# A two-scale model for the wave equation with oscillating coefficients and data

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**Abstract** We introduce a time-space two-scale transform designed to capture the high and low frequency waves in the asymptotics of the periodic homogenization of the wave equation. The asymptotical solution is the sum of the solution of known homogenized equations and of Bloch waves. We also derive the transport equations satisfied by the Bloch wave coefficients.

**Résumé** Nous introduisons une transformation à deux échelles en espace et en temps destinée à capturer à la fois les basses fréquences et les ondes de Bloch qui apparaissent lors du processus asymptotique d'homogénéisation de l'équation des ondes à coefficients périodiques. La solution du modèle qui en résulte comprend les ondes de Bloch et une contribution basse fréquence qui est la solution du modèle homogénéisé de l'équation des ondes. On établit aussi les équations de transport vérifiées par les coefficients des ondes de Bloch.

**Mots clés** Homogénéisation, Equation des ondes, Ondes de Bloch, Convergence à deux échelles
**Key words** Homogenization, Wave equation, Bloch waves, Two-scale convergence

## 1 The wave equation as a first order system

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with bounded Lipschitz boundary $\partial\Omega$, and let $I = [0, T) \subset \mathbb{R}^+$ be a finite time interval. We fix a splitting of $\partial\Omega$ into two disjoints parts $\Gamma_D$ and $\Gamma_N$ where Dirichlet and Neuman boundary conditions are applied. We consider $u^\varepsilon(t, x)$ solution to a linear scalar wave equation with periodic coefficients and oscillating data,

$$\begin{gathered}\rho^\varepsilon \partial^2_{tt} u^\varepsilon - div(a^\varepsilon \nabla u^\varepsilon) = f^\varepsilon \text{ in } I \times \Omega,\\ u^\varepsilon(t = 0) = u^\varepsilon_0,\ \partial_t u^\varepsilon(t = 0) = v^\varepsilon_0 \text{ in } \Omega,\\ u^\varepsilon = g^\varepsilon \text{ on } I \times \Gamma_D,\ a^\varepsilon \nabla u^\varepsilon.n_\Omega = h^\varepsilon \text{ on } I \times \Gamma_N.\end{gathered} \tag{1}$$

As usual in periodic homogenization theory, $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$, $\rho^\varepsilon(x) = \rho(\frac{x}{\varepsilon})$, where $a(y)$ is a symmetric matrix and $\rho(y)$ is real-valued, both being lipschitzian and $\mathbb{Z}^N$-periodic on $\mathbb{R}^N$. Moreover, we require the standard uniform positivity and ellipticity conditions to hold i.e. $0 < \rho_0 \le \rho(y) \le \rho_1 < \infty$ and $0 < a_0 \le a(y) \le a_1 < \infty$. Setting

$$U^\varepsilon = \begin{pmatrix} \sqrt{a^\varepsilon}\nabla u^\varepsilon \\ \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon \end{pmatrix},\ F^\varepsilon = \begin{pmatrix} 0 \\ f^\varepsilon/\sqrt{\rho^\varepsilon} \end{pmatrix},\ U^\varepsilon_0 = \begin{pmatrix} \sqrt{a^\varepsilon}\nabla u^\varepsilon_0 \\ \sqrt{\rho^\varepsilon} v^\varepsilon_0 \end{pmatrix},$$

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$$G^\varepsilon = \left( \begin{array}{c} \mathbb{1}_{\Gamma_D} \partial_t g^\varepsilon \sqrt{a^\varepsilon} n_\Omega \\ \mathbb{1}_{\Gamma_N} h^\varepsilon / \sqrt{\rho^\varepsilon} \end{array} \right),\ A^\varepsilon = \left( \begin{array}{cc} 0 & \sqrt{a^\varepsilon}\nabla(\frac{1}{\sqrt{\rho^\varepsilon}}.) \\ \frac{1}{\sqrt{\rho^\varepsilon}} div(\sqrt{a^\varepsilon}.) & 0 \end{array} \right),\ n_A^\varepsilon = \frac{1}{\sqrt{\rho^\varepsilon}} \left( \begin{array}{cc} 0 & \sqrt{a^\varepsilon} n_\Omega \\ \sqrt{a^\varepsilon} n_\Omega & 0 \end{array} \right),$$

