Abstract

In this paper, we present a two-scale model including an optimal active control for a one-dimensional cantilever array with regularly spaced actuators and sensors. With the purpose of implementing the control in real time, we propose an approximation that may be realized by an analog distributed electronic circuit. More precisely, our analog processor is made by Periodic Network of Resistances (PNR). The control approximation method is based on two general concepts, namely on functions of operators and on the Dunford-Schwartz representation formula. We conducted validations of the control approximation method as well as of its effect in the complete control loop.

1. Introduction

In the past decade, a number of papers have been focused on semi-decentralized distributed optimal control for systems with distributed actuators and sensors. Most of them are dealing with infinite length systems, see [1] and [10] for systems governed by partial differential equations, and [3] for discrete systems. In the papers [4] and [5] the authors have introduced an approximation of an optimal control to a finite length beam endowed with a periodic distribution of piezoelectric sensors and actuators. Even if it was giving satisfactory results, it was suffering from some limitations. In [9] it has been extended so that to cover a larger range of systems and to increase its precision and robustness. Indeed, the new method does not require that each operator of the state equation and of the cost functional be functions of a same operator but they must be only functions of a same operator up to some change of variable operators. Regarding precision, the Taylor series approximating a function of an operator has been replaced by the use of the Dunford-Schwartz representation formula followed by a quadrature rule for the contour integral.

Here we apply our new method to a recently developed and validated two-scale model of cantilever arrays, submitted in the paper [8]. It is rigorously justified thanks to an adaptation of the two-scale approximation method introduced in [6] and detailed in [7]. Its main advantage is that in the same time it requires little computing effort and it is reasonably precise.

This paper presents results from an implementation of the new semi-decentralized optimal control strategy on the two-scale model of cantilever arrays. We provide results regarding precision and cost. However our calculations have been carried out using the simplest optimal control strategy, namely a Linear Quadratic Regulator. As in [5], we also provide a realization of the semi-decentralized control scheme through a Periodic Network of Resistances (PNR), implementing a finite difference scheme for the partial differential operator in the Dunford-Schwartz formula. Finally, we quote that the entire approach can be extended to other linear optimal control problems, i.e. LQG or $H_\infty$ controls as well as to more physical actuating and sensing principles.

2. A Two-Scale Model of Cantilever Arrays

We consider a one-dimensional cantilever array comprised of an elastic base, and a number of clamped elastic cantilevers with free end, see Figure 1. Assuming that the number of cantilevers is sufficiently large, a homogenized model was derived using a two-scale approximation method. This is reported in the detailed paper [7] devoted to static regime. The corresponding model extended to dynamic regime is introduced in the letter [6]. The modelling papers were written in view of Atomic Force Microscopy application.

Fig. 1: Array of Cantilevers

After a number of simplifications, the approximate homogenized model expressed in the two-scale referential, which is a rectangle $\Omega = (0, L_B) \times (0, L_C^*)$. The parameters $L_B$ and $L_C^*$ represent respectively the base length in the macroscale $x-$direction and the scaled cantilever length in the microscale $y-$direction. The base is modelled by the line $\Gamma = \{(x, y) \mid x \in (0, L_B) \text{ and } y = 0\}$, and the rectangle $\Omega$ is filled by an infinite number of cantilevers. We describe the system motion by its bending displacement only. So, the base is governed by an Euler-Bernoulli beam equation with two kinds of distributed forces, one exerted by the attached cantilevers and the other, denoted by $u(t, x, 0)$, originating from an actuator distribution. The bending displacement, the mass per unit length, the bending coefficient and the scaled cantilever width being denoted by $w(t, x, 0)$, $\rho^B$, $R^B$ and $C^*$, the base governing equation states

$$\rho^B \partial_t^2 w + R^B \partial_x^4 w + \partial_x^6 \partial_{yy} w = u \text{ in } \Gamma. \quad (1)$$

The base is assumed to be clamped, so the boundary conditions are

$$w = \partial_x w = 0, \quad (2)$$

at its ends. Each cantilever is oriented in the $y$-direction, and its motion is governed by the Euler-Bernoulli equation
distributed along the $y$-direction. It is subjected to a control force $u(t, x, y)$ taken as distributed along each whole cantilever. It can be replaced by any other realistic force distribution. Denoting by $w(t, x, y)$, $\rho C$ and $R C$ cantilever bending displacements, the mass per unit length, and the bending coefficient, the governing equation in $(x, y) \in \Omega$ is

$$\rho C \partial_{tt}^2 w + R C \partial_y^4 w = u,$$

(3)

endowed with the boundary conditions

$$\begin{align*}
\partial_x w &= 0 \quad \text{at} \ y = 0, \\
\partial_y^2 w &= \partial_{yy} w = 0 \quad \text{at} \ y = L_C^*,
\end{align*}$$

(4)

representing an end clamped in the base, and a free end.

