \psi \, dy = \int_Y e^k_{-n} \cdot \psi \, dy = 0$ leads to

$$\int_Y \varphi \cdot \sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho}) \, dy = \int_Y \phi \cdot \phi^k_n \, dy = 0 \text{ for all } n.$$

As a consequence, ϕ is null when $k \notin L^*$ and proportional to ϕ^0_1 when $k \in L^*$. So $\nabla_y(\phi/\sqrt{\rho}) = 0$ in either case. Likewise, $div_y(\sqrt{a}\varphi) = 0$ results from the fact that $\sqrt{a}\varphi$ is orthogonal to $\nabla_y(\phi^k_n/\sqrt{\rho})$ for all n and hence to any $\nabla_y h$ with $h \in H^1_k(Y)$. This last extension, which will finally complete the proof of the membership $(\varphi, \phi) \in Ker(A_k)$, relies on the totality of the family $(\nabla_y(\phi^k_n/\sqrt{\rho}))_n$ when viewed in the subspace $\mathcal{G} \subset L^2(Y)^N$ defined by $\mathcal{G} = \nabla_y H^1_k(Y) = \nabla_y(H^1_k(Y)/\sqrt{\rho})$. To check this density property, we first point out that $\mathcal{G}$ is closed in $L^2(Y)^N$. Indeed, by the compactness of the embedding $H^1(Y) \subset L^2(Y)$, the gradient operator $\nabla_y \in \mathcal{L}(H^1_k(Y); L^2(Y)^N)$ maps every bounded closed subset of $H^1_k(Y)$ onto a bounded closed subset of $L^2(Y)^N$, and as such is an operator with closed range, see [22] Ch. IV Th. IV.1.10 p. 99. Since the multiplication operator $\sqrt{a}$ is an isomorphism of $L^2(Y)^N$, we also know that $\sqrt{a}\mathcal{G}$ is closed in $L^2(Y)^N$. Next, we recast the question as the equivalent totality of the family $(\sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho}))_n$ in $\sqrt{a}\mathcal{G}$. But, taking $h \in H^1_k(Y)$ such that $\sqrt{a}\nabla_y(h/\sqrt{\rho})$ is L^2-orthogonal in $\sqrt{a}\mathcal{G}$ to $\sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho})$ for all n, the spectral equation of ϕ^k_n yields

$$\int_Y \sqrt{a}\nabla_y(h/\sqrt{\rho}) \cdot \sqrt{a}\nabla_y(\phi^k_n/\sqrt{\rho}) \, dy = \lambda^k_n \int_Y h \cdot \phi^k_n \, dy = 0 \text{ for all } n,$$

and this again implies that h is null when $k \notin L^*$ and proportional to ϕ^0_1 when $k \in L^*$. So $\sqrt{a}\nabla_y(h/\sqrt{\rho}) = 0$ in either case. As a consequence, the L^2-orthogonal in $\mathcal{G}$ of the set made of all $\nabla_y(\phi^k_n/\sqrt{\rho})$ for varying n is null, and the stated totality is proved. ■

We end this discussion with a fundamental identity of differential calculus relating the physical operator A^ε of (9) and the spectral operator $A := A_k$ of (19). For any regular vector $\psi = \psi(x, y)$ depending on both space scales, an easy computation yields

$$A^\varepsilon\big(\psi(x, \frac{x}{\varepsilon})\big) = \Big((\frac{1}{\varepsilon}A + B)\psi\Big)(x, \frac{x}{\varepsilon}), \tag{24}$$

where the operator B is defined as the result of the formal substitution of x-derivatives for y-derivatives in A i.e.

$$B := \begin{pmatrix} 0 & \sqrt{a}\nabla_x(\frac{1}{\sqrt{\rho}}.) \\ \frac{1}{\sqrt{\rho}}div_x(\sqrt{a}.) & 0 \end{pmatrix}. \tag{25}$$

5.3 Asymptotic spectral estimates

We list here the few properties of the spectral elements (λ_n^k) and (e_n^k) to be used later on.

Lemma 11 *The order of magnitude of λ_n^k for large n is given by*

$$\alpha_0 n^{2/N} \leq \lambda_n^k \leq \alpha_1 n^{2/N},$$

where $0 < \alpha_0 \leq \alpha_1 < \infty$ are independent of $k \in Y^*$ and $n \geq 2$. Note also that the first level satisfies $\lambda_1^0 = 0 \leq \lambda_1^k \leq \alpha_1$.

Lemma 12 *The corresponding asymptotic behavior of e_n^k for large n is*

$$||e_n^k||_{H^1(Y)} \leq \alpha(1 + ||\nabla\rho||_{L^\infty(Y)} + ||\nabla a||_{L^\infty(Y)})n^{1/N},$$
$$||e_n^k||_{L^\infty(Y)} \leq \alpha(1 + ||\nabla\rho||_{L^\infty(Y)} + ||\nabla a||_{L^\infty(Y)})n,$$

where $0 < \alpha < \infty$ is independent of $k \in Y^*$ and $n \geq 1$.

The proofs will be omitted. Note also that the constants $\alpha, \alpha_0, \alpha_1$ may be chosen as functions of $(\rho_0, \rho_1, a_0, a_1)$ only, and accordingly, that the spectral estimates hold as soon as (ρ, a) is Lipschitz periodic.

We close this subsection with a preparatory result concerning the field

$$\kappa_{nm}^k := \frac{i}{2\sqrt{\lambda_{|n|}^k}} \frac{1}{|Y|} \int_Y \frac{\phi_{|m|}^k}{\sqrt{\rho}} a\nabla_y(\frac{\overline{\phi}_{|n|}^k}{\sqrt{\rho}}) - \frac{\overline{\phi}_{|n|}^k}{\sqrt{\rho}} a\nabla_y(\frac{\phi_{|m|}^k}{\sqrt{\rho}})\ dy \in \mathbb{C}^N, \tag{26}$$

whose relevance will appear more clearly after (47).

Lemma 13 *If λ_n^k is a simple eigenvalue of $-\Delta_k$ then $\kappa_{nn}^k = \nabla_k(1/\alpha_n^k)$.*

Proof. Decomposing $\phi_n^k(y) = \sqrt{\rho}\Phi_n^k(y)e^{2i\pi k.y}$ with Φ_n^k periodic on $\mathbb{R}^N$, we rewrite the eigenequation of ϕ_n^k, the normalization relation $||e_n^k|| = 1$ and the expression (26) of κ_{nn}^k as

$$\begin{gathered} 0 = E_n^k := div_y(a\nabla_y\Phi_n^k) + 2i\pi k.a\nabla_y\Phi_n^k + 2i\pi div_y(ak\Phi_n^k) + (\lambda_n^k\rho - 4\pi^2 ak.k)\Phi_n^k, \\ \frac{1}{|Y|}\int_Y \rho|\Phi_n^k|^2\ dy = 1 \text{ and } \lambda_n^k = \frac{1}{|Y|}\int_Y \Theta_n^k\ dy \\ \text{where } \Theta_n^k := a\nabla_y\Phi_n^k.\nabla_y\overline{\Phi}_n^k + 4\pi^2|\Phi_n^k|^2 ak.k + 4\pi k.\operatorname{Im}(\overline{\Phi}_n^k a\nabla_y\Phi_n^k), \\ 4\pi\sqrt{\lambda_n^k}\kappa_{nn}^k = \frac{1}{|Y|}\int_Y \Xi_n^k\ dy \text{ where } \Xi_n^k := 8\pi^2|\Phi_n^k|^2 ak + 4\pi\operatorname{Im}(\overline{\Phi}_n^k a\nabla_y\Phi_n^k). \end{gathered} \tag{27}$$

Now, a lengthy but elementary calculation (only based on iterations of the product formula $\partial(uv) = v\partial u + u\partial v$) leads to the identity

$$2\operatorname{Re}(\overline{\Phi}_n^k\nabla_k E_n^k) = 2\rho|\Phi_n^k|^2\nabla_k\lambda_n^k + \lambda_n^k\nabla_k(\rho|\Phi_n^k|^2) - \nabla_k\Theta_n^k - \Xi_n^k + \Big(2\operatorname{Re} div_y H_n^k(j)\Big)_{j=1,\ldots,N} \tag{28}$$

where

$$H_n^k(j) := \overline{\Phi}_n^k a \nabla_y \frac{\partial \Phi_n^k}{\partial k_j} + 2i\pi \overline{\Phi}_n^k a k \frac{\partial \Phi_n^k}{\partial k_j} + 2i\pi |\Phi_n^k|^2 a \frac{\partial k}{\partial k_j}$$

is periodic (like Φ_n^k and $\nabla_k \Phi_n^k$). Integrating (28) over $y \in Y$ and taking advantage of (27), we get $\nabla_k \lambda_n^k = 4\pi \sqrt{\lambda_n^k} \kappa_{nn}^k$ or equivalently $\kappa_{nn}^k = \nabla_k (1/\alpha_n^k)$. This concludes the proof provided that we legitimate the formal derivation ∇_k used above. In fact, given any fixed value of the parameter k_0 such that $\lambda_n^{k_0}$ is simple, there exists a neighborhood $\mathcal{N}$ of k_0 for which the simple eigenvalue $k \in \mathcal{N} \mapsto \lambda_n^k \in \mathbb{R}$ is analytic, and for which the corresponding eigenvector $k \in \mathcal{N} \mapsto \phi_n^k \in L^2(Y)$ may be *chosen* analytic. Due to the simplicity assumption, this property may be considered as easy in Kato's perturbation theory of analytically dependent operators.

More specifically, we refer the reader to [24] Ch. II Section 4 p 98 for the construction of a regular parametrization of ϕ_n^k. ■

6 Two-scale transforms

Let us start with the construction of the space two-scale transform. We first split the physical domain Ω into a large number of εY-cells up to a small left-over region $\Omega - \Omega_\varepsilon$ around the boundary $\partial\Omega$ by setting $\Omega_\varepsilon := \bigcup \mathcal{C}_\varepsilon$, where $\mathcal{C}_\varepsilon := \{\varepsilon\ell + \varepsilon Y \mid \ell \in L, \, \varepsilon\ell + \varepsilon Y \subset \Omega\}$ is the set of all cells fully contained in Ω. For any $k \in Y^*$, we then define $S_k^\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y)$ by the formula

$$S_k^\varepsilon u(x,y) := \sum_{\omega_\varepsilon \in \mathcal{C}_\varepsilon} u(\varepsilon\ell_{\omega_\varepsilon} + \varepsilon y) e^{-2i\pi k.\ell_{\omega_\varepsilon}} \mathbb{1}_{\omega_\varepsilon}(x), \tag{29}$$

where $\varepsilon\ell_{\omega_\varepsilon} \in \varepsilon L$ stands for the unique node of ω_ε. We check at once the pseudo-isometric property

$$||S_k^\varepsilon u||^2_{L^2(\Omega\times Y)} = \frac{1}{|Y|} \int_{\Omega\times Y} |S_k^\varepsilon u|^2 \, dydx = \int_{\Omega_\varepsilon} |u|^2 \, dx = ||u||^2_{L^2(\Omega_\varepsilon)} \leq ||u||^2_{L^2(\Omega)} \tag{30}$$

for all $u \in L^2(\Omega)$. A straightforward consequence is that $S_k^\varepsilon u^\varepsilon \in L^2(\Omega \times Y)$ has limit points in the weak convergence of $L^2(\Omega \times Y)$ as ε vanishes, whenever $u^\varepsilon \in L^2(\Omega)$ remains uniformly bounded. Such a limit point is nothing else but a two-scale limit in the sense of [1], as is made clear by our two-scale conversion lemma:

Lemma 14 *Let $\varphi \in L^2(Y; H^\nu(\Omega))$ and $k \in Y^*$. If $\varphi := \varpi_k \varphi$ is k-quasiperiodically extended on $\mathbb{R}^N$, then*

$$\int_\Omega |\varphi|^2 (x, \frac{x}{\varepsilon}) \, dx \leq C(N, \partial\Omega) ||\delta_x^\nu \varphi||^2_{L^2(\Omega\times Y)} \tag{31}$$

where $\delta_x := (1 - \Delta_x)^{1/2}$. Moreover, for any $u \in L^2(\Omega)$,

$$\left| \frac{1}{|Y|} \int_{\Omega\times Y} S_k^\varepsilon u \cdot \varphi \, dydx - \int_\Omega u(x) \cdot \varphi(x, \frac{x}{\varepsilon}) \, dx \right| \leq ||u||_{L^2(\Omega)} R_\varepsilon(\varphi) \tag{32}$$

where $R_\varepsilon(\varphi)$ tends to zero with ε.

Remark 15 *(i) The operator δ_x could be replaced by any differential operator with constant coefficients controlling the $H^1(\Omega)$-norm uniformly on Y.*

(ii) Given any regular function $\varphi(x,y)$ such that $|\varphi|^2(x,y)$ is periodic in y, the usual integral $\int_\Omega |\varphi|^2(x, \frac{x}{\varepsilon}) \, dx$ can be estimated in many ways, see [1] and [30] among others. For instance,

the following straightforward inequality holds true with $r_\varepsilon \to 0$ depending on the geometry of $\partial\Omega$ only:

$$\int_\Omega |\varphi|^2(x, \frac{x}{\varepsilon})\ dx \le \int_\Omega ||\varphi(., \frac{x}{\varepsilon})||^2_{L^\infty(\Omega)}\ dx \le |\Omega|(1+r_\varepsilon)\frac{1}{|Y|}\int_Y ||\varphi(., y)||^2_{L^\infty(\Omega)}\ dy.$$

The resulting bound is essentially equivalent to (31) in its principle but involves a non-hilbertian norm $||.||_{L^\infty}$, and only applies to measure-bounded domains Ω. We devised (31) precisely to avoid this inconvenience, even if the counterpart is a logically higher regularity assumption in x (but not in y, which is fundamental in the sequel).

(iii) For any regular k-quasiperiodic function φ, the convergence

$$\int_\Omega \varphi(x, \frac{x}{\varepsilon})\ dx \to \frac{1\!\!1_{L^*}(k)}{|Y|}\int_{\Omega\times Y}\varphi(x, y)\ dxdy$$

is readily checked, for instance as a particular case of (32). Re-interpreting the integral in the righthand side, we could even drop $1\!\!1_{L^*}(k)$, since in the theory [17] of almost periodic functions the L^*-mean value of a k-quasiperiodic function is zero for $k \notin L^*$.

Proof. (i) We first prove (31). Thanks to the Lipschitz regularity of the bounded boundary $\partial\Omega$, there exists an extension operator $J \in \mathcal{L}(L^2(Y; H^\nu(\Omega)); L^2(Y; H^\nu(\mathbb{R}^N)))$ with respect to the x-variable, see [29] Ch. II Sect. 3.6 - 3.7. We set $\widetilde{\varphi} := \varpi_k(J\varphi) \in L^2_{loc}(\mathbb{R}^N; H^\nu(\mathbb{R}^N))$ and compute

$$\int_\Omega |\varphi|^2(x, \frac{x}{\varepsilon})\ dx = \int_\Omega |\widetilde{\varphi}|^2(x, \frac{x}{\varepsilon})\ dx \le \int_{\mathbb{R}^N} |\widetilde{\varphi}|^2(x, \frac{x}{\varepsilon})\ dx$$

$$= \sum_{\omega_\varepsilon}\frac{1}{|Y|}\int_Y |\omega_\varepsilon|\ |J\varphi|^2(\varepsilon\ell_{\omega_\varepsilon} + \varepsilon y, y)\ dy \le \sum_{\omega_\varepsilon}\frac{1}{|Y|}\int_Y |\omega_\varepsilon|\ ||J\varphi(., y)||^2_{L^\infty(\omega_\varepsilon)}\ dy.$$

But the Sobolev embedding for a unit cell asserts after dilation that

$$|\omega|\ ||\cdot||^2_{L^\infty(\omega)} \le C_N||\cdot||^2_{H^\nu(\omega)}$$

holds true with a constant C_N depending only on N provided that $|\omega| \le 1$. Applying this estimate to $J\varphi(., y)$ yields

$$\int_\Omega |\varphi|^2(x, \frac{x}{\varepsilon})\ dx \le C_N\sum_{\omega_\varepsilon}\frac{1}{|Y|}\int_Y ||J\varphi(., y)||^2_{H^\nu(\omega_\varepsilon)}\ dy = \frac{C_N}{|Y|}\int_Y ||J\varphi(., y)||^2_{H^\nu(\mathbb{R}^N)}\ dy$$

$$\le \frac{||J||C_N}{|Y|}\int_Y ||\varphi(., y)||^2_{H^\nu(\Omega)}\ dy \le \frac{||J||C_N}{|Y|}\int_Y ||\delta^\nu_x\varphi(., y)||^2_{L^2(\Omega)}\ dy = ||J||C_N||\delta^\nu_x\varphi||^2_{L^2(\Omega\times Y)},$$

which completes the proof of (i) since $||J||$ may only depend on the geometry of $\partial\Omega$.

(ii) We now establish (32) for any $\varphi \in L^2(Y; H^{\nu+1}(\Omega))$. Simple calculations and changes of variables lead to

$$\frac{1}{|Y|}\int_{\Omega\times Y} S^\varepsilon_k u\cdot\varphi\ dydx = \int_{\Omega_\varepsilon} u(x)\left(\frac{1}{|\omega^x_\varepsilon|}\int_{\omega^x_\varepsilon}\overline{\varphi}(X, \frac{x}{\varepsilon})\ dX\right)dx, \tag{33}$$

$$\frac{1}{|\omega^x_\varepsilon|}\int_{\omega^x_\varepsilon}\left(\varphi(X, \frac{x}{\varepsilon}) - \varphi(x, \frac{x}{\varepsilon})\right)\ dX = \frac{\varepsilon}{|Y|}\int_0^1\int_{\frac{\omega^x_\varepsilon - x}{\varepsilon}} y'.\nabla_x\varphi(x + \varepsilon s y', \frac{x}{\varepsilon})\ dy'ds, \tag{34}$$

where $\omega_\varepsilon^x \in \mathcal{C}_\varepsilon$ denotes the ε-cell in Ω containing x. Combining (33) and (34) we obtain

$$\left|\frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon u \cdot \varphi \; dydx - \int_{\Omega_\varepsilon} u(x)\cdot \varphi(x,\frac{x}{\varepsilon}) \; dx\right|$$
$$\leq \; C_Y \varepsilon ||u||_{L^2(\Omega)} \left(\int_{\Omega_\varepsilon} ||\nabla_x \varphi(.,\frac{x}{\varepsilon})||^2_{L^\infty(\omega_\varepsilon^x)} \; dx\right)^{1/2}.$$

Now we apply the above-mentioned Sobolev inequality to $\nabla_x\varphi(.,x/\varepsilon)$ on $\omega = \omega_\varepsilon^x$ for any fixed $x \in \Omega_\varepsilon$ and remark that

$$\int_{\Omega_\varepsilon} \frac{1}{|\omega_\varepsilon^x|} ||\nabla_x\varphi(.,\frac{x}{\varepsilon})||^2_{H^\nu(\omega_\varepsilon^x)} \; dx = \frac{1}{|Y|}\int_Y ||\nabla_x \varphi(.,y)||^2_{H^\nu(\Omega_\varepsilon)} \; dy.$$

This yields

$$\left|\frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon u \cdot \varphi \; dydx - \int_{\Omega_\varepsilon} u(x)\cdot\varphi(x,\frac{x}{\varepsilon}) \; dx\right| \leq C\varepsilon ||u||_{L^2(\Omega)} ||\nabla_x\varphi||_{L^2(Y;H^\nu(\Omega))}.$$

It remains to estimate the integral of $|\varphi|^2$ on $\Omega - \Omega_\varepsilon \overset{ae}{=} \Omega \cap \bigcup \mathcal{C}'_\varepsilon$ with $\mathcal{C}'_\varepsilon := \{\varepsilon\ell + \varepsilon Y \mid \ell \in L$, $\varepsilon\ell + \varepsilon Y \not\subset \Omega$ and $(\varepsilon\ell+\varepsilon Y)\cap\Omega \neq \varnothing\}$. But

$$\int_{\Omega-\Omega_\varepsilon} |\varphi|^2(x,\frac{x}{\varepsilon}) \; dx = \sum_{\omega_\varepsilon\in\mathcal{C}'_\varepsilon} \int_{\Omega\cap\omega_\varepsilon} |J\varphi|^2(x,\frac{x}{\varepsilon}) \; dx \leq \sum_{\omega_\varepsilon\in\mathcal{C}'_\varepsilon}\int_{\omega_\varepsilon} ||J\varphi(.,\frac{x}{\varepsilon})||^2_{L^\infty(\omega_\varepsilon)} \; dx$$

$$= \sum_{\omega_\varepsilon\in\mathcal{C}'_\varepsilon} \frac{1}{|Y|}\int_Y |\omega_\varepsilon| \; ||J\varphi(.,y)||^2_{L^\infty(\omega_\varepsilon)} \; dy \leq \frac{C_N}{|Y|}\int_Y ||J\varphi(.,y)||^2_{H^\nu(\cup\mathcal{C}'_\varepsilon)} \; dy$$

tends to zero because the Lebesgue measure of $\bigcup\mathcal{C}'_\varepsilon \subset \mathbb{R}^N$ is small with ε.