we recast the scalar wave equation (1) as a first order system of size $N+1$,

$$\begin{gathered} (\partial_t - A^\varepsilon) U^\varepsilon = F^\varepsilon \text{ in } I \times \Omega, \\ U^\varepsilon(t=0) = U_0^\varepsilon \text{ in } \Omega, \\ (n_A^\varepsilon U^\varepsilon - G^\varepsilon)_{1,..,N} = 0 \text{ on } I \times \Gamma_N \text{ and } (n_A^\varepsilon U^\varepsilon - G^\varepsilon)_{N+1} = 0 \text{ on } I \times \Gamma_D, \end{gathered}$$

which may be understood in the weak sense

$$\int_{I\times\Omega} (F^\varepsilon.\psi + U^\varepsilon.(\partial_t - A^\varepsilon)\psi)\ dtdx + \int_\Omega U_0^\varepsilon.\psi(t=0)\ dx + \int_{I\times\partial\Omega} G^\varepsilon.\psi\ dtds(x) = 0 \tag{2}$$

for all admissible functions $\psi \in H^1(I\times\Omega)^{N+1}$ such that $\psi(t,.) \in D(A^\varepsilon) = \{(\varphi, \phi) \in L^2(\Omega)^N \times L^2(\Omega) \mid \sqrt{a^\varepsilon}\varphi \in H^{div}(\Omega),\ \phi/\sqrt{\rho^\varepsilon} \in H^1(\Omega),\ \sqrt{a^\varepsilon}\varphi.n_\Omega = 0 \text{ on } \Gamma_N \text{ and } \phi/\sqrt{\rho^\varepsilon} = 0 \text{ on } \Gamma_D\}$ a.e. in $t \in I$ and $\psi(T,.) = 0$.

**Theorem 1** *For any fixed $\varepsilon$, the weak formulation (2) has a unique solution $U^\varepsilon \in L^2(I\times\Omega)^{N+1}$ for any $U_0^\varepsilon \in L^2(\Omega)^{N+1}$, $F^\varepsilon \in L^2(I\times\Omega)^{N+1}$, $\partial_t g^\varepsilon \in H^1(I; H^{1/2}(\Gamma_D))$, $h^\varepsilon \in H^1(I; H^{-1/2}(\Gamma_N))$. Moreover, $U^\varepsilon$ satisfies the estimate*

$$||U^\varepsilon||_{L^2(I\times\Omega)} \le C(||F^\varepsilon||_{L^2(I\times\Omega)} + ||U_0^\varepsilon||_{L^2(\Omega)} + ||\partial_t g^\varepsilon||_{H^1(I;H^{1/2}(\Gamma_D))} + ||h^\varepsilon||_{H^1(I;H^{-1/2}(\Gamma_N))})$$

*uniformly in $\varepsilon$.*

In the sequel, we assume that the data are bounded as

$$||F^\varepsilon||_{L^2(I\times\Omega)} + ||U_0^\varepsilon||_{L^2(\Omega)} + ||\partial_t g^\varepsilon||_{H^1(I;H^{1/2}(\Gamma_D))} + ||h^\varepsilon||_{H^1(I;H^{-1/2}(\Gamma_N))} \le C, \tag{3}$$

so that the solution $U^\varepsilon$ is also bounded in $L^2(I\times\Omega)^{N+1}$.

# 2 The wave two-scale transform

Let $Y = Y^* = (0,1)^N$ be a unit cell of the $N$-dimensional lattice $L = L^* = \mathbb{Z}^N$. Given $K \in \mathbb{N}^*$, we observe that the dual lattices $KL$ and $\frac{L^*}{K}$ satisfy $L = L_K + KL$ and $\frac{L^*}{K} = L^* + L_K^*$ for some fundamental subsets $L_K \subset L$ and $L_K^* \subset Y^*$ of cardinal $K^N$, such that $L_K \cap (KL) = \{0\}$ and $L_K^* \cap L^* = \{0\}$. Also, we introduce a set $Y_K$ made of $K^N$ cells indexed by $L_K$ and translated from $Y$, such that $Y_K$ tends to cover $\mathbb{R}^N$ when $K$ increases.