The weak formulation associated to (1-4) states as

$$\begin{align*}
&\int_0^{L_C} (\rho \partial_t^2 w v + R \partial_y^4 w \partial_y^2 v) \, dx \\
&+ \int_0^{L_C} \rho C \partial_y^3 w \partial_y^2 v + R C \partial_y^5 w \partial_y^3 v \, d y d x \\
&= \int_0^{L_C} (u \, v) \, d x + \int_0^{L_C} \partial_t w \, v \, d y d x,
\end{align*}$$

(5)

for any regular function $v$, satisfying in particular the conditions: $v = \partial_y v = 0$ at both end of the base and $\partial_y v = 0$ at $y = 0$ at the junction.

3. Model Reformulation

To simplify the model, but keeping its distributed feature, we discretize in the $y$-direction projecting on a basis $K_n(y) = \int_0^{L_C} y T_n(y) \, d y$, where $T_n(y)$ is the basis of Chebyshev polynomial. We define the approximations of the displacement and of the control

$$\begin{align*}
w(t, x, y) &\approx \sum_{n=1}^{N} w_n(t, x) K_n(y), \\
\partial_t w(t, x, y) &\approx \sum_{n=1}^{N} \partial_t w_n(t, x) K_n(y),
\end{align*}$$

(6)

where $w_n(t, x)$ and $u_n(t, x)$ are the polynomial coefficients in the approximation of $w$ and $u$ respectively. We also choose $v \approx \sum_{n=1}^{N} v_n(t, x) K_n(y)$, so we find that $(w_n(t, x))_{n=1,2,\ldots,N}$ are the solutions to a set of equations posed on $\Gamma$.

$$\begin{align*}
&\sum_{n, m=1}^{N} M_{m,n} \partial_x^2 w_n + K_{m,n} \partial_x^4 w_n \\
&+ R C K_{m,n} w_n = \sum_{n=1}^{N} \bar{B}_{m,n} u_n \text{ in } [0, \infty) \times \Gamma.
\end{align*}$$

(6)

The boundary conditions are

$$\begin{align*}
&\sum_{n=1}^{N} \partial_x w_n(t, 0) K_n(0) = 0 \quad \text{and} \quad \sum_{n=1}^{N} \partial_x w_n(t, L_B) K_n(0) = 0, \\
&\sum_{n=1}^{N} \partial_y w_n(t, L_B) K_n(0) = 0.
\end{align*}$$

In (6), we use the notations

$$\begin{align*}
M_{m,n} &= \rho \partial_t^2 (K_n K_n) \, |_{\Gamma} + \int_0^{L_C} \rho C \partial_y^4 \partial_y K_n K_n \, d y, \\
K_{m,n} &= R \partial_t^2 (K_n K_n) \, |_{\Gamma}, \\
R C K_{m,n} &= \int_0^{L_C} \partial_y^3 \partial_y K_n \partial_y^2 K_n \, d y, \\
\bar{B}_{m,n} &= (K_n K_n) \, |_{\Gamma} + \int_0^{L_C} \partial_y^2 K_n K_n \, d y.
\end{align*}$$

The LQR problem is set for control variables $(u_n)_{n=1,2,\ldots,N} \in L^2(\Gamma)^N$ and for the cost functional

$$\begin{align*}
J = \int_0^t \sum_{n=1}^{N} \| \partial_x^2 w_n(t, x) \|^2_{L^2(\Gamma)} \\
+ \| u_n(t, x) \|^2_{L^2(\Gamma)} \, d t.
\end{align*}$$

(7)

The choice of the functional is related to vibration stabilization of the microcantilever array.

4. Classical Formulation of the LQR Problem

Now, we write the above LQR problem in a classical abstract setting, see [2], even if we do not detail the functional framework. We set $z^T = (u_n)_{n=1,2,\ldots,N}$ the state variable, $w^T = (u_n)_{n=1,2,\ldots,N}$ the control variable, $A = (M^{-1}(R^B \partial_x^2 + K^C))_{N \times N}$ the state operator, $C = (\partial_x^2 I_{N \times N})_{0 \times N}$ the control operator, $B = (0_{N \times N})_{0 \times N}$ the observation operator, and $S = I_{N \times N}$ the weight operator. Consequently, the LQR problem, consisting in minimizing the functional under the constraint (6), may be written under its usual form as

$$\partial_t z(t, x) = A z(t) + B u(t)$$

for $t > 0$ and $z(0) = z_0$, (8)

with the minimized cost functional (7). We know that $(A, B)$ is stabilizable and that $(A, C)$ is detectable, in the sense that the system is controllable and observable. It follows that for each $z_0$, the LQR problem (8) admits a unique solution

$$u^* = -K z,$$

(9)

where $K = S^{-1} B^* P$ and $P$ is the unique self-adjoint non-negative solution to the operational Riccati equation

$$A^* P + PA - 2B^* P B + C^* C = 0.$$  (10)