(iii) We end the proof with a regularization step. So far (32) has been proved for any $\varphi_c \in L^2(Y;H^{\nu+1}(\Omega))$. To extend it by density to any $\varphi \in L^2(Y;H^\nu(\Omega))$, it is enough to let ε go to zero and then φ_c to φ in the following easy estimate

$$\left|\frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon u\cdot\varphi \; dydx - \int_\Omega u(x)\varphi(x,\frac{x}{\varepsilon}) \; dx\right| \leq C(N,\partial\Omega)||u||_{L^2(\Omega)}||\delta_x^\nu(\varphi-\varphi_c)||_{L^2(\Omega\times Y)}$$
$$+\left|\frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon u\cdot\varphi_c \; dydx - \int_\Omega u(x)\cdot\varphi_c(x,\frac{x}{\varepsilon}) \; dx\right|$$

based on (30) and (31). Altogether, this completes the construction of a negligible upper bound of the type $||u||_{L^2(\Omega)}R_\varepsilon(\varphi)$ in (32). ■

Let us adapt what precedes to the definition of the time two-scale transform. Taking $\mathbb{Z}\subset\mathbb{R}$ as a canonical lattice and $\Lambda = (0,1)$ as a unit cell, we set $I_\varepsilon := \bigcup\mathcal{C}^+_\varepsilon$, where $\mathcal{C}^+_\varepsilon := \{\varepsilon\ell+\varepsilon\Lambda \mid \ell\in\mathbb{Z}$, $\varepsilon\ell+\varepsilon\Lambda\subset I\}$ is the family of all $\varepsilon\Lambda$-cells contained in I, and we define $T^\varepsilon : L^2(I)\to L^2(I\times\Lambda)$ by

$$T^\varepsilon u(t,\tau) := \sum_{\theta_\varepsilon\in\mathcal{C}^+_\varepsilon} u(\varepsilon\ell_{\theta_\varepsilon}+\varepsilon\tau)\mathbb{1}_{\theta_\varepsilon}(t), \tag{35}$$

where $\varepsilon\ell_{\theta_\varepsilon}\in\varepsilon\mathbb{Z}$ stands for the left end point of θ_ε. Note that the subdivision of I_ε has been adjusted to form an exact partition of I around $0\in I$. The time version of the two-scale conversion lemma then reads:

Lemma 16 *For any $\varphi \in L^2(\Lambda; H^1(I))$ periodically extended in $L^2_{loc}(\mathbb{R}; H^1(I))$,*

$$\int_I |\varphi|^2(t, \frac{t}{\varepsilon})\ dt \le C||\delta_t \varphi||^2_{L^2(I\times\Lambda)}$$

where $\delta_t := (1 - \partial^2_{tt})^{1/2}$. Moreover, for any $u \in L^2(I)$,

$$\left| \int_{I\times\Lambda} (T^\varepsilon u)\ \varphi\ d\tau dt - \int_I u(t)\varphi(t, \frac{t}{\varepsilon})\ dt \right| \le ||u||_{L^2(I)} R_\varepsilon(\varphi)$$

where $R_\varepsilon(\varphi)$ tends to zero with ε.

To conclude this part, we create a mixture of time-space two-scale transforms and spectral analysis, by defining for any $k \in Y^*$ a pseudo-isometric operator $W^\varepsilon_k : L^2(I \times \Omega)^{N+1} \to L^2(I \times \Lambda \times \Omega \times Y)^{N+1}$ acting in all time and space variables

$$W^\varepsilon_k := (1 - \Pi^k)S^\varepsilon_k + \sum_{n\in\mathbb{M}^k} T^{\varepsilon\alpha^k_n} \Pi^k_n S^\varepsilon_k, \tag{36}$$

where $\mathbb{M}^k$, Π^k, Π^k_n and α^k_n have been introduced in (18) and in Theorem 9. Extending by quasiperiodicity the images of each W^ε_k from $L^2(Y)$ to $L^2(Y_K)$ also yields a multi-fibered wave two-scale transform

$$W^\varepsilon := \sum_{k\in L^*_K} \varpi_k W^\varepsilon_k. \tag{37}$$

Remark 17 *The kernel and non-kernel parts of S^ε_k in (36) may seem to have been treated differently. In fact, up to an artificial choice of a sequence of one-dimensional projectors π^k_n decomposing $1 - \Pi^k = \sum_n \pi^k_n$, the kernel term $(1 - \Pi^k)S^\varepsilon_k$ could very well be obtained as a sum of $T^{\varepsilon\alpha}\pi^k_n S^\varepsilon_k$, with the consistent convention $T^{\varepsilon\alpha} := 1$ when $\alpha = +\infty$ (the appropriate 'period' for kernel-waves). Viewing the kernel as a whole appears more logical.*

As a matter of fact, W^ε_k and W^ε are contractions as composite functions of contractions:

$$||W^\varepsilon_k U||^2_{L^2(I\times\Lambda\times\Omega\times Y)} \le ||U||^2_{L^2(I\times\Omega)} \text{ and } ||W^\varepsilon U||^2_{L^2(I\times\Lambda\times\Omega\times Y_K)} \le ||U||^2_{L^2(I\times\Omega)}, \tag{38}$$

where we recall that $||W^\varepsilon U||^2_{L^2(I\times\Lambda\times\Omega\times Y_K)} := \dfrac{1}{|Y_K|}\displaystyle\int_{Y_K} |W^\varepsilon U|^2 dy$ with $|Y_K| = K^N|Y|$.

Let us check in detail the first inequality of (38) which is fundamental. The second one will then ensue by the k-orthogonal decomposition of Theorem 7. To do so, we start with applying orthogonality relations w.r.t. y in the formula of the wave two-scale transform

$$W^\varepsilon_k = (1 - \Pi^k)S^\varepsilon_k + \sum_{n\in\mathbb{M}^k} T^{\varepsilon\alpha^k_n}\Pi^k_n S^\varepsilon_k = (1 - \Pi^k)S^\varepsilon_k + \sum_{n\in\mathbb{M}^k} \Pi^k_n (T^{\varepsilon\alpha^k_n} S^\varepsilon_k)$$

in order to obtain its squared y-norm for a.e. (t, τ, x) under the form

$$||W^\varepsilon_k U||^2_{L^2(Y)} = ||(1 - \Pi^k)S^\varepsilon_k U||^2_{L^2(Y)} + \sum_{n\in\mathbb{M}^k} ||T^{\varepsilon\alpha^k_n}\Pi^k_n S^\varepsilon_k U||^2_{L^2(Y)}.$$

We then perform a partial integration w.r.t. (t, τ) to get rid of each of the contractions $T^{\varepsilon\alpha^k_n} : L^2(I) \to L^2(I \times \Lambda)$ i.e.

$$||W^\varepsilon_k U||^2_{L^2(I\times\Lambda\times Y)} \le ||(1 - \Pi^k)S^\varepsilon_k U||^2_{L^2(I\times Y)} + \sum_{n\in\mathbb{M}^k} ||\Pi^k_n S^\varepsilon_k U||^2_{L^2(I\times Y)} = ||S^\varepsilon_k U||^2_{L^2(I\times Y)}.$$

In particular, after x-integration we recover the estimate

$$||W^\varepsilon_k U||^2_{L^2(I\times\Omega\times\Lambda\times Y)} \le ||S^\varepsilon_k U||^2_{L^2(I\times\Omega\times Y)},$$

in which $S^\varepsilon_k : L^2(\Omega) \to L^2(\Omega \times Y)$ is a contraction independent of $t \in I$. The conclusion is that $W^\varepsilon_k : L^2(I \times \Omega) \to L^2(I \times \Omega \times \Lambda \times Y)$ is a contraction as expected.

7 The wave two-scale model

Assuming the data bounded as in (13) and fixing $K \in \mathbb{N}^*$, we know by (38) that the bounded solutions U^ε of Theorem 3 give rise to bounded wave two-scale transforms $W^\varepsilon U^\varepsilon$. Denoting by U the weak limit in $L^2(I \times \Lambda \times \Omega \times Y_K)^{N+1}$ of any [3] of its converging subsequence, we investigate the structure of U, and search for equations satisfied by U.
Our result will be rigorously stated in Subsection 7.2 by means of a weak formulation as a homogenized hyperbolic system, but presented first more explicitly in Theorem 19 of Subsection 7.1 through the strong form obtained after disintegration by parts. Besides, the results of Subsection 7.2 have an abstract nature, since they remain at the level of the global homogenized system satisfied by U taken as a whole, without any insight into its spectral structure. They are probably less illuminating than the corresponding ones of Subsection 7.1, which highlight the band structure of the model, by putting forward the local discoupled equations satisfied by the different modal components (or band-coefficients) of U. Yet, both subsections are essentially equivalent, the link between them being well-detailled in Subsection 7.9.

7.1 The homogenized model in strong form

The classical expressions of the homogenized coefficients of (6) are

$$\widehat{\rho} := \frac{1}{|Y|}\int_Y \rho \, dy \text{ and } \widehat{a} := \frac{1}{|Y|}\int_Y a(1-P)\, dy, \tag{39}$$

where the projector

$$P \in \mathcal{L}(L^2(Y)^N) \tag{40}$$

with range $\nabla H^1_\sharp(Y)$ maps any $\theta \in L^2(Y)^N$ on $\nabla_y w \in L^2(Y)^N$, with w the unique solution in $H^1_\sharp(Y)/\mathbb{C}$ to the so-called cell problem $div_y(a\nabla_y w) = div_y(a\theta)$ understood in a variational sense, see for instance [7] or formula (2.6) of [1].

Remark 18 *Another equivalent definition of the homogenized matrix is*

$$\widehat{a}_{ij} := \frac{1}{|Y|}\int_Y a(\epsilon_j - \nabla_y \chi_j).\epsilon_i \, dy = \frac{1}{|Y|}\int_Y a(\epsilon_j - \nabla_y\chi_j).(\epsilon_i - \nabla_y \chi_i)\, dy,$$

where $(\chi_j)_{1\le j\le N}$ *denote the solutions to the cell problems associated with the canonical basis* $(\epsilon_j)_{1\le j\le N}$ *of* $\mathbb{R}^N$ *in such a way that* $\nabla_y \chi_j = P\epsilon_j$.

After extraction of a subsequence, we introduce the weak limits of the data (note that ∇u_0 and $\partial_t g$ are well-defined because the property of being a gradient is preserved by weak convergences)

$$\begin{gathered} f := \lim_\varepsilon f^\varepsilon \in L^2(I\times\Omega), \\ \nabla u_0 := \lim_\varepsilon \nabla u_0^\varepsilon \in L^2(\Omega)^N \text{ and } v_0 := \lim_\varepsilon \rho^\varepsilon v_0^\varepsilon/\widehat{\rho} \in L^2(\Omega), \\ \partial_t g := \lim_\varepsilon \partial_t g^\varepsilon \in H^1(I; H^{1/2}(\Gamma_D)) \text{ and } h := \lim_\varepsilon h^\varepsilon \in H^1(I; L^2(\Gamma_N)), \end{gathered} \tag{41}$$

and the weak limits of the relevant projections along e^k_n for any $n \in \mathbb{M}^k$

$$F^k_n := \lim_\varepsilon \int_\Lambda e^{-2i\pi s_n \tau}\frac{1}{|Y|}\int_Y W^\varepsilon_k F^\varepsilon \cdot e^k_n \, dy d\tau \in L^2(I\times\Omega), \tag{42}$$

$$U^k_{0,n} := \lim_\varepsilon \frac{1}{|Y|}\int_Y S^\varepsilon_k U^\varepsilon_0 \cdot e^k_n \, dy \in L^2(\Omega). \tag{43}$$

[3] The convergence of the whole sequence eventually results from the uniqueness of the limit problem proved in Proposition 36 (for $\Omega = \mathbb{R}^N$ only).

Note that (λ_n^k, e_n^k) has been defined in Theorem 9 and $\mathbb{M}^k$ in (18). We also recall the description of L_K^* in Subsection 5.1 and the expression of κ_{nm}^k given in (26) in order to state:

Theorem 19 *Suppose the coefficients regular as in (5). Then, for any fixed $K \in \mathbb{N}^*$ and any bounded data as in (13), any weak limit U of the bounded wave two-scale transforms $W^\varepsilon U^\varepsilon \in L^2(I \times \Lambda \times \Omega \times Y_K)^{N+1}$ of the solutions U^ε to (11) takes the form*

$$U(t,\tau,x,y) = U_H(t,x,y) + \sum_{k\in L_K^*} \sum_{n\in\mathbb{M}^k} U_n^k(t,x) e^{2i\pi s_n \tau} e_n^k(y), \tag{44}$$

$$U_H = \begin{pmatrix} \sqrt{a}(1-P)\nabla_x u \\ \sqrt{\rho}\partial_t u \end{pmatrix}, \tag{45}$$

where $u = u(t,x)$ solves the well-posed scalar problem

$$\begin{gathered} \widehat{\rho}\partial_{tt}^2 u - div_x(\widehat{a}\nabla_x u) = f \text{ in } I\times\Omega, \\ u(t=0) = u_0 \text{ and } \partial_t u(t=0) = v_0 \text{ in } \Omega, \\ u = g \text{ on } I\times\Gamma_D, \text{ and } \widehat{a}\nabla_x u.n_\Omega = h \text{ on } I\times\Gamma_N, \end{gathered} \tag{46}$$

and where $U_n^k = U_n^k(t,x)$ solves for any $n \in \mathbb{M}^k$ the first-order hyperbolic system

$$\begin{gathered} \partial_t U_n^k - s_n \sum_{m\in\mathbb{M}_n^k} \kappa_{nm}^k.\nabla_x U_m^k = F_n^k \text{ in } I\times\Omega, \\ U_n^k(t=0) = U_{0,n}^k \text{ in } \Omega, \end{gathered} \tag{47}$$

of size the multiplicity of the corresponding energy $\lambda_{|n|}^k$. Here the sum runs over the set $\mathbb{M}_n^k$ of all $m \in \mathbb{M}^k$ with the same sign as n and such that $\lambda_{|m|}^k = \lambda_{|n|}^k$.

Concerning the low frequency part U_H of the model, we recognize in (46) nothing else but the homogenized wave equation (3.20) of [8] and [20]. Its well-posedness is discussed in Sect. 3 of [8] in the typical case of homogeneous boundary conditions. Note also here that g and u_0 have only been defined up to some constants by the values (41) of their derivatives. An immaterial compatibility relation between g and u_0 is obviously in order to tie these two constants, see e.g. (105). But since u only appears through (45), it is in the nature of things that u (and accordingly u_0) be meaningful up to some constant only.

In parallel, the high frequency part involving U_n^k shows a decoupling of all modes with different eigenvalues $\lambda_{|n|}^k$ and different sign s_n. When the eigenvalue λ_n^k of $-\Delta_k$ is simple, we recover as a particular case of (47) that U_n^k is solution to a single transport equation $\partial_t U_n^k - s_n\kappa_{nn}^k.\nabla_x U_n^k = F_n^k$ understood in a distributional sense in $I \times \Omega$. Unfortunately, no boundary condition on $\partial\Omega$ has been derived ensuring that U_n^k is uniquely determined by its initial value $U_{0,n}^k$. This remark will explain the need to take $\Omega = \mathbb{R}^N$ in some further results.

Remark 20 *The expression of κ_{nn}^k given in Lemma 13 for a simple eigenvalue has obvious common points with the homogenized equations (2.17), (2.12) and (4.45) obtained in [21] for the Wigner measure in an energy band. Note that the assumption of isolation is essential to guarantee the differentiability of the eigenvalue λ_n^k as a function of k, see the proof of Lemma 13. A usual way to circumvent it is to perform a local study in the set of points k where λ_n^k remains simple (or of constant multiplicity). This is the point of view adopted in [21], but it then becomes very difficult, if not impossible, to give a simple description of the behavior of the individual scalar equations when they reach a crossing point. The reader familar with the problem of band crossings will be happy to see how our transport equations (47) have been derived in full generality (without any isolation assumptions or any regularity restrictions on the spectral values).*

In Section 8 we will deduce from Theorem 19 an approximation result of the type

$$U^\varepsilon(t,x) \approx U_H(t,x,\frac{x}{\varepsilon}) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{M}^k} U_n^k(t,x)e^{is_n\sqrt{\lambda_{|n|}^k}t/\varepsilon}e_n^k(\frac{x}{\varepsilon})$$

in the strong sense. Furthermore, a formal limit $K\to\infty$ (i.e. $Y_K\to\mathbb{R}^N$ and $L_K^*\to L^*$) can then be performed to recover the complete set of Bloch waves.

7.2 Weak formulation of the homogenized model as a system

In this subsection, we rephrase the preceding discussion in terms of systems and of weak formulations. As before, assuming the data bounded by (13) and fixing $K\in\mathbb{N}^*$, we extract from the (multi-fibered) wave two-scale transform $W^\varepsilon U^\varepsilon$ defined in (37) a weakly converging subsequence in $L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1}$, and decompose its limit U as

$$U = \sum_{k\in L_K^*}\varpi_k U^k \in L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1}, \tag{48}$$

where U^k is the weak limit of the (single-fibered) wave two-scale transform $W_k^\varepsilon U^\varepsilon$ defined in (36) and where ϖ_k is the k-quasiperiodic extension operator of (40). In the same way, we introduce on Y_K the L_K^*-sums

$$\begin{aligned} F &:= \sum_{k\in L_K^*}\varpi_k F^k \in L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1},\\ U_0 &:= \sum_{k\in L_K^*}\varpi_k U_0^k \in L^2(\Omega\times Y_K)^{N+1},\\ G &:= \sum_{k\in L_K^*}\varpi_k G^k = \varpi_0 G^0, \end{aligned} \tag{49}$$

of the k-quasiperiodic extensions of the two-scale weak limits [4] of the data $(F^\varepsilon, U_0^\varepsilon, G^\varepsilon)$ occuring in the physical problem (11) i.e.

$$\begin{aligned} F^k &:= \lim_\varepsilon W_k^\varepsilon F^\varepsilon \in L^2(I\times\Lambda\times\Omega\times Y)^{N+1},\\ U_0^k &:= \lim_\varepsilon S_k^\varepsilon U_0^\varepsilon \in L^2(\Omega\times Y)^{N+1},\\ G^k &:= \mathbb{1}_{L^*}(k)\lim_\varepsilon\begin{pmatrix}\mathbb{1}_{\Gamma_D}\partial_t g^\varepsilon\sqrt{a}n_\Omega\\ \mathbb{1}_{\Gamma_N}h^\varepsilon/\sqrt{\rho}\end{pmatrix} = \mathbb{1}_{L^*}(k)\begin{pmatrix}\mathbb{1}_{\Gamma_D}\partial_t g\sqrt{a}n_\Omega\\ \mathbb{1}_{\Gamma_N}h/\sqrt{\rho}\end{pmatrix}. \end{aligned} \tag{50}$$

Note that the boundary terms $\partial_t g$ and h have already been discussed in (41).
Next, we define an integro-differential operator $\mathcal{B}$ acting on smooth functions $(p,q)=(p,q)(x,y)$ of the macro-micro space variables (x,y) by setting

$$\mathcal{B}\begin{pmatrix}p\\ q\end{pmatrix} := \frac{1}{\widehat{\rho}}\begin{pmatrix}\sqrt{a}(1-P)\nabla_x\left(\frac{1}{|Y|}\int_Y\sqrt{\rho}q\ dy\right)\\ \sqrt{\rho}div_x\left(\frac{1}{|Y|}\int_Y a(1-P)a^{-1/2}p\ dy\right)\end{pmatrix}, \tag{51}$$

where $\widehat{\rho}$ and P have been introduced in (39) and (40). Roughly speaking, $\mathcal{B}$ is the operator obtained as the composition of the derivation (w.r.t. x) issued from B of (25) with the projection (w.r.t. y) onto the kernel of the operator A_0 of (19) in the case $k=0$ of periodic conditions.