Now, for any $k$ in $Y^*$, we define the functional space $L_k^2 = \{u \in L_{loc}^2(\mathbb{R}^N) \mid u(x+\ell) = u(x)e^{2i\pi k.\ell}$ a.e. for all $\ell \in L\}$ of $k$-quasiperiodic functions, and for any $s \ge 0$ we denote by $H_k^s(Y)$ the Sobolev space of functions on $Y$ whose $k$-quasiperiodic extension in $L_k^2$ belongs to $H_{loc}^s(\mathbb{R}^N)$. As in [2], the set $L_0^2(Y_K)$ of all $KL$-periodic $L_{loc}^2(\mathbb{R}^N)$-functions can be described as the hilbertian sum of $K^N$ subspaces $L_0^2(Y_K) = \bigoplus_{k\in L_K^*}^\perp L_k^2$, the norm in $L^2(Y_K)$ being chosen as $v \mapsto (\frac{1}{|Y_K|}\int_{Y_K} |v|^2\ dy)^{1/2}$.

Next, for any $k \in Y^*$, we introduce the elliptic operators $\Delta_k = \frac{1}{\sqrt{\rho}} div_y(a\nabla_y \frac{1}{\sqrt{\rho}}.)$ associated with the wave equation on a microscopic scale, with domains $D(\Delta_k) = \{\phi \in L^2(Y) \mid \phi/\sqrt{\rho} \in H_k^2(Y)\}$.

These non-negative self-adjoint operators with compact resolvent on $L^2(Y)$ govern the high frequency spectral analysis of the problem. Each $-\Delta_k$ is reduced by a spectral hilbertian basis $(\phi_n^k)$ of $L^2(Y)$ such that

$$\phi_n^k \in H_k^2(Y) \text{ and } -\Delta_k\phi_n^k = \lambda_n^k\phi_n^k,$$

the sequence of repeated eigenvalues $(\lambda_n^k)$ being non-negative and non-decreasing. The kernel of $-\Delta_k$ is null for $k \notin L^*$ and one-dimensional (generated by $\phi_1^0$) otherwise. We will enumerate the spectral elements $(\phi_n^k)$ by $n \in \mathbb{M}_k^+$ where $\mathbb{M}_k^+ = \mathbb{N}^*$ for $k \notin L^*$ and $\mathbb{M}_k^+ = \mathbb{N}^* - \{1\}$ otherwise, so that in either case $\phi_n^k \notin Ker(A_k)$ if $n \in \mathbb{M}_k^+$. We extend these sets by symmetry, i.e. $\mathbb{M}_k = \mathbb{Z}^*$ for $k \notin L^*$ and $\mathbb{M}_k = \mathbb{Z}^* - \{-1, 1\}$ otherwise. The eigenvectors (kernel excepted) of the microscopic scale operator

$$A = \begin{pmatrix} 0 & \sqrt{a}\nabla_y(\frac{1}{\sqrt{\rho}}.) \\ \frac{1}{\sqrt{\rho}}div_y(\sqrt{a}.) & 0 \end{pmatrix} \text{ are } e_n^k = \frac{1}{\sqrt{2}}\begin{pmatrix} -is_n\frac{\sqrt{a}}{\sqrt{\lambda_{|n|}^k}}\nabla_y(\phi_{|n|}^k/\sqrt{\rho}) \\ \phi_{|n|}^k \end{pmatrix}$$

for $n$ in $\mathbb{M}_k$ and $s_n$ denoting the sign of $n$. The orthogonal projectors in $L^2(Y)^{N+1}$ onto $span(e_n^k)$ and onto $span(\{e_n^k\}_{n\in\mathbb{M}_k})$ are denoted by $\Pi_n^k$ and $\Pi^k$. Viewed as a family of quasi-periodic functions, $\{e_n^k \mid k \in L_K^*,\, n \in \mathbb{M}_k\}$ constitutes a hilbertian basis of $L^2(Y_K)^{N+1}$.

Let us introduce our space two-scale transform. We first split the physical domain $\Omega$ into a number of $\varepsilon Y$-cells $\omega$ up to a small left-over region $\Omega - \Omega_\varepsilon$ near the boundary $\partial\Omega$ by setting $\Omega_\varepsilon = \bigcup \mathcal{C}_\varepsilon$, where $\mathcal{C}_\varepsilon = \{\varepsilon\ell + \varepsilon Y \mid \ell \in L,\, \varepsilon\ell + \varepsilon Y \subset \Omega\}$ is the set of all cells fully contained in $\Omega$. For any $k \in Y^*$, we then define the modulated space two-scale transform $S_k^\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y)$ by

$$S_k^\varepsilon u(x,y) = \sum_{\omega\in\mathcal{C}_\varepsilon} u(\varepsilon\ell_\omega + \varepsilon y)e^{-2i\pi k.\ell_\omega}\mathbb{1}_\omega(x),$$