[4] The quasiperiodizations of the weak limits and the weak limits of the quasiperiodizations coincide.

We also introduce the orthogonal projector $\widetilde{\Pi}_n^k \in \mathcal{L}(L^2(Y_K)^{N+1})$ defined for any $n \in \mathbb{M}^k$ by

$$\widetilde{\Pi}_n^k : \psi \mapsto \left(\frac{1}{|Y_K|}\int_{Y_K} \psi \cdot e_n^k \, dy\right) e_n^k,$$

together with the corresponding global projector $\widetilde{\Pi}^k := \sum_{n\in\mathbb{M}^k} \widetilde{\Pi}_n^k \in \mathcal{L}(L^2(Y_K)^{N+1})$, where e_n^k has been identified with its k-quasiperiodic extension on $\mathbb{R}^N$ according to Remark 10 (i).

Remark 21 *The notation suggests that $\widetilde{\Pi}_n^k \in \mathcal{L}(L^2(Y_K)^{N+1})$ is in some sense an extension of $\Pi_n^k \in \mathcal{L}(L^2(Y)^{N+1})$ defined in Theorem 9. This is the case insofar as $\widetilde{\Pi}_n^k = \Pi_n^k$ if $K = 1$.*

We then define an integro-differential wave operator $\mathcal{A}$ acting on both space scales by setting

$$\mathcal{A} := \mathcal{B} + \sum_{k\in L_K^*} \left(\sum_{n,m\in\mathbb{M}_+^k \text{ s.t. } \lambda_n^k=\lambda_m^k} \widetilde{\Pi}_{+m}^k B \widetilde{\Pi}_{+n}^k + \widetilde{\Pi}_{-m}^k B \widetilde{\Pi}_{-n}^k \right) \tag{52}$$

on

$$D(\mathcal{A}) := \{(\varphi, \phi) \in H_\sharp^1(Y_K; H^1(\Omega))^N \times H_\sharp^1(Y_K; H^1(\Omega)) \\ \text{s.t. } \varphi = 0 \text{ on } \Gamma_N \times Y_K \text{ and } \phi = 0 \text{ on } \Gamma_D \times Y_K\} \subset L^2(\Omega \times Y_K)^{N+1}.$$

It will be proved later on in Corollary 35 that $i\mathcal{A}$ is essentially self-adjoint on $L^2(\Omega \times Y_K)^{N+1}$, and not only symmetric as it is readily seen from the formal expression of $\mathcal{A}$ (and $\mathcal{B}$). We continue to comment on $\mathcal{A}$ with now a few words about the summation in (52). First, for any fixed $n \in \mathbb{M}_+^k$, the sum over m is actually finite (because $\lambda_m^k \to \infty$ as $m \to \infty$) and non void (because the diagonal case $m = n$ always appears). Second, when $n \in \mathbb{M}_+^k$ varies, the different terms in (52) give rise to a direct sum of operators (because of the presence of orthogonal projectors on the left and on the right of B) indexed by the eigenvalue $\lambda^k = \lambda_n^k$. But, using λ^k instead of (m, n) to reindex the sum reveals a band structure, in the sense that $\mathcal{A} - \mathcal{B}$ is the direct sum of the block operators

$$\left(\sum_{n,m\in\mathbb{M}_+^k \text{ s.t. } \lambda_n^k=\lambda_m^k=\lambda^k} \widetilde{\Pi}_{+m}^k B \widetilde{\Pi}_{+n}^k + \widetilde{\Pi}_{-m}^k B \widetilde{\Pi}_{-n}^k \right)_{\lambda^k} \tag{53}$$

when λ^k varies over $\mathbb{R}_+$, these blocks being pairwise independent. Of course, there are as many terms in (53) as the (squared) multiplicity of λ^k as an eigenvalue of $-\Delta_k$. In case of simplicity, the block in (53) reduces to "two opposite modes" $\widetilde{\Pi}_{+n}^k B \widetilde{\Pi}_{+n}^k + \widetilde{\Pi}_{-n}^k B \widetilde{\Pi}_{-n}^k$, which will give rise to two directions of propagation in opposite sense ($+\kappa_{nn}^k$ and $-\kappa_{nn}^k$) in the transport equations (47) of the final model. Finally, the fact that the different "components" $\widetilde{\Pi}_m^k B \widetilde{\Pi}_n^k$ of $\mathcal{A}$ only interacts for $(m, n) \in \mathbb{M}^k \times \mathbb{M}^k$ in a common energy level ($\lambda_{|n|}^k = \lambda_{|m|}^k$) is a key feature of our model. This discoupling phenomenon will be exhibited as the result of a destructive interference between exponentials with high oscillations in time:

$$e^{\pm i\sqrt{\lambda_{|n|}^k}t/\varepsilon} e^{\mp i\sqrt{\lambda_{|m|}^k}t/\varepsilon} \rightharpoonup 0 \text{ when } \lambda_{|n|}^k \neq \lambda_{|m|}^k.$$

We are now in a position to formulate our multi-fibered asymptotic wave two-scale model written as a system. This model is comprised of microscopic equations imposing strong constraints on the τ-dependence i.e.

$$\partial_\tau\big((1 - \widetilde{\Pi}^k)U\big) = 0 \text{ and } \partial_\tau\big(e^{-2i\pi s_n \tau}\widetilde{\Pi}_n^k U\big) = 0 \text{ for all } n \in \mathbb{M}^k \text{ and } k \in L_K^*, \tag{54}$$

and of the following macroscopic equation

$$\int_{I\times\Lambda\times\Omega\times Y_K} F\cdot\psi\; dydxd\tau dt+\int_{I\times\Lambda\times\Omega\times Y_K} U\cdot(\partial_t-\mathcal{A})\psi\; dydxd\tau dt$$
$$+\int_{\Omega\times Y_K} U_0\cdot\psi(t=0,\tau=0)\; dydx+\int_{I\times\Lambda\times\partial\Omega\times Y_K} G\cdot(1-\widetilde{\Pi}^0)\psi\; dyd\sigma d\tau dt=0 \tag{55}$$

valid for any admissible test function ψ in

$$\mathcal{W}:=\{\psi\in H^1_\sharp(\Lambda\times Y_K;H^1(I\times\Omega))^{N+1}\mid\psi\text{ solves (54)},$$
$$\psi(t,\tau,.,.)\in D(\mathcal{A})\text{ for every }(t,\tau)\in I\times\Lambda\text{ and }\psi(T,.,.,.)=0\}.$$

Theorem 22 *Suppose the data bounded as in (13) and the coefficients regular as in (5). Then any weak limit U of the bounded wave two-scale transforms $W^\varepsilon U^\varepsilon\in L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1}$ of the solutions U^ε to (11) takes the form (44)-(45) and satisfies (54)-(55).*

Remark 23 *(i) The microscopic equation (54) completely determines the way U depends on τ. And this dependence is essentially trivial since it only involves the three elementary functions $e^{-2i\pi\tau},1,e^{+2i\pi\tau}$. Accordingly, the microscopic equation (54) can be read again in the special form (44) taken by U. For instance, the fact that the low frequency part U_H only depends on (t,x,y) is a reminiscence of the fact that $(1-\Pi^k)U$ is constant in τ for all k.*

(ii) Because of our choice of time cells in (35), the time interval I is exactly subdivided around the origin $0\in I$, whose image in the macro-micro variables (t,τ) is precisely the point $(t=0,\tau=0)$ occuring in (55). This technicality solely explains the appearance of the value $\tau=0$. Note that this condition should not be interpreted as a Cauchy condition in the periodic variable τ, since (55) is not an evolution equation in τ (no τ-derivative being involved) and since the admissible test functions ψ in (55) describe a finite-dimensional space from the standpoint of the τ-dependence.

(iii) It will be seen a posteriori that the lack of boundary conditions in (47) originates from the stringent condition "$\psi(t,\tau,.,.)\in D(\mathcal{A})$ for every τ" imposed by $\mathcal{W}$ on the test function. In the same respect, refer to Remark 27. Unfortunately, we did not manage to enlarge the space $\mathcal{W}$ by weakening this condition.

We finish with an equivalent version of Theorem 22 devoted to the case of only one fiber k. In this context, the operator $\mathcal{A}_k$ analogous to $\mathcal{A}$ is just

$$\mathcal{A}_k:=\mathbb{1}_{L^*}(k)\mathcal{B}+\sum_{n,m\in\mathbb{M}^k_+\text{ s.t. }\lambda^k_n=\lambda^k_m}\Pi^k_{+m}B\Pi^k_{+n}+\Pi^k_{-m}B\Pi^k_{-n}. \tag{56}$$

Theorem 24 *The assumptions are those of the preceding result. For any $k\in Y^*$, the component U^k of U in (48) satisfies the one-fibered wave two-scale model comprised of the microscopic equation (58) and of the following macroscopic equation*

$$\int_{I\times\Lambda\times\Omega\times Y} F^k\cdot\psi\; dydxd\tau dt+\int_{I\times\Lambda\times\Omega\times Y} U^k\cdot(\partial_t-\mathcal{A}_k)\psi\; dydxd\tau dt$$
$$+\int_{\Omega\times Y} U^k_0\cdot\psi(t=0,\tau=0)\; dydx+\int_{I\times\Lambda\times\partial\Omega\times Y} G^k\cdot(1-\Pi^k)\psi\; dydxd\tau dt=0 \tag{57}$$

for all $\psi\in\mathcal{W}_k$ defined in (63).

The easy equivalence between Theorem 22 and Theorem 24 will be checked in Subsection 7.10. Independently, Theorem 19 will be deduced from Theorem 24 in Subsection 7.9, where the limiting local equations of (46) — wave equation on the low frequency part — and of (47) — transport equations on the modal coefficients — will be pulled out from the global operator $\mathcal{A}$ of (52). The remaining parts of Section 7 are devoted to the proof of Theorem 24.

7.3 Microscopic equations

For any fixed $k \in Y^*$, we start with the derivation of the one-fibered microscopic equations

$$(\partial_\tau - 2i\pi\Pi_s^k)U^k = 0 \quad \text{with } \Pi_s^k := \sum_{n\in\mathbb{M}^k} s_n \Pi_n^k \in \mathcal{L}(L^2(Y)^{N+1}), \tag{58}$$

where the orthogonal projectors Π_n^k are defined in Theorem 9.

Lemma 25 *For any $\psi \in H_\sharp^1(\Lambda; L^2(I\times\Omega\times Y))^{N+1}$ and any $k \in Y^*$,*

$$\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot (\partial_\tau - 2i\pi\Pi_s^k)\psi \; dydxd\tau dt \to 0.$$

Moreover, any weak limit U^k of $W_k^\varepsilon U^\varepsilon$ is solution to the microscopic equations (58), which may be detailed as

$$\partial_\tau\big((1-\Pi^k)U^k\big) = 0 \textit{ and } \partial_\tau\big(e^{-2i\pi s_n\tau}\Pi_n^k U^k\big) = 0 \textit{ for all } n \in \mathbb{M}^k. \tag{59}$$

Proof. (i) We first establish the convergence

$$\begin{aligned}&\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot (\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt \\ &-|Y|\sum_n \int_{I\times\Omega} U^\varepsilon(t,x)\cdot(\partial_\tau - 2i\pi\Pi_s^k)\Pi_n^k\psi_c(t,\frac{t}{\varepsilon\alpha_n^k},x,\frac{x}{\varepsilon})\; dxdt \to 0\end{aligned} \tag{60}$$

for any $\psi_c \in H_\sharp^1(\Lambda; L^2(I\times\Omega\times Y))^{N+1}$ taken as a finite sum of the type

$$\psi_c := (1-\Pi^k)\psi_0(t,\tau,x,y) + \sum_n \psi_n(t,\tau,x)e_n^k(y),$$

where $\psi_n \in \mathcal{C}^\infty(I\times\Lambda\times\Omega)$ is Λ-periodic with respect to τ and compactly supported in the interior of $I\times\Omega$ and $\psi_0 \in H_\sharp^1(\Lambda; L^2(I\times\Omega\times Y))^{N+1}$. By the very definition of W_k^ε, we compute

$$\begin{aligned}&\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot (\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt \\ =& \int_{I\times\Lambda\times\Omega\times Y} (1-\Pi^k)S_k^\varepsilon U^\varepsilon\cdot(\partial_\tau - 2i\pi\Pi_s^k)\psi_c + \sum_n \Pi_n^k T^{\varepsilon\alpha_n^k}S_k^\varepsilon U^\varepsilon\cdot(\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt \\ =& \int_{I\times\Lambda\times\Omega\times Y} (1-\Pi^k)S_k^\varepsilon U^\varepsilon\cdot\partial_\tau\psi_c + \sum_n T^{\varepsilon\alpha_n^k}S_k^\varepsilon U^\varepsilon\cdot\Pi_n^k(\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt \\ =& \int_{I\times\Lambda\times\Omega\times Y} \partial_\tau\big((1-\Pi^k)S_k^\varepsilon U^\varepsilon\cdot\psi_c\big) + \sum_n T^{\varepsilon\alpha_n^k}S_k^\varepsilon U^\varepsilon\cdot(\partial_\tau - 2i\pi\Pi_s^k)\Pi_n^k\psi_c \; dydxd\tau dt.\end{aligned}$$

Therein, the first term is trivially null by the assumed τ-periodicity of ψ_c, and the remaining finite sum can be converted into an integral over $I\times\Omega$ by applying the two-scale conversion Lemmas 14 and 16 in both space and time variables to the test functions $(\partial_\tau - 2i\pi\Pi_s^k)\Pi_n^k\psi_c$, which are regular enough by our choice of ψ_n. This yields (60) as claimed.

(ii) For each n, we now establish the equality

$$\begin{aligned}&\int_{I\times\Omega} U^\varepsilon(t,x)\cdot(\partial_\tau - 2i\pi\Pi_s^k)\Pi_n^k\psi_c(t,\frac{t}{\varepsilon\alpha_n^k},x,\frac{x}{\varepsilon})\; dxdt \\ =\; & -\varepsilon\int_{I\times\Omega} \alpha_n^k\big(F^\varepsilon\cdot\Pi_n^k\psi_c + U^\varepsilon\cdot(\partial_t - B)\Pi_n^k\psi_c\big)(t,\frac{t}{\varepsilon\alpha_n^k},x,\frac{x}{\varepsilon})\; dxdt.\end{aligned} \tag{61}$$

Taking

$$\psi_n^\varepsilon(t,x) := \Pi_n^k \psi_c(t, \frac{t}{\varepsilon \alpha_n^k}, x, \frac{x}{\varepsilon}) = \psi_n(t, \frac{t}{\varepsilon \alpha_n^k}, x) e_n^k(\frac{x}{\varepsilon}) \in \mathcal{V}^\varepsilon$$

as a test function [5] in the physical problem (11), we get

$$\int_{I\times\Omega} F^\varepsilon \cdot \psi_n^\varepsilon + U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_n^\varepsilon \; dxdt = 0,$$

where

$$\begin{aligned}(\partial_t - A^\varepsilon)\psi_n^\varepsilon &= \Big((\frac{1}{\varepsilon\alpha_n^k}\partial_\tau - \frac{1}{\varepsilon}A)\Pi_n^k\psi_c + (\partial_t - B)\Pi_n^k\psi_c \Big)(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}) \\ &= \Big(\frac{1}{\varepsilon\alpha_n^k}(\partial_\tau - 2i\pi\Pi_s^k)\Pi_n^k\psi_c + (\partial_t - B)\Pi_n^k\psi_c \Big)(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}) \qquad (62)\end{aligned}$$

in virtue of the fundamental differential identity (24) applied to ψ_n^ε. Equality (61) follows.

(iii) To complete the proof, we incorporate (61) into (60) and recast it as

$$\begin{aligned}&\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot (\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt \\ &+\varepsilon|Y|\sum_n \int_{I\times\Omega} \alpha_n^k \big(F^\varepsilon \cdot \Pi_n^k\psi_c + U^\varepsilon\cdot(\partial_t - B)\Pi_n^k\psi_c\big)(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}) \; dxdt \to 0.\end{aligned}$$

In particular, the convergence stated in the first part of the proposition holds true for the class of test functions ψ_c used so far. By the density of this class in $H^1_\sharp(\Lambda; L^2(I\times\Omega\times Y))^{N+1}$, the case of general ψ's follows at once from the easy estimate

$$\begin{aligned}\left|\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon\cdot(\partial_\tau - 2i\pi\Pi_s^k)\psi \; dydxd\tau dt\right| &\le \left|\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon\cdot(\partial_\tau - 2i\pi\Pi_s^k)\psi_c \; dydxd\tau dt\right| \\ +||W_k^\varepsilon U^\varepsilon||_{L^2(I\times\Lambda\times\Omega\times Y)} &\big(2\pi||\psi-\psi_c||_{L^2(I\times\Lambda\times\Omega\times Y)} + ||\partial_\tau\psi - \partial_\tau\psi_c||_{L^2(I\times\Lambda\times\Omega\times Y)}\big)\,.\end{aligned}$$

Finally, the distributional equations $(\partial_\tau - 2i\pi s_n)\Pi_n^k U^k = 0$ stated in the second part of the lemma are obtained by passing to the limit. ■

7.4 Construction of admissible test functions

Let $\mathcal{W}_k$ denote the space of all test functions ψ satisfying:

$$\begin{cases} \psi \in \mathcal{C}^\infty(\overline{I\times\Lambda\times\Omega\times Y})^{N+1} \text{ is } k\text{-quasiperiodic in } y, \\ (\partial_\tau - 2i\pi\Pi_s^k)\psi = 0, \\ \psi(t,\tau,.,y) \in D \text{ for all } t,\tau,y, \\ \psi(.,\tau,x,y) \text{ has compact support in } I \text{ for all } \tau,x,y, \end{cases} \qquad (63)$$

where D has been defined in (12) and Π_s^k in (58). We agree to endow $\mathcal{W}_k$ with a sufficiently restrictive norm, for instance the norm of $H^{\nu+1}$ in all variables (t,τ,x,y) will do the job. Note that the most restrictive condition in (63) is certainly the microscopic equation inherited from Subsection 7.3, since it prescribes the τ-dependence of $\psi\in\mathcal{W}_k$ in a trivial manner.

[5] Note that $e_n^k \in H^1_{loc}$ makes ψ_n^ε regular enough by Lemma 12.

According to the microscopic equation of (63), the n^{th} projection $\Pi_n^k\psi$ and the kernel projection $(1-\Pi^k)\psi$ of a given $\psi \in \mathcal{W}_k$ read as

$$\left\{\begin{array}{l} \Pi_n^k\psi(t,\tau,x,y) = e^{2i\pi s_n\tau}\psi_n(t,x)e_n^k(y) \\ (1-\Pi^k)\psi(t,x,y) = \int_\Lambda (1-\Pi^k)\psi(t,\tau,x,y)\ d\tau = \int_\Lambda \psi(t,\tau,x,y)\ d\tau \end{array}\right. \tag{64}$$

where $\psi_n(t,x) := e^{-2i\pi s_n\tau}\int_Y \psi\cdot e_n^k\ dy$. Consistently, we introduce a family of waves

$$\left\{\begin{array}{l} \psi_n^{k,\varepsilon}(t,x) := \Pi_n^k\psi(t,\dfrac{t}{\varepsilon\alpha_n^k},x,\dfrac{x}{\varepsilon}) = e^{2i\pi s_n t/\varepsilon\alpha_n^k}\psi_n(t,x)e_n^k(\dfrac{x}{\varepsilon}) \\ \psi_{1-\Pi}^{k,\varepsilon}(t,x) := (1-\Pi^k)\psi(t,x,\dfrac{x}{\varepsilon}) = \displaystyle\int_\Lambda \psi(t,\tau,x,\dfrac{x}{\varepsilon})\ d\tau \end{array}\right. \tag{65}$$

on which the physical problem (11) will be legitimately tested:

Proposition 26 *If $\psi \in \mathcal{W}_k$ then $\psi_{1-\Pi}^{k,\varepsilon}$and $\psi_{+n}^{k,\varepsilon}+\psi_{-n}^{k,\varepsilon}$ are admissible test functions in (11) in the sense that they belong to $\mathcal{V}^\varepsilon$.*

Proof. The admissibility of $\psi_{1-\Pi}^{\varepsilon,k}$ is straightforward (in fact $\psi_{1-\Pi}^{\varepsilon,k} \in \mathcal{V} \subset \mathcal{V}^\varepsilon$) because $\psi_{1-\Pi}^{\varepsilon,k}$ is a mere τ-average of ψ, see (65), in particular $\psi(t,\tau,.,y) \in D$ implies $\psi_{1-\Pi}^{\varepsilon,k}(t,.) \in D$. As for n-waves, the key point is to check the boundary conditions for the special combinations $\psi_{+n}^{\varepsilon,k}+\psi_{-n}^{\varepsilon,k}$, since the regularity $\psi_n^{\varepsilon,k} \in H^1(I\times\Omega)^{N+1}$ is guaranteed by Lemma 12. By means of an orthogonal change of basis in the range of $\Pi_{+n}^k+\Pi_{-n}^k$, we turn $\{e_{+n}^k, e_{-n}^k\}$ into $\{v_n^k, w_n^k\}$ defined in (20):

$$(\Pi_{+n}^k+\Pi_{-n}^k)\psi = \left(\frac{1}{|Y|}\int_Y \psi\cdot v_n^k\ dy\right)\ v_n^k + \left(\frac{1}{|Y|}\int_Y \psi\cdot w_n^k\ dy\right)\ w_n^k. \tag{66}$$

Therein, $\psi = (\varphi,\phi) \in \mathcal{W}_k$ satisfies $\psi\cdot v_n^k = 0$ on $I\times\Lambda\times\Gamma_N\times Y$ and $\psi\cdot w_n^k = 0$ on $I\times\Lambda\times\Gamma_D\times Y$. Particularizing to the special value $\tau = t/\varepsilon\alpha_n^k$ of the microscopic time variable, it follows that $(\Pi_{+n}^k+\Pi_{-n}^k)\psi(t,t/\varepsilon\alpha_n^k,.,y)$ meets the boundary restrictions laid down by D, in particular $(\psi_{+n}^{\varepsilon,k}+\psi_{-n}^{\varepsilon,k})(t,.) \in D(A^\varepsilon)$ for every $t \in I$ as expected. ∎

Remark 27 *There is an important point to be mentioned about the behavior of $\psi_{\pm n}^{\varepsilon,k}$ on the boundary $\partial\Omega$. Apparently, the definition of $\mathcal{W}_k$ only prescribes Dirichlet-Neumann conditions on $\partial\Omega$. But we must draw the reader's attention to the fact that it is not so simple, because in (63) the boundary conditions laid down by D for all τ are coupled with the microscopic equation governing the τ-dependence. For this reason, the expression "for all τ" induces an unpleasant restriction on the boundary values of ψ, which eventually explains why the space $\mathcal{W}_k$ of test functions is not so large as it may seem, and why consequently some boundary conditions are lacking in the final model. Let us detail this point. Coming back to $\psi_{+n}^{k,\varepsilon}+\psi_{-n}^{k,\varepsilon}$ of (65) through (66) and decomposing exponentials in sines and cosines, we find with more care that the microscopic equation of $\psi \in \mathcal{W}_k$ enforces the equality*

$$\begin{aligned}(\psi_{+n}^{k,\varepsilon}+\psi_{-n}^{k,\varepsilon})(t,x) = v_n^k(\frac{x}{\varepsilon})\frac{1}{|Y|}\int_Y \psi(t,\tau,x,y)\cdot\left(v_n^k\cos(2\pi\tau-\frac{2\pi t}{\varepsilon\alpha_n^k}) - w_n^k\sin(2\pi\tau-\frac{2\pi t}{\varepsilon\alpha_n^k})\right)\ dy \\ +w_n^k(\frac{x}{\varepsilon})\frac{1}{|Y|}\int_Y \psi(t,\tau,x,y)\cdot\left(w_n^k\cos(2\pi\tau-\frac{2\pi t}{\varepsilon\alpha_n^k}) + v_n^k\sin(2\pi\tau-\frac{2\pi t}{\varepsilon\alpha_n^k})\right)\ dy\end{aligned}$$

independently of $\tau \in \Lambda$. When the spectral element $w_n^k(x/\varepsilon)$ is not responsible for any cancellation effect on the boundary $\partial\Omega$, the Dirichlet condition $[\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon}]_D = 0$ on $I \times \Gamma_D$ can only be met if the factor in front of $w_n^k(x/\varepsilon)$ vanishes for all $(t, \tau, x) \in I \times \Lambda \times \Gamma_D$ and all ε, whence

$$\left(\int_Y \psi \cdot w_n^k \, dy\right)(t,\tau,x)\cos(2\pi\tau - \frac{2\pi t}{\varepsilon\alpha_n^k}) + \left(\int_Y \psi \cdot v_n^k \, dy\right)(t,\tau,x)\sin(2\pi\tau - \frac{2\pi t}{\varepsilon\alpha_n^k}) = 0$$

for $x \in \Gamma_D$. This restriction is so strong that it practically implies $\psi_{+n} = \psi_{-n} = 0$ on $I \times \Gamma_D$. The same being true of the Neumann part, we conclude that $\psi \in \mathcal{W}_k$ vanishes identically on the boundary $\partial\Omega$ in most cases (i.e. unless $w_n^k(x/\varepsilon)$ and/or $v_n^k(x/\varepsilon)$ vanish on the boundary $\partial\Omega$).

7.5 Uniform bounds with respect to n-wave summation

We now proceed to project the physical problem (11) onto the set of waves (65) and convert the resulting system of equations into a single synthetic sum. More specifically, we will solve the problem of summing over $n \in \mathbb{M}_+^k$ the infinitely-many relations

$$\int_{I\times\Omega} F^\varepsilon \cdot \psi_{1-\Pi}^{k,\varepsilon} + U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_{1-\Pi}^{k,\varepsilon} \, dxdt + \int_\Omega U_0^\varepsilon \cdot \psi_{1-\Pi}^{k,\varepsilon} \, dx + \int_{I\times\partial\Omega} G^\varepsilon \cdot \psi_{1-\Pi}^{k,\varepsilon} \, d\sigma dt = 0 \quad (67)$$

and

$$\int_{I\times\Omega} F^\varepsilon \cdot (\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon}) + U^\varepsilon \cdot (\partial_t - A^\varepsilon)(\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon}) \, dxdt$$
$$+ \int_\Omega U_0^\varepsilon \cdot (\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon})(t=0) \, dx + \int_{I\times\partial\Omega} G^\varepsilon \cdot (\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon}) \, d\sigma dt = 0, \quad (68)$$

while in the same time, we will justify the process of wave-wise convergence by proving the inversion of limits $\lim_\varepsilon \sum_n = \sum_n \lim_\varepsilon$. This is the main purpose of:

Proposition 28 *Under assumption (5) each of the four series appearing in (68) converges. Moreover, the convergences are uniform with respect to $0 < \varepsilon < 1$ and to $\psi \in \mathcal{P}$ for any fixed precompact subset $\mathcal{P} \subset \mathcal{W}_k$.*

Proof. Taking advantage of (68), we only focus on the series involving F^ε, U^ε and U_0^ε respectively:

$$\left|\int_{I\times\Omega} F^\varepsilon \cdot \psi_n^{\varepsilon,k} \, dxdt\right| \leq ||F^\varepsilon||_{L^2(I\times\Omega)} ||\psi_n^{\varepsilon,k}||_{L^2(I\times\Omega)},$$

$$\left|\int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k} \, dxdt\right| \leq ||U^\varepsilon||_{L^2(I\times\Omega)} ||(\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k}||_{L^2(I\times\Omega)},$$

$$\left|\int_\Omega U_0^\varepsilon \cdot \psi_n^{\varepsilon,k}(t=0) \, dx\right| \leq ||U_0^\varepsilon||_{L^2(\Omega)} ||\psi_n^{\varepsilon,k}(t=0)||_{L^2(\Omega)}.$$

Since $||U_0^\varepsilon||$, $||F^\varepsilon||$ and $||U^\varepsilon||$ are uniformly bounded, it suffices here to exhibit a summable sequence $(c_n^\psi)_n$ enjoying some ψ-continuity property in the norm of $\mathcal{W}_k$ and satisfying

$$||\psi_n^{\varepsilon,k}||_{L^2(I\times\Omega)} + ||(\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k}||_{L^2(I\times\Omega)} + ||\psi_n^{\varepsilon,k}(t=0)||_{L^2(\Omega)} \leq c_n^\psi \text{ for all } n$$

uniformly in ε. The construction will be carried through for $(\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k}$ only. Estimating the other terms in much the same way is easier and left to the reader.

Since ψ satisfies the microscopic equation, (62) provides a simplified expression of

$$(\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k} = \left(\partial_t \Pi_n^k \psi - B\Pi_n^k \psi\right)(t, \frac{t}{\varepsilon \alpha_n^k}, x, \frac{x}{\varepsilon}),$$

in which $\partial_t \Pi_n^k \psi$ and $B\Pi_n^k \psi$ can be dealt with separately. So, Lemma 11 together with Lemmas 14 and 16 applied in whatever order leads to

$$\begin{aligned} ||(\partial_t \Pi_n^k \psi)(t, \frac{t}{\varepsilon \alpha_n^k}, x, \frac{x}{\varepsilon})||_{L^2(I\times\Omega)} &\leq C||\delta_x^\nu \delta_t^2 \Pi_n^k \psi||_{L^2(I\times\Lambda\times\Omega\times Y)} = C||\Pi_n^k(\delta_x^\nu \delta_t^2 \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)} \\ &= C||\frac{1}{\lambda_n^{\nu/2}}\Pi_n^k(A^\nu \delta_x^\nu \delta_t^2 \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)} \leq \frac{C}{n^{\nu/N}}||\Pi_n^k(A^\nu \delta_x^\nu \delta_t^2 \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)} \end{aligned}$$

and

$$\begin{aligned} ||(B\Pi_n^k \psi)(t, \frac{t}{\varepsilon \alpha_n^k}, x, \frac{x}{\varepsilon})||_{L^2(I\times\Omega)} &\leq C||\delta_x^{\nu+1} \delta_t \Pi_n^k \psi||_{L^2(I\times\Lambda\times\Omega\times Y)} = C||\Pi_n^k(\delta_x^{\nu+1} \delta_t \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)} \\ &= C||\frac{1}{\lambda_n^{\nu/2}}\Pi_n^k(A^\nu \delta_x^{\nu+1} \delta_t \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)} \leq \frac{C}{n^{\nu/N}}||\Pi_n^k(A^\nu \delta_x^{\nu+1} \delta_t \psi)||_{L^2(I\times\Lambda\times\Omega\times Y)}, \end{aligned}$$

where the righthand sides are ε-independent n-summable terms ($\sum n^{-2\nu/N} < \infty$) whenever $\psi \in H_t^2 L_\tau^2 H_x^{\nu+1} H_y^\nu$, because the regularity assumption (5) ensures that $A^\nu \in \mathcal{L}(H^\nu(Y)^{N+1}; L^2(Y)^{N+1})$. This yields a suitable $(c_n^\psi)_n$. $\blacksquare$

At this stage, we know that we can confine ourselves to the study of only one of the n-waves $\psi_n^{k,\varepsilon}$ or kernel waves $\psi_{1-\Pi}^{k,\varepsilon}$ defined in (65), and then recollect the information over all of them. After application of the two-scale conversion Lemmas 14 and 16 to

$$\left\{ \begin{aligned} &\int_\Omega U_0^\varepsilon \cdot \psi_n^{k,\varepsilon}(t=0)\, dx - \frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon U_0^\varepsilon \cdot \Pi_n^k \psi(t=0, \tau=0)\, dydx \to 0, \\ &\int_\Omega U_0^\varepsilon \cdot \psi_{1-\Pi}^{k,\varepsilon}(t=0)\, dx - \frac{1}{|Y|}\int_{\Omega\times Y} S_k^\varepsilon U_0^\varepsilon \cdot (1-\Pi^k)\psi(t=0)\, dydx \to 0, \end{aligned} \right. \tag{69}$$

$$\left\{ \begin{aligned} &\int_{I\times\Omega} F^\varepsilon \cdot \psi_n^{k,\varepsilon}\, dxdt - \frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon F^\varepsilon \cdot \Pi_n^k \psi\, dydxd\tau dt \to 0, \\ &\int_{I\times\Omega} F^\varepsilon \cdot \psi_{1-\Pi}^{k,\varepsilon}\, dxdt - \frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon F^\varepsilon \cdot (1-\Pi^k)\psi\, dydxd\tau dt \to 0, \end{aligned} \right. \tag{70}$$

we see that $\partial_t - A^\varepsilon$ and G^ε are the only terms of (67)-(68) still to be analyzed. This will be achieved in Proposition 29 of subsection 7.6 and in Proposition 31 of subsection 7.7.

7.6 First step to homogenized operators

We now investigate the asymptotic behavior of the differential part $\partial_t - A^\varepsilon$ by means of extra operators to be simplified later on

$$B_n^k := \sum_{m\in\mathbb{M}^k \text{ s.t. } \alpha_n^k/\alpha_m^k \in \mathbb{N}^*} e^{-2i\pi s_m \tau \alpha_n^k/\alpha_m^k} e^{2i\pi s_m \tau} \Pi_m^k B \Pi_n^k. \tag{71}$$

Proposition 29 *For any $\psi \in \mathcal{W}_k$ and any $k \in Y^*$,*

$$\int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_n^{k,\varepsilon}\, dxdt \tag{72}$$
$$-\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot \big(\partial_t - (1-\Pi^k)B - B_n^k\big)\Pi_n^k\psi\; dydxd\tau dt \to 0$$

and

$$\int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_{1-\Pi}^{k,\varepsilon}\, dxdt \tag{73}$$
$$-\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} W_k^\varepsilon U^\varepsilon \cdot \big(\partial_t - (1-\Pi^k)B\big)(1-\Pi^k)\psi\; dydxd\tau dt \to 0.$$

Proof. (i) Let us start with the non-kernel part (72). Recalling the simplified expression (62) of $(\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k}$ for $\psi \in \mathcal{W}_k$ solution to the microscopic equation of (63), successive applications of Lemmas 14 and 16 yield

$$\int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_n^{\varepsilon,k}\, dxdt - \frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon \cdot (\partial_t - B)\Pi_n^k\psi\; dydxd\tau dt \to 0.$$

Besides, orthogonality relations in the expression (36) of W_k^ε easily give

$$\begin{aligned}\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} & T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon \cdot (\partial_t - B)\Pi_n^k\psi - W_k^\varepsilon U^\varepsilon \cdot \big(\partial_t - (1-\Pi^k)B\big)\Pi_n^k\psi\; dydxd\tau dt \\ &= -\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} \big(T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon - (1-\Pi^k)S_k^\varepsilon U^\varepsilon\big)\cdot B\Pi_n^k\psi\; dydxd\tau dt,\end{aligned}$$

where $T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon - (1-\Pi^k)S_k^\varepsilon U^\varepsilon$ can be asymptotically expressed by means of $W_k^\varepsilon U^\varepsilon$ only in virtue of Proposition 30 below. This leads to (71) and (72) as claimed.

(ii) Let us now deal with the kernel part (73). By (24) applied to $(1-\Pi^k)\psi$, the construction (65) of $\psi_{1-\Pi}^{\varepsilon,k}$ for $\psi \in \mathcal{W}_k$ solution to the microscopic equation of (63) yields

$$\begin{aligned}(\partial_t - A^\varepsilon)\psi_{1-\Pi}^{\varepsilon,k} &= \left(\frac{1}{\varepsilon}(\partial_\tau - A)(1-\Pi^k)\psi + (\partial_t - B)(1-\Pi^k)\psi\right)(t,x,\frac{x}{\varepsilon}) \\ &= \Big((\partial_t - B)(1-\Pi^k)\psi\Big)(t,x,\frac{x}{\varepsilon}),\end{aligned}$$

so

$$\int_{I\times\Omega} U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi_{1-\Pi}^{\varepsilon,k}\, dxdt - \frac{1}{|Y|}\int_{I\times\Omega\times Y} S_k^\varepsilon U^\varepsilon \cdot (\partial_t - B)(1-\Pi^k)\psi\; dydxdt \to 0$$

thanks to Lemma 14 (only). Again, orthogonality relations allow

$$\frac{1}{|Y|}\int_{I\times\Omega\times Y} S_k^\varepsilon U^\varepsilon \cdot (\partial_t - B)(1-\Pi^k)\psi - W_k^\varepsilon U^\varepsilon \cdot (\partial_t - (1-\Pi^k)B)(1-\Pi^k)\psi\; dydxdt$$

$$= -\frac{1}{|Y|}\int_{I\times\Omega\times Y} \Pi^k S_k^\varepsilon U^\varepsilon \cdot B(1-\Pi^k)\psi\; dydxdt.$$

To conclude with (73), it remains to show that $\Pi^k S_k^\varepsilon U^\varepsilon \to 0$ weakly in $L^2(I\times\Omega\times Y)^{N+1}$, or alternatively that $\int_\Omega \zeta\Pi^k S_k^\varepsilon U^\varepsilon\, dx \to 0$ weakly in $L^2(I\times Y)^{N+1}$ for any fixed $n \in \mathbb{M}^k$ and

$\zeta \in C_c^\infty(\Omega)$. But following (83), this amounts to say that $e^{+2i\pi s_n t/\varepsilon\alpha_n^k} V_n^\varepsilon \to 0$ weakly in $L^2(I)$, which is a plain consequence of the precompacity of V_n^ε proved in the course of Proposition 30 below, see (ii). ■

The following proposition (with $\alpha := \alpha_n^k$) expresses $T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon - (1-\Pi^k) S_k^\varepsilon U^\varepsilon$ as a combination of $\Pi_m^k W_k^\varepsilon U^\varepsilon$ for varying m such that α_n^k/α_m^k is an integer.

Proposition 30 *For any fixed $\alpha > 0$ and any $k \in Y^*$,*

$$T^{\varepsilon\alpha} S_k^\varepsilon U^\varepsilon - (1-\Pi^k) S_k^\varepsilon U^\varepsilon - \sum_{m \in \mathbb{M}^k \ s.t.\ \alpha/\alpha_m^k \in \mathbb{N}^*} e^{-2i\pi s_m \tau} e^{2i\pi s_m \tau\alpha/\alpha_m^k} \Pi_m^k W_k^\varepsilon U^\varepsilon \to 0$$

weakly in $L^2(I \times \Lambda \times \Omega \times Y)^{N+1}$.

Proof. Through points (i)-(v) of this proof, we let x vary in a compact subset of Ω and we restrict ε to be small enough to ensure that the ε-cell $\omega_\varepsilon^x := \varepsilon\ell_x + \varepsilon Y$ containing x is wholly included in Ω. Once for all, we fix two regular functions $\theta \in C_c^\infty(]0,T[)$ and $\zeta \in C_c^\infty(\Omega)$ with compact supports, boundary conditions playing no role here.

(i) Equation on $S_k^\varepsilon U^\varepsilon$ in a distributional sense.
We begin with converting the physical problem (11) into a distributional evolution equation on $S_k^\varepsilon U^\varepsilon$ to be projected later on:

$$(\partial_t - \frac{1}{\varepsilon} A) S_k^\varepsilon U^\varepsilon = S_k^\varepsilon F^\varepsilon \text{ in } \mathcal{D}'(]0,T[\times\Omega\times Y)^{N+1}. \tag{74}$$

With this aim, given $\eta \in C_c^\infty(Y)^{N+1}$, we take $\psi(t,X) := \theta(t)\eta(\frac{X}{\varepsilon} - \ell_x)$ as a test function in the physical problem (11) written in the (t,X)-variables, and we express the resulting integral in terms of $S_k^\varepsilon U^\varepsilon(t,x,y)$ and $S_k^\varepsilon F^\varepsilon(t,x,y)$ only. For any fixed x, this yields

$$\begin{aligned}
0 &= \frac{1}{\varepsilon^N} \int_{I\times\Omega} F^\varepsilon \cdot \psi + U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi \; dX dt \\
&= \int_{I\times Y} F^\varepsilon(t, \varepsilon\ell_x + \varepsilon y) \cdot \theta(t)\eta(y) + U^\varepsilon(t,\varepsilon\ell_x + \varepsilon y) \cdot \left(\theta'(t)\eta(y) - \frac{1}{\varepsilon}\theta(t) A\eta(y)\right) \; dydt \\
&= e^{2i\pi k.\ell_x} \int_{I\times Y} S_k^\varepsilon F^\varepsilon \cdot \theta\eta + S_k^\varepsilon U^\varepsilon \cdot \left(\partial_t - \frac{1}{\varepsilon} A\right) \theta\eta \; dydt,
\end{aligned}$$

which is nothing else than equation (74).