where $\varepsilon\ell_\omega \in \varepsilon L$ stands for the unique node of $\omega$. Likewise we introduce a time two-scale transform. Taking $\mathbb{Z} \subset \mathbb{R}$ as a canonical lattice and $\Lambda = [0,1)$ as a canonical unit cell, we set $I_\varepsilon = \bigcup \mathcal{C}_\varepsilon^+$, where $\mathcal{C}_\varepsilon^+ = \{\varepsilon\ell + \varepsilon\Lambda \mid \ell \in \mathbb{Z},\, \varepsilon\ell + \varepsilon\Lambda \subset I\}$ is the family of all $\varepsilon\Lambda$-cells contained in $I$, and we define our time two-scale transform $T^\varepsilon : L^2(I) \to L^2(I \times \Lambda)$ by

$$T^\varepsilon u(t,\tau) = \sum_{\theta\in\mathcal{C}_\varepsilon^+} u(\varepsilon\ell_\theta + \varepsilon\tau)\mathbb{1}_\theta(t),$$

where $\varepsilon\ell_\theta \in \varepsilon\mathbb{Z}$ stands for the left end point of $\theta$. Finally, we combine the space two-scale transform and the spectral decomposition of $L^2(Y)^{N+1}$ together with the (parameterized) time two-scale transform, to define our one-fibered wave two-scale transform $W_k^\varepsilon : L^2(I \times \Omega)^{N+1} \to L^2(I \times \Lambda \times \Omega \times Y)^{N+1}$ by

$$W_k^\varepsilon = \mathbb{1}_{L^*}(k)(1 - \Pi^0)S_0^\varepsilon + \sum_{n\in\mathbb{M}_k} T^{2\pi\varepsilon/\sqrt{\lambda_{|n|}^k}}\Pi_n^k S_k^\varepsilon.$$

Extending by quasi-periodicity the images of each $W_k^\varepsilon$ from $L^2(Y)$ to $L^2(Y_K)$ yields our multi-fibered wave two-scale transform $W^\varepsilon = \sum_{k\in L_K^*} W_k^\varepsilon$.

**Lemma 2** *The transforms $W_k^\varepsilon$ and $W^\varepsilon$ are contractions, in the sense*

$$||W_k^\varepsilon U||_{L^2(I\times\Lambda\times\Omega\times Y)}^2 \leq ||U||_{L^2(I\times\Omega)}^2 \text{ and } ||W^\varepsilon U||_{L^2(I\times\Lambda\times\Omega\times Y_K)}^2 \leq ||U||_{L^2(I\times\Omega)}^2.$$

A straightforward consequence is that $W^\varepsilon U^\varepsilon \in L^2(I \times \Lambda \times \Omega \times Y_K)^{N+1}$ has limit points $U_K$ in the weak convergence of $L^2(I \times \Lambda \times \Omega \times Y_K)^{N+1}$ as $\varepsilon$ tends towards zero, because $U^\varepsilon \in L^2(I \times \Omega)^{N+1}$ remains uniformly bounded in $\varepsilon$.

## 3 The two scale model for waves

Consider the macroscopic field $u^0$ solution to the homogenized scalar problem of [3],

$$\rho^0\partial^2_{tt}u^0 - div_x(a^0\nabla_x u^0) = f^0 \text{ in } I\times\Omega,$$
$$u^0(t=0) = u_0 \text{ and } \partial_t u^0(t=0) = v_0 \text{ in } \Omega,$$
$$u^0 = g \text{ on } I\times\Gamma_D,\ a^0\nabla_x u^0.n_\Omega = h \text{ on } I\times\Gamma_N,$$

with $\rho^0 = \int_Y \rho\ dy$, $f^0 = \lim_\varepsilon \int_Y S_0^\varepsilon f^\varepsilon\ dy$ weakly in $L^2(\Omega)$, $u_0 = \lim_\varepsilon u_0^\varepsilon$, $v_0 = \lim_\varepsilon \int_Y \rho S_0^\varepsilon v_0^\varepsilon\ dy/\int_Y \rho\ dy$ weakly in $L^2(\Omega)$, $g = \lim_\varepsilon g^\varepsilon$ in $L^2(\Gamma_D)$, $h = \lim_\varepsilon h^\varepsilon$ in $L^2(\Gamma_N)$ and with the usual definition of the homogenized matrix $a^0$. We set $q^0 = \sqrt{\rho}\partial_t u^0$ and $p^0 = \sqrt{a}(\nabla_x u^0 + \nabla_y u^1)$, where $u^1$ is the usual corrector in the homogenization method, so that $\nabla_y u^1$ is uniquely defined from $\nabla_x u^0$. For each $k$ and $n$, we denote by $U_n^k \in L^2(I\times\Omega)$ the amplitude of the Bloch wave $e_n^k(y)e^{2i\pi s_n\tau}$. It is a (non unique) solution to the first order system