(ii) Equation on the modal coefficient $\int_Y S_k^\varepsilon U^\varepsilon \cdot e_n^k \, dy$.
Following the same lines, we choose a new test $\psi(t,X) := \theta(t) e_n^k(\frac{X}{\varepsilon})\eta(\frac{X}{\varepsilon} - \ell_x)$ for any real-valued $\eta \in C_c^\infty(Y)$ to get the relation

$$\begin{aligned}
0 &= \frac{1}{\varepsilon^N} \int_{I\times\Omega} F^\varepsilon \cdot \psi + U^\varepsilon \cdot (\partial_t - A^\varepsilon)\psi \; dX dt \\
&= e^{-2i\pi k.\ell_x} \int_{I\times Y} F^\varepsilon(t,\varepsilon\ell_x + \varepsilon y) \cdot e_n^k(y)\theta(t)\eta(y) \\
&\quad + U^\varepsilon(t,\varepsilon\ell_x + \varepsilon y) \cdot \left(\theta'(t)\eta(y) e_n^k(y) - \frac{1}{\varepsilon}\theta(t) A(\eta e_n^k)(y)\right) \; dydt \\
&= \int_{I\times Y} \eta S_k^\varepsilon F^\varepsilon \cdot e_n^k \theta + \eta S_k^\varepsilon U^\varepsilon \cdot e_n^k \left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)\theta - \frac{1}{\varepsilon} S_k^\varepsilon U^\varepsilon \cdot d_n^\eta \theta \; dydt,
\end{aligned} \tag{75}$$

where $d_n^\eta = d_n^\eta(y)$ is a commutator term defined using the periodization $\widetilde{\eta} := \varpi_0 \eta$ of η by

$$d_n^\eta := [\,A; \widetilde{\eta}\,] e_n^k = \begin{pmatrix} \sqrt{\frac{a}{\rho}}[e_n^k]_D \nabla\widetilde{\eta} \\ \sqrt{\frac{a}{\rho}}[e_n^k]_N . \nabla\widetilde{\eta} \end{pmatrix}.$$

At this stage our idea is to let η approach a constant value on Y in order to recover the projection of (74) along e_n^k. But this approximation procedure is by no means straightforward since $S_k^\varepsilon U^\varepsilon$ does not evolve in the domain of self-adjointness of A_k defined in (19). The main obstacle here is a lack of k-quasiperiodicity for $y \mapsto S_k^\varepsilon U^\varepsilon(x, y)$. To overcome this difficulty without involving boundary integrals over ∂Y, we extend the definition (29) of $S_k^\varepsilon U^\varepsilon$ to the whole of $\mathbb{R}^N \times \mathbb{R}^N$ by setting

$$S_k^\varepsilon U^\varepsilon(x, y) = \sum_{\ell \in L} (U^\varepsilon \mathbb{1}_{\Omega_\varepsilon})(\varepsilon\ell + \varepsilon y) e^{-2i\pi k.\ell} \mathbb{1}_{\varepsilon\ell+\varepsilon Y}(x)$$

for all $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$, where $U^\varepsilon \mathbb{1}_{\Omega_\varepsilon}$ is the trivial extension of U^ε by zero outside Ω_ε. This leads to a pseudo k-quasiperiodicity property

$$S_k^\varepsilon U^\varepsilon(x, y + \ell) = S_k^\varepsilon U^\varepsilon(x + \varepsilon\ell, y) e^{2i\pi k.\ell} \text{ for all } \ell \in L, \tag{76}$$

which will prove sufficient for our purpose. We now tackle the critical commutator term thanks to a computational trick requiring a fixed non-negative truncation function $\chi \in C_c^\infty(\mathbb{R}^N)$ of constant periodization [6] i.e.

$$\sum_{\ell \in L} \chi(y + \ell) = 1 \text{ for all } y \in \mathbb{R}^N. \tag{77}$$

More specifically, we write

$$\begin{aligned}
\int_{\mathbb{R}^N \times (\ell+Y)} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x, y)\zeta(x)\chi(y)\, dydx &= \int_{\mathbb{R}^N \times Y} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x, y + \ell)\zeta(x)\chi(y + \ell)\, dydx \\
&= \int_{\mathbb{R}^N \times Y} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x + \varepsilon\ell, y)\zeta(x)\chi(y + \ell)\, dydx \qquad \text{by (76)} \\
&= \int_{\mathbb{R}^N \times Y} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x, y)\zeta(x - \varepsilon\ell)\chi(y + \ell)\, dydx \\
&= \int_{\Omega \times Y} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x, y)\zeta(x)\chi(y + \ell)\, dydx + \varepsilon \int_{\mathbb{R}^N \times Y} S_k^\varepsilon U^\varepsilon \cdot [\,A; \widetilde{\eta}\,]\iota_\ell^\varepsilon\, dydx,
\end{aligned}$$

where

$$\iota_\ell^\varepsilon(x, y) := e_n^k(y)\frac{\zeta(x - \varepsilon\ell) - \zeta(x)}{\varepsilon}\chi(y + l).$$

Summing over $\ell \in L$ thanks to (77), we obtain this way a new expression of the commutator term in (75), namely:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega \times Y} S_k^\varepsilon U^\varepsilon \cdot d_n^\eta \zeta\, dydx \\
= \; &\frac{1}{\varepsilon}\int_{\mathbb{R}^N \times \mathbb{R}^N} (S_k^\varepsilon U^\varepsilon \cdot d_n^\eta)(x, y)\zeta(x)\chi(y)\, dydx - \int_{\mathbb{R}^N \times Y} S_k^\varepsilon U^\varepsilon \cdot \sum_{\ell \in L} [\,A; \widetilde{\eta}\,]\iota_\ell^\varepsilon\, dydx,
\end{aligned}$$

[6] Such a χ is usually introduced in harmonic analysis when identifying the periodic distributions on $\mathbb{R}^N$ with the distributions on the N-dimensional torus.

which can be simplified by inserting the identity

$$\theta[A;\eta]\iota_\ell^\varepsilon = \varepsilon\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)(\theta\eta\iota_\ell^\varepsilon) - \varepsilon\left(\partial_t - \frac{1}{\varepsilon}A\right)(\theta\eta\iota_\ell^\varepsilon) - \theta\eta\left(A - \frac{2i\pi s_n}{\alpha_n^k}\right)\iota_\ell^\varepsilon.$$

As a consequence,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{I\times\Omega\times Y} S_k^\varepsilon U^\varepsilon\cdot d_n^\eta\theta\zeta\; dydxdt &= \frac{1}{\varepsilon}\int_{I\times\mathbb{R}^N\times\mathbb{R}^N}(S_k^\varepsilon U^\varepsilon\cdot d_n^\eta)(x,y)\theta(t)\zeta(x)\chi(y)\; dydxdt \\ &+\varepsilon\int_{I\times\mathbb{R}^N\times Y} S_k^\varepsilon U^\varepsilon\cdot\left(\partial_t - \frac{1}{\varepsilon}A\right)(\theta\eta\iota^\varepsilon) - \eta S_k^\varepsilon U^\varepsilon\cdot\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)(\theta\iota^\varepsilon)\; dydxdt \\ &+\int_{I\times\mathbb{R}^N\times Y}\eta S_k^\varepsilon U^\varepsilon\cdot\theta\left(A - \frac{2i\pi s_n}{\alpha_n^k}\right)\iota^\varepsilon\; dydxdt,\end{aligned} \tag{78}$$

where

$$\iota^\varepsilon := \sum_{\ell\in L}\iota_\ell^\varepsilon = e_n^k(y)\sum_{\ell\in L}\frac{\zeta(x-\varepsilon\ell)-\zeta(x)}{\varepsilon}\chi(y+\ell). \tag{79}$$

According to our preliminary step (74), it so happens that

$$\int_{I\times\mathbb{R}^N\times Y} S_k^\varepsilon U^\varepsilon\cdot\left(\partial_t - \frac{1}{\varepsilon}A\right)(\theta\eta\iota^\varepsilon)\; dydxdt = -\int_{I\times\mathbb{R}^N\times Y}\eta S_k^\varepsilon F^\varepsilon\cdot\theta\iota^\varepsilon\; dydxdt. \tag{80}$$

Combining (75) and (78) together with (80), we get

$$\begin{aligned}-\int_{I\times\Omega\times Y}\eta S_k^\varepsilon U^\varepsilon\cdot e_n^k\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)\theta\zeta\; dydxdt - \varepsilon\int_{I\times\mathbb{R}^N\times Y}\eta S_k^\varepsilon U^\varepsilon\cdot\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)\theta\iota^\varepsilon\; dydxdt \\ = \int_{I\times\Omega\times Y}\eta S_k^\varepsilon F^\varepsilon\cdot e_n^k\theta\zeta\; dydxdt + \varepsilon\int_{I\times\mathbb{R}^N\times Y}\eta S_k^\varepsilon F^\varepsilon\cdot\theta\iota^\varepsilon\; dydxdt \\ -\int_{I\times\mathbb{R}^N\times Y}\eta S_k^\varepsilon U^\varepsilon\cdot\theta\left(A - \frac{2i\pi s_n}{\alpha_n^k}\right)\iota^\varepsilon\; dydxdt \\ -\frac{1}{\varepsilon}\int_{I\times\mathbb{R}^N\times\mathbb{R}^N}(S_k^\varepsilon U^\varepsilon\cdot d_n^\eta)(x,y)\theta(t)\zeta(x)\chi(y)\; dydxdt.\end{aligned} \tag{81}$$

This relation is a simple evolution equation on $\int_Y \eta S_k^\varepsilon U^\varepsilon\cdot e_n^k\; dy$ made intricate by several technical quantities. When $0\le\eta\le 1$ varies over an increasing sequence of limit 1 on Y, each term of (81) except the last one converges trivially. As for this last term, we can remark that it cancels in the limit because it takes a div_y-form [7] integrated on a domain ($\mathbb{R}^N$) with no boundary:

$$\begin{aligned}(S_k^\varepsilon U^\varepsilon\cdot d_n^\eta)(x,y)\chi(y) &= div_y\Big(\widetilde{\eta}\chi\left([S_k^\varepsilon U^\varepsilon]_N\sqrt{\frac{a}{\rho}}\overline{[e_n^k]_D} + [S_k^\varepsilon U^\varepsilon]_D\sqrt{\frac{a}{\rho}}\overline{[e_n^k]_N}\right)\Big) \\ &-\widetilde{\eta}\; div_y\Big(\chi\left([S_k^\varepsilon U^\varepsilon]_N\sqrt{\frac{a}{\rho}}\overline{[e_n^k]_D} + [S_k^\varepsilon U^\varepsilon]_D\sqrt{\frac{a}{\rho}}\overline{[e_n^k]_N}\right)\Big).\end{aligned}$$

As a result (81) holds true with $\eta = 1$ i.e.

$$-\int_{I\times\Omega\times Y} S_k^\varepsilon U^\varepsilon\cdot e_n^k\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)\theta\zeta\; dydxdt - \varepsilon\int_{I\times\mathbb{R}^N\times Y} S_k^\varepsilon U^\varepsilon\cdot\left(\partial_t - \frac{2i\pi s_n}{\varepsilon\alpha_n^k}\right)\theta\iota^\varepsilon\; dydxdt$$

[7] Even if a regularizing procedure on U^ε may be needed here, it does not require any uniformity in ε.

$$= \int_{I\times\Omega\times Y} S_k^\varepsilon F^\varepsilon \cdot e_n^k \theta\zeta \, dydxdt + \varepsilon \int_{I\times\mathbb{R}^N\times Y} S_k^\varepsilon F^\varepsilon \cdot \theta\iota^\varepsilon \, dydxdt$$

$$- \int_{I\times\mathbb{R}^N\times Y} S_k^\varepsilon U^\varepsilon \cdot \theta \left(A - \frac{2i\pi s_n}{\alpha_n^k} \right) \iota^\varepsilon \, dydxdt. \tag{82}$$

(iii) Equation on the non-oscillating part of $\int_Y S_k^\varepsilon U^\varepsilon \cdot e_n^k \, dy$.
Changing $\theta(t)$ into $\theta(t)e^{2i\pi s_n t/\varepsilon\alpha_n^k}$ in (82), we thus obtain a distributional evolution equation on I for

$$V_n^\varepsilon := e^{-2i\pi s_n t/\varepsilon\alpha_n^k} \frac{1}{|Y|} \int_Y S_k^\varepsilon U^\varepsilon \cdot e_n^k \, dy. \tag{83}$$

Likewise, a similar sequence of transformations, applied for instance to a basis of $Ker(A_k)$, would lead to a distributional evolution equation for $V_{1-\Pi}^\varepsilon := (1-\Pi^k)S_k^\varepsilon U^\varepsilon$. To sum up:

$$\begin{cases} \int_\Omega \zeta V_n^\varepsilon \, dx = \alpha_n^\varepsilon + \varepsilon\beta_n^\varepsilon(t) + \int_0^t \gamma_n^\varepsilon(s) \, ds, \\ \int_\Omega \zeta V_{1-\Pi}^\varepsilon \, dx = \alpha_{1-\Pi}^\varepsilon + \varepsilon\beta_{1-\Pi}^\varepsilon(t) + \int_0^t \gamma_{1-\Pi}^\varepsilon(s) \, ds, \end{cases} \tag{84}$$

with

$$\beta_n^\varepsilon(t) := -\frac{1}{|Y|} \int_{\mathbb{R}^N\times Y} e^{-2i\pi s_n t/\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon \cdot \iota^\varepsilon \, dydx, \tag{85}$$

$$\gamma_n^\varepsilon(t) := \frac{1}{|Y|} \int_{\mathbb{R}^N\times Y} e^{-2i\pi s_n t/\varepsilon\alpha_n^k} \left(\zeta S_k^\varepsilon F^\varepsilon \cdot e_n^k + \varepsilon S_k^\varepsilon F^\varepsilon \cdot \iota^\varepsilon - S_k^\varepsilon U^\varepsilon \cdot \left(A - \frac{2i\pi s_n}{\alpha_n^k} \right) \iota^\varepsilon \right) \, dydx, \tag{86}$$

and similar $(\beta_{1-\Pi}^\varepsilon, B_{1-\Pi}^\varepsilon)$. Here α_n^ε and $\alpha_{1-\Pi}^\varepsilon$ are constants (at $t=0$) playing no special role afterwards.

(iv) Bounds on the coefficients of (84).
We now check that $(\beta_n^\varepsilon, \gamma_n^\varepsilon)$ and $(\beta_{1-\Pi}^\varepsilon, \gamma_{1-\Pi}^\varepsilon)$ in (85)-(86) remain uniformly bounded in $L^2(I)$ for any fixed n. As a result, both terms in (84) will belong to precompact subsets of $L^2(I)$, since they are sum of a strongly convergent term ($\varepsilon\beta_n^\varepsilon$ or $\varepsilon\beta_{1-\Pi}^\varepsilon$) in $L^2(I)$ and of a bounded term in $H^1(I)$. To do so, we provide explicit estimates for any of the quantities paired with $S_k^\varepsilon U^\varepsilon$ and $S_k^\varepsilon F^\varepsilon$ in (85)-(86) by checking $||\iota^\varepsilon||_{L^2(\mathbb{R}^N\times Y)} + ||A\iota^\varepsilon||_{L^2(\mathbb{R}^N\times Y)} \le C||\nabla\zeta||_{L^2(\Omega)}$. Indeed, when y varies in a bounded set (in Y here) the sum in (79) is actually finite, so

$$\iota^\varepsilon = -e_n^k(y) \sum_{\ell\in L} \chi(y+\ell) \int_0^1 \ell.\nabla\zeta(x-\varepsilon s\ell) ds$$

can be roughly estimated in the L^2-norm of the x-variable by

$$||\iota^\varepsilon||_{L^2(\mathbb{R}^N)} \le C|e_n^k|(y) \sum_{\ell\in L} \chi(y+\ell) ||\nabla\zeta||_{L^2(\mathbb{R}^N)} = C||\nabla\zeta||_{L^2(\Omega)} |e_n^k|(y).$$

Likewise

$$\left(A - \frac{2i\pi s_n}{\alpha_n^k} \right) \iota^\varepsilon = -\sum_{\ell\in L} [\, A; \chi(y+\ell)\,] e_n^k \int_0^1 \ell.\nabla\zeta(x-\varepsilon s\ell) ds$$

$$\le C|e_n^k|(y) \sum_{\ell\in L} |\ell| \, |\nabla\chi|(y+\ell) \int_0^1 |\nabla\zeta|(x-\varepsilon s\ell) ds$$

and

$$|| \left(A - \frac{2i\pi s_n}{\alpha_n^k} \right) \iota^\varepsilon ||_{L^2(\mathbb{R}^N)} \le C|e_n^k|(y) ||\nabla\zeta||_{L^2(\mathbb{R}^N)} \sum_{\ell\in L} |\ell| \, |\nabla\chi|(y+\ell) \le C||\nabla\zeta||_{L^2(\Omega)} |e_n^k|(y).$$

We conclude the estimate of ι^ε and $A\iota^\varepsilon$ by an obvious integration over Y.

(v) Coherence and incoherence according to α/α_n^k.

Recalling (83) and the construction of $T^{\varepsilon\alpha}$, we now decompose

$$T^{\varepsilon\alpha}S_k^\varepsilon U^\varepsilon - (1-\Pi^k)S_k^\varepsilon U^\varepsilon = (T^{\varepsilon\alpha}-1)V_{1-\Pi}^\varepsilon + \sum_n T^{\varepsilon\alpha}(e^{2i\pi s_n t/\varepsilon\alpha_n^k}V_n^\varepsilon)e_n^k$$
$$= (T^{\varepsilon\alpha}-1)V_{1-\Pi}^\varepsilon + \sum_n e^{2i\pi s_n\tau\alpha/\alpha_n^k}E_n^{\varepsilon\alpha}(T^{\varepsilon\alpha}-T^{\varepsilon\alpha_n^k})V_n^\varepsilon e_n^k + \sum_n e^{2i\pi s_n\tau\alpha/\alpha_n^k}E_n^{\varepsilon\alpha}T^{\varepsilon\alpha_n^k}V_n^\varepsilon e_n^k,$$

where $E_n^{\varepsilon\alpha}$ denotes the multiplication operator by the step-wise unitary exponential

$$E_n^{\varepsilon\alpha}(t) := \sum_{m\in\mathbb{N}} e^{2i\pi m s_n\alpha/\alpha_n^k}\mathbb{1}_{(m\varepsilon\alpha, m\varepsilon\alpha+\varepsilon\alpha)}(t).$$

Note also that $T^{\varepsilon\alpha_n^k}V_n^\varepsilon e_n^k = e^{-2i\pi s_n\tau}\Pi_n^k W_k^\varepsilon U^\varepsilon$ and that $E_n^{\varepsilon\alpha}=1$ whenever $\alpha/\alpha_n^k\in\mathbb{N}^*$. So, Proposition 30 is eventually concerned with the weak convergence of $r^\varepsilon + R^\varepsilon \to 0$ where

$$\begin{cases} r^\varepsilon := (T^{\varepsilon\alpha}-1)V_{1-\Pi}^\varepsilon + \sum_n e^{2i\pi s_n\tau\alpha/\alpha_n^k}E_n^{\varepsilon\alpha}(T^{\varepsilon\alpha}-T^{\varepsilon\alpha_n^k})V_n^\varepsilon e_n^k, \\ R^\varepsilon := \displaystyle\sum_{n \text{ s.t. } \alpha/\alpha_n^k\notin\mathbb{N}^*} e^{2i\pi s_n\tau\alpha/\alpha_n^k}E_n^{\varepsilon\alpha}T^{\varepsilon\alpha_n^k}V_n^\varepsilon e_n^k. \end{cases}$$

Since every term in $r^\varepsilon + R^\varepsilon$ remains uniformly bounded in $L^2(I\times\Lambda\times\Omega\times Y)^{N+1}$, for instance

$$||\sum_n e^{2i\pi s_n\tau\alpha/\alpha_n^k}E_n^{\varepsilon\alpha}(T^{\varepsilon\alpha}-T^{\varepsilon\alpha_n^k})V_n^\varepsilon e_n^k||^2_{L^2(I\times\Lambda\times\Omega\times Y)} = \sum_n ||(T^{\varepsilon\alpha}-T^{\varepsilon\alpha_n^k})V_n^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega)}$$
$$\leq 2\sum_n ||T^{\varepsilon\alpha}V_n^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega)} + ||T^{\varepsilon\alpha_n^k}V_n^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega)} \leq 4\sum_n ||V_n^\varepsilon||^2_{L^2(I\times\Omega)} = 4||\Pi^k S_k^\varepsilon U^\varepsilon||^2_{L^2(I\times\Omega)},$$

we are reduced to testing $r^\varepsilon + R^\varepsilon$ on a total family of $L^2(I\times\Lambda\times\Omega\times Y)^{N+1}$ only. This allows us to fix n once for all and to investigate $\langle r^\varepsilon + R^\varepsilon|\theta(t)\xi(\tau)\zeta(x)e_n^k(y)\rangle$ for a given (θ,ξ,ζ).

First part : We check that $r^\varepsilon\to 0$ by using the fact that $V_{1-\Pi}^\varepsilon$ and V_n^ε do not contain any highly oscillating factors in time.

Since (84) yields

$$\begin{aligned}(T^{\varepsilon\alpha}-1)\int_\Omega \zeta V_n^\varepsilon\,dx &= \varepsilon(T^{\varepsilon\alpha}-1)\beta_n^\varepsilon + (T^{\varepsilon\alpha}-1)\int_0^t \gamma_n^\varepsilon(s)\,ds \\ &= \varepsilon(T^{\varepsilon\alpha}-1)\beta_n^\varepsilon + \sum_{\theta_\varepsilon}\mathbb{1}_{\theta_\varepsilon}(t)\int_t^{m\varepsilon\alpha+\varepsilon\alpha\tau}\gamma_n^\varepsilon(s)\,ds \\ &\leq \varepsilon|(T^{\varepsilon\alpha}-1)\beta_n^\varepsilon| + \sum_{\theta_\varepsilon}\mathbb{1}_{\theta_\varepsilon}(t)|\theta_\varepsilon|^{1/2}\left(\int_{\theta_\varepsilon}|\gamma_n^\varepsilon|^2(s)\,ds\right)^{1/2}\end{aligned}$$

where $\theta_\varepsilon = (m\varepsilon\alpha, m\varepsilon\alpha+\varepsilon\alpha)$ runs over all $\varepsilon\alpha$-cells contained in I, we finally get

$$\begin{aligned}||(T^{\varepsilon\alpha}-1)\int_\Omega\zeta V_n^\varepsilon\,dx||_{L^2(I)} &\leq 2\varepsilon||\beta_n^\varepsilon||_{L^2(I)} + \left(\sum_{\theta_\varepsilon}|\theta_\varepsilon|^2\int_{\theta_\varepsilon}|\gamma_n^\varepsilon|^2(s)\,ds\right)^{1/2} \\ &\leq 2\varepsilon||\beta_n^\varepsilon||_{L^2(I)} + \varepsilon\alpha||\gamma_n^\varepsilon||_{L^2(I)}. \end{aligned}\tag{87}$$

Combining $(T^{\varepsilon\alpha}-T^{\varepsilon\alpha_n^k})V_n^\varepsilon = (T^{\varepsilon\alpha}-1)V_n^\varepsilon - (T^{\varepsilon\alpha_n^k}-1)V_n^\varepsilon$ and (87) applied to both α and α_n^k, we recover $r^\varepsilon\to 0$ after intermediate partial integrations on $I\times\Lambda\times\Omega\times Y$. A similar estimate on $(T^{\varepsilon\alpha}-1)V_{1-\Pi}^\varepsilon$ is left to the reader.

SECOND PART : We check that $R^\varepsilon \to 0$ by using the weak convergence of $E_n^{\varepsilon\alpha}$ to zero when the ratio α/α_n^k is not an integer.
Setting $\theta_n(t,\tau) := e^{2i\pi s_n \tau\alpha/\alpha_n^k}\xi(\tau)\theta(t)$, we have

$$\langle R^\varepsilon|\theta(t)\xi(\tau)\zeta(x)e_n^k(y)\rangle = \int_{I\times\Lambda} T^{\varepsilon\alpha_n^k}\left(\int_\Omega \zeta V_n^\varepsilon\ dx\right)(t,\tau)E_n^{\varepsilon\alpha}(t)\theta_n(t,\tau)\ d\tau dt, \tag{88}$$

where $E_n^{\varepsilon\alpha}\theta_n \to 0$ weakly in $L^2(I\times\Lambda)$ whenever $\alpha/\alpha_n^k \notin \mathbb{N}^*$ by the well-known criterion (see [19] Ex. IV.13.27 p. 342 and Th. IV.8.20 p. 298) of weak (*) convergence $E_n^{\varepsilon\alpha} \to 0$ in $L^\infty(I)$, namely:

$$\left|\int_0^t E_n^{\varepsilon\alpha}(s)\ ds\right| = \varepsilon\alpha\left|(\frac{t}{\varepsilon\alpha} - [\frac{t}{\varepsilon\alpha}])e^{2i\pi s_n[\frac{t}{\varepsilon\alpha}]\alpha/\alpha_n^k} + \frac{\sin(\pi[\frac{t}{\varepsilon\alpha}]\alpha/\alpha_n^k)}{\sin(\pi\alpha/\alpha_n^k)}\right| \to 0 \text{ for all } t\in I.$$

Besides, $T^{\varepsilon\alpha_n^k}(\int_\Omega \zeta V_n^\varepsilon\ dx)$ varies in a precompact subset of $L^2(I\times\Lambda)$ according to (ii), so the weak-strong product in (88) tends to zero as claimed. ■

7.7 Convergence of boundary terms

In this subsection we take a closer look at boundary terms. It turns out that most of them actually play no role in the homogenization process because our original setting excludes fast oscillations of the boundary data $(g^\varepsilon, h^\varepsilon)$. Indeed, the pertaining quantities $\partial_t g^\varepsilon$ and h^ε are assumed in (13) to have a bounded derivative as ε tends to zero.

Proposition 31 *For G^ε build in (8) with $(\partial_t g^\varepsilon, h^\varepsilon)$ bounded by (13) and for any $(\psi_n^{k,\varepsilon}, \psi_{1-\Pi}^{k,\varepsilon})$ as in (65),*

$$\int_{I\times\partial\Omega} G^\varepsilon\cdot(\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon})\ d\sigma dt \to 0, \tag{89}$$

and

$$\int_{I\times\partial\Omega} G^\varepsilon\cdot\psi_{1-\Pi}^{k,\varepsilon}\ d\sigma dt \tag{90}$$
$$-\frac{1}{|Y|}\int_{I\times Y}\left(\int_{\Gamma_D} n_\Omega\partial_t g^\varepsilon\cdot\sqrt{a}\varphi_{1-\Pi}^k\ d\sigma + \int_{\Gamma_N} h^\varepsilon\cdot\frac{\phi_{1-\Pi}^k}{\sqrt{\rho}}\ d\sigma\right)\ dydt \to 0,$$

where $(\varphi_{1-\Pi}^k, \phi_{1-\Pi}^k) := (1-\Pi^k)\psi$.

Proof. (i) We begin with the non-kernel part (89). Recalling (64) and the admissibility of $\psi_{+n}^{\varepsilon,k} + \psi_{-n}^{\varepsilon,k}$ proved in Proposition 26 for any $n\in\mathbb{M}_+^k$, we obtain

$$\int_{I\times\partial\Omega} G^\varepsilon\cdot(\psi_{+n}^{k,\varepsilon} + \psi_{-n}^{k,\varepsilon})\ d\sigma dt = \int_I G_+^\varepsilon(t)e^{-2i\pi t/\varepsilon\alpha_n^k}dt + \int_I G_-^\varepsilon(t)e^{+2i\pi t/\varepsilon\alpha_n^k}dt,$$

where

$$G_\pm^\varepsilon(t) := \int_{\partial\Omega} G^\varepsilon(t,x)\cdot\psi_{\pm n}(t,x)e_n^k(\frac{x}{\varepsilon})d\sigma$$

remains bounded in $H^1(I)$ in view of (13) and of the straightforward estimate [8]

$$|G_\pm^\varepsilon| + |\partial_t G_\pm^\varepsilon| \le C||e_n^k||_{L^\infty}\left(||G^\varepsilon||_{L^2(\partial\Omega)} + ||\partial_t G^\varepsilon||_{L^2(\partial\Omega)}\right)\left(||\psi_{\pm n}||_{L^2(\partial\Omega)} + ||\partial_t\psi_{\pm n}||_{L^2(\partial\Omega)}\right).$$

[8] Note that $e_n^k \in L^\infty$ by Lemma 12.

According to the compactness of the embedding $H^1(I) \subset L^2(I)$, the family $\{G_\pm^\varepsilon \mid 0 < \varepsilon < 1\}$ is a precompact subset of $L^2(I)$. So the weak-strong products of $G_\pm^\varepsilon(t)$ with the oscillatory factors $e^{\mp 2i\pi t/\varepsilon\alpha_n^k}$ cancel [9] in the limit.

(ii) As for the kernel part, we prove (90) starting from

$$\int_{I\times\partial\Omega} G^\varepsilon \cdot \psi_{1-\Pi}^{\varepsilon,k}\, d\sigma dt = \int_{I\times\Gamma_D} n_\Omega \partial_t g^\varepsilon \cdot \sqrt{a}\varphi_{1-\Pi}^k(t,x,\frac{x}{\varepsilon})\, d\sigma dt + \int_{I\times\Gamma_N} h^\varepsilon \cdot \frac{\phi_{1-\Pi}^k}{\sqrt{\rho}}(t,x,\frac{x}{\varepsilon})\, d\sigma dt,$$

where $div_y(\sqrt{a}\varphi_{1-\Pi}^k) = 0$ and $\nabla_y(\phi_{1-\Pi}^k/\sqrt{\rho}) = 0$ by the very definition of $Ker(A_k)$. To conclude the proof, we extend $\partial_t g^\varepsilon$ (only defined on Γ_D) to the whole of $\partial\Omega$ by zero, and we apply Lemma 32 to the resulting precompact sequence of $L^2(I\times\partial\Omega)$ to obtain

$$\int_{I\times\Gamma_D} n_\Omega \partial_t g^\varepsilon \cdot \sqrt{a}\varphi_{1-\Pi}^k(t,x,\frac{x}{\varepsilon})\, d\sigma dt - \frac{1}{|Y|}\int_{I\times\Gamma_D\times Y} n_\Omega \partial_t g^\varepsilon \cdot \sqrt{a}\varphi_{1-\Pi}^k(t,x,y)\, dy d\sigma dt \to 0,$$

the function $\chi := \sqrt{a}\varphi_{1-\Pi}^k$ being regular enough by (64). As for h^ε-terms, the Y-average may or may not be inserted before $h^\varepsilon \cdot \phi_{1-\Pi}^k/\sqrt{\rho}$ in (90), since $\phi_{1-\Pi}^k/\sqrt{\rho}$ is independent of y. ∎

Lemma 32 *If $\{\kappa^\varepsilon \mid 0 < \varepsilon < 1\} \subset L^2(\partial\Omega)$ is precompact and if $\chi \in L^2(\partial\Omega; \mathcal{C}^0(\mathbb{R}^N))^N$ is k-quasiperiodic in y with $div_y(\chi) = 0$ then*

$$\int_{\partial\Omega} \kappa^\varepsilon(x)\chi(x,\frac{x}{\varepsilon}).n_\Omega(x)\, d\sigma - \frac{\mathbb{1}_{L^*}(k)}{|Y|}\int_{\partial\Omega\times Y} \kappa^\varepsilon(x)\chi(x,y).n_\Omega(x)\, dy d\sigma \to 0.$$

Proof. Suppose the convergence did not hold true for $k = 0$. Then, up to an extraction $\kappa^\varepsilon \to \kappa$ in $L^2(\partial\Omega)$, at least one limit point κ would satisfy

$$\left|\int_{\partial\Omega} \kappa(x)\chi(x,\frac{x}{\varepsilon}).n_\Omega(x)\, d\sigma - \frac{1}{|Y|}\int_{\partial\Omega\times Y} \kappa(x)\chi(x,y).n_\Omega(x)\, dy d\sigma\right| > C > 0 \qquad (91)$$

for ε small enough. Approximating χ by a div_y-free $\chi' \in \mathcal{C}^\infty(\overline{\Omega}) \otimes \mathcal{C}_\sharp^\infty(Y)^N$ in $L^2(\partial\Omega; \mathcal{C}_\sharp^0(\overline{Y}))^N$ and κ by $\kappa' \in \mathcal{C}^\infty(\overline{\Omega})$ in $L^2(\partial\Omega)$, inequality (91) would hold with (κ', χ') in place of (κ, χ). But this is impossible, because

$$\int_{\partial\Omega} \kappa'(x)\chi'(x,\frac{x}{\varepsilon}).n_\Omega(x)\, d\sigma = \int_\Omega div\big(\kappa'(x)\chi'(x,\frac{x}{\varepsilon})\big)\, dx$$

$$= \int_\Omega (div_x + \frac{1}{\varepsilon}div_y)\big(\kappa'(x)\chi'(x,y)\big)(x,\frac{x}{\varepsilon})\, dx = \int_\Omega div_x(\kappa'\chi')(x,\frac{x}{\varepsilon})\, dx$$

$$\to \frac{1}{|Y|}\int_{\Omega\times Y} div_x(\kappa'\chi')(x,y)\, dy dx = \frac{1}{|Y|}\int_{\partial\Omega\times Y} \kappa'(x)\chi'(x,y).n_\Omega(x)\, dy d\sigma$$

in virtue of Remark 15 (iii), whence a contradiction to (91). The case $k \notin L^*$ is similar. ∎

7.8 Second step to homogenized operators

This subsection is devoted to the derivation of the one-fibered wave two-scale operator $\mathcal{A}_k$ defined in (56), and from which derives the formula previously set for $\mathcal{A}$ in (52).

[9] Let us recall that (i) is a triviality in most cases in view of Remark 27.

Remark 33 *The convergence of the series defining $\mathcal{A}_k$ is a by-product of our previous results, and especially of the proof of Proposition 28, but it can also be double-checked directly:*

$$||\mathcal{A}_k\psi||^2_{L^2(I\times\Lambda\times\Omega\times Y)} \leq C\left(||\psi||^2_{L^2(I\times\Lambda\times\Omega\times Y)} + ||\nabla_x\psi||^2_{L^2(I\times\Lambda\times\Omega\times Y)}\right).$$

Indeed, taking advantage of orthogonality and adjonction relations, we can see that

$$\begin{aligned}
||\mathcal{A}_k\psi||^2_{L^2(Y)} &\leq ||(1-\Pi^k)B(1-\Pi^k)\psi||^2_{L^2(Y)} + ||\sum \Pi^k_m B\Pi^k_n\psi||^2_{L^2(Y)} \\
&= ||(1-\Pi^k)B(1-\Pi^k)\psi||^2_{L^2(Y)} + ||\Pi^k B\Pi^k\psi||^2_{L^2(Y)} \\
&\leq ||(1-\Pi^k)B(1-\Pi^k)\psi||^2_{L^2(Y)} + 2||\Pi^k B(1-\Pi^k)\psi||^2_{L^2(Y)} + 2||\Pi^k B\psi||^2_{L^2(Y)} \\
&\leq 2||B(1-\Pi^k)\psi||^2_{L^2(Y)} + 2||\Pi^k B\psi||^2_{L^2(Y)} \\
&\leq 2||\int_\Lambda B\psi\ d\tau||^2_{L^2(Y)} + 2||B\psi||^2_{L^2(Y)} \qquad \text{by (64)},
\end{aligned}$$

with moreover $(1-\Pi^k)B(1-\Pi^k) = 1\!\!1_{L^*}(k)$ according to Lemma 37.

In order to discuss the plausibility of the uniqueness property in problem (57), it will be of interest to know that the integro-differential operator $\mathcal{A}_k$ (when properly defined) inherits the self-adjoint character of B:

Proposition 34 *The operator $i\mathcal{A}_k$ with domain*

$$\begin{gathered}
D(\mathcal{A}_k) := \{(\varphi,\phi) \in H^1_k(Y;H^1(\Omega))^N \times H^1_k(Y;H^1(\Omega)) \\
s.t.\ \varphi = 0 \ on\ \Gamma_N\times Y \ and\ \phi = 0\ on\ \Gamma_D\times Y\}
\end{gathered}$$

is essentially [10] *self-adjoint on $L^2(\Omega\times Y)^{N+1}$.*

Proof. Obviously, $i\mathcal{A}_k$ is a densely defined symmetric operator. Consequently, the essential self-adjointness statement is equivalent to the fact that the operators $\pm i - i\mathcal{A}_k$ both have dense ranges, see [24] Problems p. 269. In fact, taking advantage of orthogonality relations, it suffices to show that the ranges of the restrictions to $D(\mathcal{A}_k)$ of $\Pi^k_{+n}(\pm i - iB)\Pi^k_{+n} + \Pi^k_{-n}(\pm i - iB)\Pi^k_{-n}$ and $(1-\Pi^0)(\pm i - iB)(1-\Pi^0)$ are dense in the ranges of the projections $\Pi^k_{+n}+\Pi^k_{-n}$ and $1-\Pi^0$ respectively.

(i) First case ($\Pi^k_{+n}+\Pi^k_{-n}$). Choosing $\psi \in D(\mathcal{A}_k)$ of the type $\psi(x,y) := v(x)v^k_n(y) + w(x)w^k_n(y)$ for varying (v,w) in

$$\hat{D} := \{(v,w)\in\mathcal{C}^\infty(\overline{\Omega})\times\mathcal{C}^\infty(\overline{\Omega}) \mid v = 0 \text{ on } \Gamma_N \text{ and } w = 0 \text{ on } \Gamma_D\}, \tag{92}$$

we compute [11]

$$\begin{aligned}
&\big(\Pi^k_{+n}(\pm i - iB)\Pi^k_{+n} + \Pi^k_{-n}(\pm i - iB)\Pi^k_{-n}\big)\psi \\
&= \pm i(vv^k_n + ww^k_n) + \big(a^k_n(vw^k_n - wv^k_n) + w^k_n(b^k_n.\nabla_x v) - v^k_n(b^k_n.\nabla_x w)\big)
\end{aligned}\tag{93}$$

thanks to (25) and (20) by means of

$$\left\{\begin{aligned}
a^k_n &:= \frac{-i}{2|Y|}\int_Y \frac{\overline{\phi^k_n}}{\sqrt{\rho}} div_x\Big(\frac{a}{\sqrt{\lambda^k_n}}\nabla_y\big(\frac{\phi^k_n}{\sqrt{\rho}}\big)\Big) - \frac{a}{\sqrt{\lambda^k_n}}\nabla_y\big(\frac{\overline{\phi^k_n}}{\sqrt{\rho}}\big).\nabla_x\big(\frac{\phi^k_n}{\sqrt{\rho}}\big)\ dy \in \mathbb{C}, \\
b^k_n &:= \frac{1}{\sqrt{\lambda^k_n}}\,\mathrm{Im}\left(\frac{1}{|Y|}\int_Y \frac{\overline{\phi^k_n}}{\sqrt{\rho}} a\nabla_y\big(\frac{\phi^k_n}{\sqrt{\rho}}\big)\ dy\right) \in \mathbb{R}^N.
\end{aligned}\right.$$

[10] The formulation of the exact domain of self-adjointness would be tedious.

[11] Hint : it is simpler to use $\{e^k_{+n}, e^k_{-n}\}$ and only turn back to $\{v^k_n, w^k_n\}$ in the end.