$$\partial_t U_n^k - s_n \sum_{m\in\mathbb{M}_k,\ \lambda_m^k=\lambda_n^k,\ s_n=s_m} \kappa_{nm}.\nabla_x U_m^k = F_n^k \text{ in } I\times\Omega,$$
$$U_n^k(t=0) = U_{0n}^k \text{ in } \Omega,$$

with coefficients, right hand side and initial condition given by

$$\begin{aligned}
\kappa_{nm} &= \frac{i}{2\sqrt{\lambda^k_{|n|}}}\int_Y \frac{\phi^k_{|m|}}{\sqrt{\rho}} a\nabla_y\Big(\frac{\overline{\phi}^k_{|n|}}{\sqrt{\rho}}\Big) - \frac{\overline{\phi}^k_{|n|}}{\sqrt{\rho}} a\nabla_y\Big(\frac{\phi^k_{|m|}}{\sqrt{\rho}}\Big)\ dy \in \mathbb{C}^N,\\
F_n^k &= \lim_\varepsilon \int_\Lambda e^{-2i\pi s_n\tau}\int_Y W_k^\varepsilon F^\varepsilon.\overline{e_n^k} dy d\tau,\ \text{ and } U_{0n}^k = \lim_\varepsilon \int_Y S_k^\varepsilon U_0.\overline{e_n^k} dy,
\end{aligned}$$

the limits being understood respectively in $L^2(I\times\Lambda\times\Omega\times Y)$ and $L^2(I\times\Omega\times Y)$ weak.

**Theorem 3** *Fix $K\in\mathbb{N}^*$. If the data are bounded as in (3) then the sequence $W^\varepsilon U^\varepsilon$ derived from the unique solution $U^\varepsilon \in L^2(I\times\Omega)$ to the weak formulation (2) is uniformly bounded. The limit of any of its weakly converging subsequence has the form*

$$U_K(t,\tau,x,y) = \begin{pmatrix} p^0 \\ q^0 \end{pmatrix}(t,x,y) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{M}_k} U_n^k(t,x)e^{2i\pi s_n\tau}e_n^k(y) \in L^2(I\times\Lambda\times\Omega\times Y_K)^{N+1}.$$

From the two-scale limit $U_K$ we get an approximation of the actual physical field

$$U^\varepsilon(t,x) \approx \begin{pmatrix} p^0 \\ q^0 \end{pmatrix}(t,x,\frac{x}{\varepsilon}) + \sum_{k\in L_K^*}\sum_{n\in\mathbb{M}_k} U_n^k(t,x)e^{is_n\sqrt{\lambda^k_{|n|}}t/\varepsilon}e_n^k(\frac{x}{\varepsilon}) \tag{4}$$

in the sens of Theorem 3.

The proof of the theorem relies on the classical two-scale testing method of [1] applied to derivatives and on the use of our two-scale transform to deal with projections and more general kernel operators. The thrust of our paper is precisely to mix both techniques in order to deal with fully integro-differential equations.

**Remarks** *(i) So far, we have not been able to derive boundary conditions for the Bloch wave coefficients $U_n^k$. However, it is always possible to formulate approximate boundary conditions out of* *approximation (4). In case where we retain only one Bloch wave and its companion propagating in the opposite sense, we find*

$$\partial_t U_n^k - \partial_t U_{-n}^k = 0 \text{ on } I \times \Gamma_D \text{ and } \partial_t U_n^k + \partial_t U_{-n}^k = 0 \text{ on } I \times \Gamma_N.$$

*(ii) Our results are stated for any fixed $K$.* A *formal limit $K \to \infty$ (i.e. $Y_K \to \mathbb{R}^N$) can be performed in the model in order to recover the complete set of Bloch waves.*
*(iii) The idea of this paper originated in [4] and [5].*

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