We check at once the relation $div_x(b_n^k) = 2\,\mathrm{Re}(a_n^k)$ of formal self-adjointness for $H := i(a_n^k + b_n^k.\nabla_x)$. As a consequence, we are led through a unitary equivalence to show that the operators

$$\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \pm i \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} 0 & -iH \\ +iH & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{94}$$

with domain $\hat{D}$ both have dense ranges in $L^2(\Omega) \times L^2(\Omega)$, or equivalently that the matrix-valued differential operator of (94) is essentially self-adjoint on $\hat{D} \subset L^2(\Omega) \times L^2(\Omega)$. But this plainly results from the balanced boundary conditions expressed in $\hat{D}$ and from the relation $div_x(b_n^k) = 2\,\mathrm{Re}(a_n^k)$ of formal self-adjointness for H, the field b_n^k being real-valued.

(ii) Second case $(1-\Pi^0)$. Suppose that $(p,q) \in Range(1-\Pi^0) = Ker(A_0)$ is orthogonal to the range of $(1-\Pi^0)(\pm i - iB)(1-\Pi^0)$ restricted to $D(\mathcal{A}_0)$. In particular,

$$(p,q) \in L^2(\Omega \times Y)^{N+1} \mid div_y(\sqrt{a}p) = 0 \text{ and } \nabla_y(q/\sqrt{\rho}) = 0,$$

satisfies

$$\pm \int_{\Omega\times Y} \frac{1}{\sqrt{a}} p \cdot \varphi + \sqrt{\rho} q \cdot \phi \; dydx = \int_{\Omega\times Y} \sqrt{a} p \cdot \nabla_x \phi + \frac{q}{\sqrt{\rho}} \cdot div_x(\varphi) \; dydx \tag{95}$$

for all $(\varphi, \phi) \in \mathcal{C}^\infty(\overline{\Omega \times Y})^{N+1}$ such that

$$\begin{cases} (\varphi, \phi) \text{ is periodic in } y, \\ div_y(\varphi) = 0 \text{ and } \nabla_y \phi = 0 \text{ in } \Omega \times Y, \\ \varphi = 0 \text{ on } \Gamma_N \times Y \text{ and } \phi = 0 \text{ on } \Gamma_D \times Y. \end{cases}$$

Our goal here is to infer that $(p,q) = 0$.

(a) Taking $\varphi \in \mathcal{C}_c^\infty(\Omega)^N$ independent of $y \in Y$ and $\phi = 0$, we read that $q/\sqrt{\rho} \in H^1(\Omega)$ only depends on x, with besides

$$\nabla_x(\frac{q}{\sqrt{\rho}}) = \mp \frac{1}{|Y|} \int_Y \frac{1}{\sqrt{a}} p \; dy \quad \text{in } \Omega.$$

Owing to this, an extension to any $\varphi \in \mathcal{C}^\infty(\overline{\Omega})^N$ null on Γ_N yields at once $\gamma(q/\sqrt{\rho}) = 0$ on Γ_D. Finally, integrating (95) by parts, we see that

$$\int_{\Omega\times Y} \left(\nabla_x(\frac{q}{\sqrt{\rho}}) \pm \frac{1}{\sqrt{a}} p \right) \cdot \varphi \; dydx = 0 \tag{96}$$

holds true for any y-periodic $\varphi \in \mathcal{C}^\infty(\overline{\Omega \times Y})^N$ such that $div_y(\varphi) = 0$ in $\Omega \times Y$ and $\varphi = 0$ on $\Gamma_N \times Y$. By density, this orthogonality relation even extends to any div_y-free field $\varphi \in L^2(\Omega \times Y)^N$, and in particular to $\sqrt{a}p$.

(b) Likewise, taking $\varphi = 0$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ independent of $y \in Y$, we read that $\frac{1}{|Y|}\int_Y \sqrt{a}p \; dy \in H^{div}(\Omega)$, with besides

$$div_x \left(\frac{1}{|Y|} \int_Y \sqrt{a} p \; dy \right) = \mp \left(\frac{1}{|Y|} \int_Y \rho \; dy \right) \frac{q}{\sqrt{\rho}} \quad \text{in } \Omega. \tag{97}$$

As before, an extension to any $\phi \in \mathcal{C}^\infty(\overline{\Omega})$ null on Γ_D yields at once $\gamma_n(\frac{1}{|Y|}\int_Y \sqrt{a}p \; dy) = 0$ on Γ_N.

(c) As a consequence of the above discussion, the sum of (96) with $\varphi = \sqrt{a}p$ and (97) times $\overline{q}/\sqrt{\rho}$ yields

$$
\begin{aligned}
&||p||^2_{L^2(\Omega\times Y)} + \left(\frac{1}{|Y|}\int_Y \rho\ dy\right) ||\frac{q}{\sqrt{\rho}}||^2_{L^2(\Omega\times Y)} \\
= &\mp \left\langle \gamma_n \left(\frac{1}{|Y|}\int_Y \sqrt{a}p\ dy\right) | \gamma(\frac{\overline{q}}{\sqrt{\rho}}) \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)},
\end{aligned}
$$

where the trace of $q/\sqrt{\rho}$ on Γ_D and the normal trace of the Y-average of $\sqrt{a}p$ on Γ_N have already been identified to zero before. This concludes the proof of $(p, q) = 0$. ■

A straightforward consequence of Proposition 34 is that:

Corollary 35 *The operator $i\mathcal{A}$ is essentially self-adjoint on $L^2(\Omega \times Y_K)^{N+1}$.*

We now turn to the proof of Theorem 24:

Proof. By (67)-(68), (69)-(70), (72)-(73), (89)-(90), and the definitions (49), the following infinite dimensional system of decoupled equations is found to be satisfied for any $\psi \in \mathcal{W}_k$ in the limit

$$
\begin{aligned}
&\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} F^k \cdot (\Pi^k_{+n} + \Pi^k_{-n})\psi + U^k \cdot \big(\partial_t - (1-\Pi^k)B\big)(\Pi^k_{+n} + \Pi^k_{-n})\psi \\
&\qquad\qquad -U^k \cdot (B^k_{+n}\Pi^k_{+n} + B^k_{-n}\Pi^k_{-n})\psi\ dydxd\tau dt \\
&+\frac{1}{|Y|}\int_{\Omega\times Y} U^k_0 \cdot (\Pi^k_{+n} + \Pi^k_{-n})\psi(t=0,\tau=0)\ dydx = 0 \text{ for all } n \in \mathbb{M}^k_+,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{|Y|}\int_{I\times\Lambda\times\Omega\times Y} F^k \cdot (1-\Pi^k)\psi + U^k \cdot \big(\partial_t - (1-\Pi^k)B(1-\Pi^k)\big)(1-\Pi^k)\psi\ dydxd\tau dt \qquad (98)\\
&+\frac{1}{|Y|}\int_{\Omega\times Y} U^k_0 \cdot (1-\Pi^k)\psi(t=0,\tau=0)\ dydx + \frac{1}{|Y|}\int_{I\times\partial\Omega\times Y} G^k \cdot (1-\Pi^k)\psi\ dyd\sigma dt = 0.
\end{aligned}
$$

Furthermore, taking advantage of the τ-average, we can exhibit additional cancellation effects due to the symmetric role played by U^k and $\psi \in \mathcal{W}_k$ in the limit. For instance,

$$
\int_{\Lambda\times Y} U^k \cdot (1-\Pi^k)B\Pi^k_n\psi\ dyd\tau = \int_{\Lambda\times Y} (1-\Pi^k)U^k \cdot B\Pi^k_n\psi\ dyd\tau = 0 \text{ for all } n \in \mathbb{M}^k,
$$

because $(1-\Pi^k)U^k$ is independent of τ while $\Pi^k_n\psi$ varies like $e^{2i\pi s_n\tau}$, see (64). In the same fashion, any term in the definition (71) of B^k_n indexed by $m \in \mathbb{M}^k$ such that $(s_n, \alpha^k_n) \neq (s_m, \alpha^k_m)$ finally disappears, because the balance of τ-exponentials in $\Pi^k_m U^k \cdot B^k_n\Pi^k_n\psi$ leads to the relation $s_m = s_m(1-\alpha^k_n/\alpha^k_m)+s_n$, or equivalently $s_n\alpha^k_m = s_m\alpha^k_n$. This explains how (56) originates from (71). Besides, the special term $\mathbb{1}_{L^*}(k)\mathcal{B}$ results from the simplification of $(1-\Pi^k)B(1-\Pi^k)$ due to Lemma 37 below. ■

We close the discussion of Theorem 24 with a remark about the uniqueness of our homogenized solutions U in a restricted case:

Proposition 36 *If $\Omega = \mathbb{R}^N$ then the solution $U \in L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1}$ to the micro-macro equations (54)-(55) is unique.*

The detailed proof will be omitted. The essential ideas are to be found in Theorem 3, since the evolution equation $(\partial_t - \mathcal{A}_k)U^k = F^k$ obtained for U^k in the limit is governed by an essentially self-adjoint operator, as in Section 4. The key point here is to realize that the space $\mathcal{W}_k$ of all admissible test functions in the weak formulation (57) with $\Omega = \mathbb{R}^N$ is a core of $D(\mathcal{A}_k)$ when the time variables are fixed. Indeed, the boundary conditions laid down by D in (12) and by $\hat{D}$ in (92) disappear for $\Omega = \mathbb{R}^N$, and Proposition 34 proves that $i\mathcal{A}_k$ is essentially self-adjoint on $H^1_k(Y; H^1(\mathbb{R}^N))^{N+1} \subset L^2(\mathbb{R}^N \times Y)^{N+1}$. So is $i\mathcal{A}$ as a finite direct sum of $i\mathcal{A}_k$'s.

7.9 Local formulation

In this subsection we extract the local PDEs of the homogenized model stated in Theorem 19 from the formal expressions of the operators $\mathcal{A}$ and $\mathcal{B}$ used so far.

We first point out some simplifications due to the special forms of the original problem (10) and of the initial data (8).

Lemma 37 *The operator $\mathcal{B}$ is the part of B in the kernel of A_k when the fiber $k \in L^*$ corresponds to periodic conditions only. In other words $(1-\Pi^k)B(1-\Pi^k) = 1\!\!1_{L^*}(k)\mathcal{B}$ for any $k \in Y^*$. In a similar fashion, up to an extraction*

$$(1-\Pi^k)W_k^\varepsilon F^\varepsilon = \frac{1\!\!1_{L^*}(k)}{\int_Y \rho\, dy} \begin{pmatrix} 0 \\ \sqrt{\rho}\int_Y S_0^\varepsilon f^\varepsilon\, dy \end{pmatrix} \to 1\!\!1_{L^*}(k) \begin{pmatrix} 0 \\ \sqrt{\rho} f/\widehat{\rho} \end{pmatrix}, \tag{99}$$

where f has been defined as the weak limit of f^ε in $L^2(I\times\Omega)$ and $\widehat{\rho}$ as the mean of ρ.

Proof. When $k \notin L^*$ the last component of any vector in $Range(1-\Pi^k) = Ker(A_k)$ is zero because the (kernel) equation $\nabla_y = 0$ has no other solution in $H^1_k(Y)$. So $(1-\Pi^k)B(1-\Pi^k) = 0$, and likewise $(1-\Pi^k)W_k^\varepsilon F^\varepsilon = 0$, given the very special form (7) of F^ε. This is no longer true in the periodic case ($k=0$) we shall now investigate. With this aim, setting $(\varphi,\phi) := (1-\Pi^0)(p,q)$ for a regular (p,q), we check that $1-\Pi^0$ acts component-wise as two independent projections, since $q \mapsto \phi$ is the orthogonal projection of $L^2(Y)$ onto $\mathbb{C}\sqrt{\rho} \subset L^2(Y)$, while $p \mapsto \varphi$ is the orthogonal projection of $L^2(Y)^N$ onto

$$\{\varphi \in L^2(Y)^N \mid \sqrt{a}\varphi \in H^{div}_\sharp(Y) \text{ and } div_y(\sqrt{a}\varphi) = 0\} = \left(\sqrt{a}\nabla H^1_\sharp(Y)\right)^\perp \subset L^2(Y)^N.$$

For the link between div-free fields and gradients we refer to the proof (ii) of Theorem 9. As a result, $\phi = \sqrt{\rho}(\int_Y \sqrt{\rho} q\, dy)/(\int_Y \rho\, dy)$ and $\varphi = p - \sqrt{a}\nabla_y w$, where $w \in H^1_\sharp(Y)/\mathbb{C}$ denotes the unique solution to the periodic elliptic problem $-div_y(a\nabla_y w) = -div_y(\sqrt{a}p)$, whose righthand side is (a formal notation for) the linear form $\zeta \mapsto \int_Y \sqrt{a}p.\nabla\zeta\, dy$ viewed on $H^1_\sharp(Y)$. Equivalently

$$\varphi = a^{+1/2}(1-P)a^{-1/2}p. \tag{100}$$

This insight into $1-\Pi^0$ leads us to

$$B(1-\Pi^0)\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \sqrt{a}\nabla_x\left(\frac{1}{|Y|}\int_Y \sqrt{\rho} q\, dy/\widehat{\rho}\right) \\ \frac{1}{\sqrt{\rho}} div_x\left(a(1-P)a^{-1/2}p\right) \end{pmatrix}$$

and to

$$(1-\Pi^0)B(1-\Pi^0)\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\widehat{\rho}}\begin{pmatrix} \sqrt{a}(1-P)\nabla_x\left(\frac{1}{|Y|}\int_Y \sqrt{\rho} q\, dy\right) \\ \sqrt{\rho}\, div_x\left(\frac{1}{|Y|}\int_Y a(1-P)a^{-1/2}p\, dy\right) \end{pmatrix},$$

which is exactly (51). We leave the easier relation (99) to the reader. ■

We then state a generalization of a classical lemma on the two-scale convergence of gradients (see [1]). The novelty here is that the derivative w.r.t. the macroscopic variable disappears for any fiber $k \notin L^*$ corresponding to aperiodic conditions. The proof will be omitted since it follows the same lines as in the classical case of $k \in L^*$.

Lemma 38 *If $\nabla u_0^\varepsilon \in L^2(\Omega)^N$ is a bounded sequence of gradients then after extraction*

$$S_k^\varepsilon \nabla u_0^\varepsilon \to 1\!\!1_{L^*}(k)\nabla_x u_0 + \nabla_y u_0^k \text{ weakly in } L^2(\Omega\times Y)^N,$$

where ∇u_0 denotes the weak limit of ∇u_0^ε in $L^2(\Omega)^N$ and where $u_0^k \in L^2(\Omega; H_k^1(Y))$.

We now turn to the derivation of Theorem 19 as a disintegration by parts of Theorem 24:

(i) The non-kernel part. At first, we check that $\Pi_{\pm m}^k B \Pi_{\pm n}^k$ acts [12] as

$$\Pi_{\pm m}^k B \Pi_{\pm n}^k : \psi \mapsto \pm e_{\pm m}\, \kappa_{nm}^k . \nabla_x \left(\frac{1}{|Y|}\int_Y \psi \cdot e_{\pm n}^k \, dy \right)$$

with κ_{nm}^k defined in (26). Next, recalling (56), we come back to (57) with a test function $\psi \in \mathcal{W}_k$ of the type $\psi(t,\tau,x,y) := e^{2i\pi s_n \tau}\varphi(t,x)e_n^k(y)$ for $n \in \mathbb{M}^k$ fixed and φ compactly supported in $I\times\Omega$. Given the structure (44) of U, the resulting equation reads

$$\int_{I\times\Omega} F_n^k \cdot \varphi \, dxdt + \int_{I\times\Omega} U_n^k \cdot \left(\partial_t\varphi - s_n \sum_m \kappa_{nm}^k . \nabla_x \varphi\right) dxdt + \int_\Omega U_{0,n}^k \cdot \varphi(t=0)\, dx = 0,$$

with

$$F_n^k := \int_\Lambda e^{-2i\pi s_n \tau} \frac{1}{|Y|}\int_Y F^k \cdot e_n^k \, dyd\tau \text{ and } U_{0,n}^k := \frac{1}{|Y|}\int_Y U_0^k \cdot e_n^k \, dy.$$

But these two expressions obviously coincide with the equivalent definitions given in (42)-(43). Thus the transport system (47) is established.

(ii) The kernel part for $k \notin L^*$. Let us prove that $(1-\Pi^k)U^k = 0$ disappears for $k \notin L^*$. Gathering the microscopic equation $\partial_\tau(1-\Pi^k)U^k = 0$ obtained in (59) and the macroscopic equation $\partial_t(1-\Pi^k)U^k = 0$ obtained from (57) with a test function $\psi = (1-\Pi^k)\psi \in \mathcal{W}_k$ (see Lemma 37 and definition 50 to get rid of F^k and G^k), we conclude from these two relations in $\mathcal{D}'(I\times\Lambda\times\Omega\times Y)$ that $(1-\Pi^k)U^k(t,\tau,x,y) = (1-\Pi^k)U_0^k(x,y)$ only depends on the behavior of the initial data analyzed in Lemma 38 i.e.

$$U_0^k = \lim_\varepsilon S_k^\varepsilon U_0^\varepsilon = \lim_\varepsilon \begin{pmatrix} \sqrt{a} S_k^\varepsilon \nabla u_0^\varepsilon \\ \sqrt{\rho} S_k^\varepsilon v_0^\varepsilon \end{pmatrix} = \begin{pmatrix} \sqrt{a}\nabla_y u_0^k \\ \dots \end{pmatrix}.$$

Consequently $U_0^k \in Ker(A_k)^\perp$ for $k \notin L^*$ and $(1-\Pi^k)U_0^k = 0$ as expected.

(iii) The kernel part for $k \in L^*$. In (57) specialized to $\psi = (1-\Pi^0)\psi \in \mathcal{W}_0$, namely

$$\int_{I\times\Lambda\times\Omega\times Y} (1-\Pi^0)F^0 \cdot \psi \, dydxd\tau dt + \int_{I\times\Lambda\times\Omega\times Y} (1-\Pi^0)U^0 \cdot (\partial_t - \mathcal{B})\psi \, dydxd\tau dt$$
$$+ \int_{\Omega\times Y} (1-\Pi^0)U_0^0 \cdot \psi(t=0,\tau=0)\, dydx + \int_{I\times\Lambda\times\partial\Omega\times Y} (1-\Pi^0)G^0 \cdot \psi \, dyd\sigma d\tau dt = 0, \quad (101)$$

[12] See (93) for a related identity.

we insert the expression of $1-\Pi^0$ given in (100) together with the simplifications of the data due to Lemmas 37 and 38 i.e.

$$(1-\Pi^0)F^0 = \begin{pmatrix} 0 \\ \sqrt{\rho} f/\widehat{\rho} \end{pmatrix},$$
$$(1-\Pi^0)U_0^0 = (1-\Pi^0)\lim_{\varepsilon} \begin{pmatrix} \sqrt{a} S_0^\varepsilon \nabla u_0^\varepsilon \\ \frac{1}{\sqrt{\rho}} S_0^\varepsilon \rho^\varepsilon v_0^\varepsilon \end{pmatrix} = \begin{pmatrix} \sqrt{a}(1-P)\nabla u_0 \\ \sqrt{\rho} v_0 \end{pmatrix},$$
$$(1-\Pi^0)G^0 = \begin{pmatrix} \mathbb{1}_{\Gamma_D} \partial_t g \sqrt{a}(1-P) n_\Omega \\ \mathbb{1}_{\Gamma_N} h \sqrt{\rho}/\widehat{\rho} \end{pmatrix},$$

where $(f, \partial_t g, h, u_0, v_0)$ has been defined independently of y in (41). Setting $\Phi =: (\varphi, \sqrt{\rho}\Phi)$ and $U_H = (1-\Pi^0)U^{(0)} =: (p,q)$, the microscopic equations of $\mathcal{W}_0$ and $Ker(A_0)$

$$\begin{cases} \partial_\tau \Phi = 0 \text{ and } \nabla_y \Phi = 0, \\ \partial_\tau \varphi = 0 = div_y(\sqrt{a}\varphi) \text{ hence } Pa^{-1/2}\varphi = 0 \text{ and } \sqrt{a}\varphi \in (Range\, P)^\perp, \\ div_y(\sqrt{a}p) = 0 \text{ and } \nabla_y(q/\sqrt{\rho}) = 0, \end{cases} \tag{102}$$

and the expression (51) of $\mathcal{B}$ allow to rewrite the integrals of (101) as

$$\begin{aligned} \int_{I\times\Omega} f\cdot\Phi + \Big(\int_Y p\cdot\big(\partial_t\varphi - \sqrt{a}(1-P)\nabla_x\Phi\big)\, dy\Big) + \frac{q}{\sqrt{\rho}}\cdot\Big(\widehat{\rho}\partial_t\Phi - div_x(\int_Y \sqrt{a}\varphi\, dy)\Big)\, dxdt \\ + \int_\Omega \nabla u_0\cdot(\int_Y \sqrt{a}\varphi\, dy)(t=0)\, dx + \widehat{\rho}\int_\Omega v_0\cdot\Phi(t=0)\, dx \\ + \int_{I\times\Gamma_D} \partial_t g\, n_\Omega\cdot(\int_Y \sqrt{a}\varphi\, dy)\, d\sigma dt + \int_{I\times\Gamma_N} h\cdot\Phi = 0\, d\sigma dt. \end{aligned} \tag{103}$$

Now, if we sum up the requirements (on φ) caused by the assumption $\psi = (1-\Pi^0)\psi \in \mathcal{W}_0$, we see that (103) holds for any regular φ compactly supported in I satisfying (102) and $(\sqrt{a})\varphi = 0$ on $I\times\Lambda\times\Gamma_N\times Y$. Consequently, (103) extends by density and continuity to any regular φ compactly supported in I satisfying (102) and $n_\Omega.\int_Y \sqrt{a}\varphi\, dy = 0$ on $I\times\Lambda\times\Gamma_N$. In particular, $\varphi := -\sqrt{a}(1-P)\nabla_x \int_t^T \Phi$ becomes admissible in (103) for any regular $\Phi = \Phi(t,x)$ compactly supported in I satisfying the mixed homogenized boundary conditions

$$\Phi = 0 \text{ on } I\times\Gamma_D \text{ and } n_\Omega.\widehat{a}\nabla_x\Phi = 0 \text{ on } I\times\Gamma_N, \tag{104}$$

with $\widehat{a}$ defined in (39). As a conclusion, (103) with such a φ and with $u := u_0 + \int_0^t q/\sqrt{\rho}$ reads

$$\begin{aligned} \int_{I\times\Omega} f\cdot\Phi + \widehat{\rho}\partial_t u\cdot\partial_t\Phi + \partial_t u\cdot div_x(\widehat{a}\nabla_x \int_t^T \Phi)\, dxdt \\ - \int_{I\times\Omega} \nabla u_0\cdot\widehat{a}\nabla_x\Phi\, dxdt + \widehat{\rho}\int_\Omega v_0\cdot\Phi(t=0)\, dx \\ - \int_{I\times\Gamma_D} \partial_t g\, n_\Omega\cdot\widehat{a}\nabla_x(\int_t^T \Phi)\, d\sigma dt + \int_{I\times\Gamma_N} h\cdot\Phi\, d\sigma dt = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} \int_{I\times\Omega} f\cdot\Phi - u\cdot\Big(\widehat{\rho}\partial^2_{tt}\Phi - div_x(\widehat{a}\nabla_x\Phi)\Big)\, dxdt \\ -\widehat{\rho}\int_\Omega u_0\cdot\partial_t\Phi(t=0)\, dx + \widehat{\rho}\int_\Omega v_0\cdot\Phi(t=0)\, dx - \int_{I\times\partial\Omega} u_0\cdot(\widehat{a}\nabla_x\Phi.n_\Omega)\, d\sigma dt \\ + \int_{I\times\Gamma_D} g(t=0)\cdot(\widehat{a}\nabla_x\Phi.n_\Omega)\, d\sigma dt - \int_{I\times\Gamma_D} g\cdot(\widehat{a}\nabla_x\Phi.n_\Omega)\, d\sigma dt + \int_{I\times\Gamma_N} h\cdot\Phi\, d\sigma dt = 0. \end{aligned}$$

Through a *formal* disintegration by parts, this weak formulation for test functions Φ satisfying (104) turns out to be an *interpretation* of problem (46), provided that g be defined in a compatible way to ensure that $g(t=0)=u_0$ on $I\times\Gamma_D$. This is obviously the case for

$$g := u_0 + \int_0^t \lim_{\varepsilon} \partial_t g^{\varepsilon}, \tag{105}$$

where the derivative $\partial_t g = \lim_\varepsilon \partial_t g^\varepsilon$ has been introduced in (41).

7.10 Derivation of the multi-fibered model

We deduce here Theorem 22 from Theorem 24:

Proof. To derive the multi-fibered model (55) from the mono-fibered case studied so far, we just make use of the k-quasiperiodic extension from Y to Y_K given by ϖ_k, see Section 5. In view of (17), the variational formulation obtained in (57) for the one-fibered problem set on Y reads equivalently on Y_K as

$$\begin{aligned}&\int_{I\times\Lambda\times\Omega\times Y_K} \varpi_k F^k\cdot\psi_k + \varpi_k U^k\cdot(\partial_t - \mathcal{A}_k)\psi_k \; dydxd\tau dt \\ &+\int_{\Omega\times Y_K} \varpi_k U_0^k\cdot\psi_k(t=0,\tau=0)\; dydx + \int_{I\times\Lambda\times\partial\Omega\times Y_K} \varpi_k G^k\cdot(1-\widetilde{\Pi}^k)\psi_k \; dyd\sigma d\tau dt = 0.\end{aligned} \tag{106}$$

for all admissible k-quasiperiodic test function ψ_k defined on Y_K. We now let k vary over the finite subset L_K^*. As explained in the orthogonal Bloch-wave decomposition of Theorem 7, the whole set of equations can then be encoded into a single sum over k by applying (106) to the k-quasiperiodic component ψ_k of an arbitrary given $\psi = \sum_{k\in L_K^*}\psi_k$ such that

$$\begin{cases} \psi \in \mathcal{C}^\infty(\overline{I\times\Lambda\times\Omega\times\mathbb{R}^N})^{N+1} \text{ is } Y_K\text{-periodic},\\ (\partial_\tau - 2i\pi\widetilde{\Pi}_s)\psi = 0,\\ \psi(t,\tau,.,y)\in D \text{ for all } t,\tau,y,\\ \psi(.,\tau,x,y) \text{ has compact support in } I \text{ for all } \tau,x,y,\end{cases} \tag{107}$$

where $\widetilde{\Pi}_s := \sum_{n\in\mathbb{M}^k} s_n\widetilde{\Pi}_n^k$ is defined by analogy with (58). More specifically for such a ψ, the orthogonal decomposition of Theorem 7 exhibits (55) as the result of the summation of (106) over $k\in L_K^*$, given the definition of U in (48), of F and U_0 in (49) and of $\mathcal{A}$ in (52). For instance

$$\int_{Y_K} \varpi_k F^k\cdot\psi_k\; dy = \int_{Y_K} F\cdot\psi\; dy.$$

To conclude, it remains to notice that the set of test functions defined by (107) is dense in the set $\mathcal{W}$ used in (55), and that each term of (55) extends by continuity from ψ satisfying (107) to $\psi\in\mathcal{W}$.
Note also that the comparatively simpler derivation of the multi-fibered microscopic equations (54) from the mono-fibered case (58) studied in Subsection 7.3 follows the same lines. ■

8 Approximation in the strong sense

In this section we present an a posteriori argument, which motivates the use of our wave two-scale transform W^ε to provide explicit asymptotic developments of the physical solution U^ε.

Decomposing any weak limit U of $W^\varepsilon U^\varepsilon$ as in Theorem 19, we consider the related formal expression

$$U_\varepsilon := U_H(t,x,\frac{x}{\varepsilon}) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{M}^k} e^{2i\pi s_n t/\varepsilon\alpha_n^k} U_n^k(t,x) e_n^k(\frac{x}{\varepsilon}),$$

through the usual substitution $y = x/\varepsilon$ and the less classical one $\tau = t/\varepsilon\alpha_n^k$ (wave-wise). It turns out that this expression makes sense under mild technical restrictions, for instance

$$U_H \in L^2(I\times\Omega;\mathcal{C}_\sharp^0(Y))^{N+1} \text{ and } \sum_{n\in\mathbb{M}^k} |n|\ ||U_n^k||_{L^2(I\times\Omega)} < \infty. \tag{108}$$

This kind of additional regularity on $U \in L^2(I\times\Lambda\times\Omega\times Y_K)$ is unavoidable for the substitution, as shown by the counter-examples of [1] Section 5. Moreover, the definitions (29) of S_k^ε and (36) of W_k^ε involving a subdomain $\Omega_\varepsilon \subset \Omega$ made of εY-cells (see Section 6) show that our analysis tends to ignore the possibly chaotic behavior of U^ε in the ε-vicinity of $\partial\Omega$ when $\partial\Omega$ is not flat. We face the same problem in time since I cannot be simultaneously decomposed into an exact number of $\varepsilon\alpha_n^k$-cells for all n. Putting aside these technicalities, we can now state that U_ε provides a good approximation of U^ε if and only if the convergence of $W^\varepsilon U^\varepsilon$ is strong in $L^2(I\times\Lambda\times\Omega\times Y_K)$:

Theorem 39 *Assuming (108) and* $\lim_{t\to T^-}\limsup_{\varepsilon\to 0} ||U^\varepsilon||_{L^2((t,T)\times\Omega)} = 0$, *we have*

$$||U^\varepsilon - U_\varepsilon||_{L^2(I\times\Omega_\varepsilon)} - ||W^\varepsilon U^\varepsilon - U||_{L^2(I\times\Lambda\times\Omega\times Y_K)} \to 0.$$

Proof. For the sake of simplicity, we shall content ourselves with the case $L_K^* = \{0\}$ of only one fiber $k=0$. Let us fix $\epsilon > 0$ arbitrarily small (not to be confused with $\varepsilon \to 0$). Introducing V_H and V_n^k regular approximations of U_H and U_n^k in the sense

$$||U_H - V_H||_{L^2(I\times\Omega;L^\infty(Y))} \le \epsilon \text{ and } \sum_{n\in\mathbb{M}^k} |n|\ ||U_n^k - V_n^k||_{L^2(I\times\Omega)} \le \epsilon,$$

we first check that

$$R_\varepsilon := (U_H - V_H)(t,x,\frac{x}{\varepsilon}) + \sum e^{2i\pi s_n t/\varepsilon\alpha_n^k}(U_n^k - V_n^k)(t,x)e_n^k(\frac{x}{\varepsilon})$$

satisfies $||R_\varepsilon||_{L^2(I\times\Omega)} \le C\epsilon$ thanks to the uniform spectral estimates $|e_n^k| \le C|n|$ of Lemma 12. We then split $||U^\varepsilon - U_\varepsilon||_{L^2(I\times\Omega_\varepsilon)} - ||W_k^\varepsilon U^\varepsilon - U_k||_{L^2(I\times\Lambda\times\Omega\times Y)} = \alpha+\beta+\gamma$ with

$$\left\{\begin{array}{l} \alpha := ||U^\varepsilon - U_\varepsilon||_{L^2(I\times\Omega_\varepsilon)} - ||U^\varepsilon - U_\varepsilon + R_\varepsilon||_{L^2(I\times\Omega_\varepsilon)},\\ \beta := ||S_k^\varepsilon(U^\varepsilon - U_\varepsilon + R_\varepsilon)||_{L^2(I\times\Omega\times Y)} - ||(1-\Pi^k)(W_k^\varepsilon U^\varepsilon - U_k) + \sum\sigma_n^\varepsilon||_{L^2(I\times\Omega\times Y)},\\ \gamma := ||(1-\Pi^k)(W_k^\varepsilon U^\varepsilon - U_k) + \sum\sigma_n^\varepsilon||_{L^2(I\times\Omega\times Y)} - ||W_k^\varepsilon U^\varepsilon - U_k||_{L^2(I\times\Lambda\times\Omega\times Y)}, \end{array}\right.$$

and $\sigma_n^\varepsilon := \Pi_n^k S_k^\varepsilon U^\varepsilon(t,x,y) - e^{2i\pi t/\varepsilon\alpha_n^k}U_n^k(t,x)e_n^k(y)$. Obviously,

$$S_k^\varepsilon(U^\varepsilon - U_\varepsilon + R_\varepsilon) - (1-\Pi^k)(W_k^\varepsilon U^\varepsilon - U_k) - \sum\sigma_n^\varepsilon = R_H + R_H^\varepsilon + R_\Pi + R_\Pi^\varepsilon$$

remains small since

$$\left\{\begin{array}{l} R_H := (U_H - V_H)(t,x,y) \text{ satisfies } ||R_H||_{L^2(I\times\Omega\times Y)} \le \epsilon,\\ R_H^\varepsilon := (V_H - S_k^\varepsilon[V_H(t,x,\frac{x}{\varepsilon})])(t,x,y) \to 0 \text{ in } L^2(I\times\Omega\times Y),\\ R_\Pi := \sum e^{2i\pi s_n t/\varepsilon\alpha_n^k}(U_n^k - V_n^k)(t,x)e_n^k(y) \text{ satisfies } ||R_\Pi||_{L^2(I\times\Omega\times Y)} \le \epsilon,\\ R_\Pi^\varepsilon := \sum e^{2i\pi s_n t/\varepsilon\alpha_n^k}(V_n^k - S_k^\varepsilon[V_n^k])(t,x,y)e_n^k(y) \to 0 \text{ in } L^2(I\times\Omega\times Y). \end{array}\right.$$

At this stage, we can focus on γ only, since $\alpha+\beta \leq C\epsilon$ for ε small. Using orthogonality relations in $L^2(Y)$ and isometries in time ($T^{\varepsilon\alpha_n^k}$), we get

$$
\begin{aligned}
(\gamma + ||W_k^\varepsilon U^\varepsilon - U_k||_{L^2(I\times\Lambda\times\Omega\times Y)})^2 &= ||(1-\Pi^k)(W_k^\varepsilon U^\varepsilon - U_k)||^2_{L^2(I\times\Omega\times Y)} \\
+\sum ||\sigma_n^\varepsilon||^2_{L^2(I-I_n^\varepsilon\times\Omega\times Y)} &+ ||\sum T^{\varepsilon\alpha_n^k}\sigma_n^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega\times Y)} \\
&= ||(1-\Pi^k)(W_k^\varepsilon U^\varepsilon - U_k)||^2_{L^2(I\times\Omega\times Y)} \\
+||\Pi^k(W_k^\varepsilon U^\varepsilon - U_k) + r^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega\times Y)} &+ \sum ||\sigma_n^\varepsilon||^2_{L^2(I-I_n^\varepsilon\times\Omega\times Y)} \\
&\leq ||W_k^\varepsilon U^\varepsilon - U||^2_{L^2(I\times\Lambda\times\Omega\times Y)} + ||r^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega\times Y)} \\
+2||r^\varepsilon||_{L^2(I\times\Lambda\times\Omega\times Y)}||W_k^\varepsilon U^\varepsilon - U_k||_{L^2(I\times\Lambda\times\Omega\times Y)} &+ \sum ||\sigma_n^\varepsilon||^2_{L^2(I-I^\varepsilon\times\Omega\times Y)},
\end{aligned}
$$

where $r^\varepsilon := \sum e^{2i\pi s_n\tau}(U_n^k - T^{\varepsilon\alpha_n^k}U_n^k)(t,\tau,x)e_n^k(y) \to 0$ in $L^2(I\times\Lambda\times\Omega\times Y)$ and where

$$
I_n^\varepsilon := (0, \varepsilon\alpha_{|n|}^k[T/\varepsilon\alpha_{|n|}^k]) \supset I^\varepsilon := (0, T - \varepsilon \sup_{n\in\mathbb{M}_+^k} \alpha_n^k) \text{ for all } n.
$$

As a consequence, we recover for ε small that

$$
\begin{aligned}
\gamma^2 &\leq ||r^\varepsilon||^2_{L^2(I\times\Lambda\times\Omega\times Y)} + 2||r^\varepsilon||_{L^2(I\times\Lambda\times\Omega\times Y)}||W_k^\varepsilon U^\varepsilon - U_k||_{L^2(I\times\Lambda\times\Omega\times Y)} \\
&+\sum ||\sigma_n^\varepsilon||^2_{L^2(I-I^\varepsilon\times\Omega\times Y)} \leq \epsilon^2 + \sum ||\sigma_n^\varepsilon||^2_{L^2(I-I^\varepsilon\times\Omega\times Y)} \\
&\leq \epsilon^2 + 2\sum ||\Pi_n^k S_k^\varepsilon U^\varepsilon||^2_{L^2(I-I^\varepsilon\times\Omega\times Y)} + 2\sum ||U_n^k||^2_{L^2(I-I^\varepsilon\times\Omega)} \leq 2\epsilon^2 + 2||U^\varepsilon||^2_{L^2(I-I^\varepsilon\times\Omega\times Y)}.
\end{aligned}
$$

Finally, we can make $\gamma \leq 2\epsilon$ as small as we wish, provided that the energy of the physical solution does not concentrate at the end point T of I as assumed. ■

Our last result deals specifically with strong convergences in the norm sense.

Theorem 40 *Suppose $\Omega = \mathbb{R}^N$. If the initial data and the source term two-scale converge strongly in the sense*

$$
\begin{aligned}
\sum_{k\in L_K^*} \varpi_k S_k^\varepsilon U_0^\varepsilon &\to U_0 \text{ in } L^2(\Omega\times Y_K)^{N+1} \text{ strong}, \\
W^\varepsilon F^\varepsilon &\to F \text{ in } L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1} \text{ strong},
\end{aligned}
$$

then the corresponding solution two-scale converges strongly too

$$
W^\varepsilon U^\varepsilon \to U \text{ in } L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1} \text{ strong.}
$$

Proof. We shall only sketch the argument. By Proposition 36, the uniqueness property for the final model (54)-(55) solved by U guarantees that the whole sequence $W^\varepsilon U^\varepsilon$ weakly converges, as ε goes to zero unrestrictedly. As a consequence, the strong convergence of $W^\varepsilon U^\varepsilon$ in L^2 is equivalent to the conservation of the norm:

$$
\limsup_{\varepsilon\to 0} ||W^\varepsilon U^\varepsilon||_{L^2(I\times\Lambda\times\Omega\times Y_K)} \leq ||U||_{L^2(I\times\Lambda\times\Omega\times Y_K)}.
$$

But this results from the contraction property $||W^\varepsilon U^\varepsilon||_{L^2(I\times\Lambda\times\Omega\times Y_K)} \leq ||U^\varepsilon||_{L^2(I\times\Omega)}$ and from the study of the time-behavior of the physical solution:

$$
\begin{aligned}
||U^\varepsilon||^2_{L^2(I\times\Omega)} &= T||W^\varepsilon U^\varepsilon(t=0,\tau=0)||^2_{L^2(\Omega\times Y_K)} \\
+2\,\mathrm{Re}\int_{I\times\Lambda\times\Omega\times Y_K} &(T-t)W^\varepsilon F^\varepsilon \cdot W^\varepsilon U^\varepsilon \, dydxd\tau dt + r^\varepsilon, \qquad (109)
\end{aligned}
$$

where $r^\varepsilon \to 0$ is a technical remainder. Indeed, passing to the limit in (109) thanks to the assumed two-scale convergences in the strong sense, we get

$$\limsup_{\varepsilon\to 0} ||U^\varepsilon||^2_{L^2(I\times\Omega)} \leq T||U_0||^2_{L^2(\Omega\times Y_K)} + 2\,\mathrm{Re}\int_{I\times\Lambda\times\Omega\times Y_K} (T-t)F\cdot U \; dydxd\tau dt,$$

where the righthand side is precisely equal to $||U||^2_{L^2(I\times\Lambda\times\Omega\times Y_K)}$, as easily shown from the model (54)-(55) solved by U and as mentionned in (4) in the introduction. ∎

Typical applications of Theorem 40 are offered by situations where the data are suitably prepared, for instance $f^\varepsilon \to 0$ in $L^2(I\times\Omega)$ and

$$\begin{cases} u_0^\varepsilon(x) = \varepsilon\alpha(x, x/\varepsilon) + r^\varepsilon(x), \\ v_0^\varepsilon(x) = \beta(x, x/\varepsilon) + s^\varepsilon(x), \end{cases}$$

with $r^\varepsilon \to 0$ in $H^1(\Omega)$ and $s^\varepsilon \to 0$ in $L^2(\Omega)$. Here $\alpha(x,y)$ and $\beta(x,y)$ are allowed to vary in the vector subspace of $C^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ spanned by the finite sums over k of all regular k-quasiperiodic functions in y with compact support in x, provided that K be chosen large enough in view of the number of k's involved in α and β.

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