5. Semi-Decentralized Approximation

This Section is devoted to formulate the approximation method. The mathematical derivation has been introduced in a paper [9]. We denote by $\Lambda$, the mapping: $\Lambda: f \rightarrow w$, where $w$ is the unique solution of $\partial_x^4 w = f$ in $\Gamma$ with the boundary conditions $w = \partial_y w = 0$ for $x \in \{0, L_B\}$. The spectrum $\sigma(\Lambda)$ is discrete and made up of real eigenvalues $\lambda_k$. They are solutions to the eigenvalue problem $\lambda_k \phi_k = \lambda_k \phi_k \in \| \phi_k \|_{L^2(\Gamma)} = 1$. In the sequel, $I_2 = (\sigma_{\text{min}}, \sigma_{\text{max}})$ refers to an open interval that includes the complete spectrum.

5.1 Factorization of $K$ by a Matrix of Functions of $\Lambda$

In this part, we introduce the factorization of the controller $K$ under the form of a product of a matrix of functions of $\Lambda$. To do so, we introduce the change of variable operators $\Phi_Z = \left( \begin{array}{cc} \Lambda \dagger & 0 \\ 0 & I \end{array} \right)$, $\Phi_U = I$ and $\Phi_Y = \left( \begin{array}{cc} \partial_x^2 \Lambda \dagger & 0 \\ 0 & I \end{array} \right)$, from which we introduce the matrices of functions of $\Lambda$, $a(\Lambda) = \Phi_Z^{-1} A \Phi_Z$, $b(\Lambda) = \Phi_Z^{-1} B \Phi_Z$, and $c(\Lambda) = \Phi_Z^{-1} C \Phi_Z$.
\[ \Phi^{-1}_z B \Phi_U, c(\Lambda) = \Phi^{-1}_w C \Phi_Z \] and \( s(\Lambda) = \Phi^{-1}_w S \Phi_U \), simple to implement on a semi-decentralized architecture. A straightforward calculation yield

\[
\begin{align*}
a(\lambda) &= \left( \begin{array}{c} 0 \\ M \end{array} \right), \\
b(\lambda) &= \left( \begin{array}{c} 0 \\ -M^{-1} B \end{array} \right), \\
c(\lambda) &= \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\end{align*}
\]

where \( \bar{M} = -M^{-1}(K_B \lambda^{-1/2} + K_C \lambda^{1/2}) \). From (9), the optimal controller \( \bar{K} \) admits the factorization

\[
\bar{K} = k(\Lambda) = \Phi_U q(\Lambda) \Phi^{-1}_z,
\]

where \( q(\lambda) = s^{-1}(\lambda) b^T(\lambda) p(\lambda) \), and where for all \( \lambda \in \sigma, p(\lambda) \) is the unique self-adjoint nonnegative matrix solving the algebraic Riccati equation

\[
a^T(\lambda) p + pa(\lambda) - pb(\lambda) s^{-1}(\lambda) b^T(\lambda) p + c^T(\lambda) c(\lambda) = 0.
\]

### 5.2 Approximation of the Functions of \( \Lambda \)

We build the approximation in two steps. Firstly, we use a rational approximation \( k_R(\lambda) \) of \( k(\Lambda) \), then it is approximated by another function \( k_{R,M} \) which is simple to discretize, and yields an accurate approximation. To do so, we use the Dunford-Schwartz formula, see [12], representing a function of an operator, because it involves only the operator \( (\zeta I - \Lambda)^{-1} \) which may be simply and accurately approximated. Since the function \( k(\Lambda) \) is not known, the spectrum \( \sigma(\Lambda) \) cannot be easily determined, so we approximate \( k(\Lambda) \) by a highly accurate rational approximation \( k(\Lambda) \), then the Dunford-Schwartz formula is applied to \( k(\Lambda) \) with a path tracing out ellipses including \( I_\sigma \) but no poles. Since the interval \( I_\sigma \) is bounded, for each function \( k_{ij}(\lambda) \) have a rational approximation over \( I_{\sigma} \), we write under a global formulation, (which may be understood component-wise)

\[
k_{R}(\lambda) = \frac{\sum_{m=0}^{R_n} d_{m}\lambda^m}{\sum_{m=0}^{R_D} d_{m}'\lambda^m}, \tag{12}
\]

where \( d_m, d_m' \) are matrices of coefficients and \( R = (R^N, R^D) \) is the couple comprised of the matrices \( R^N \) of numerator polynomial degrees and the matrices \( R^D \) of denominator polynomial degrees. The path \( \mathcal{C} \), in the Dunford-Schwartz formula,

\[
k_{R}(\Lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} k_{R}(\zeta) (\zeta I - \Lambda)^{-1} d\zeta,
\]

is chosen to be an ellipse parameterized by \( \zeta(\theta) = \zeta_1(\theta) + i\zeta_2(\theta) \), with \( \theta \in [0, 2\pi] \). The parameterization is used as a change of variable, so the integral can be approximated by a quadrature formula involving \( M \) nodes \( \theta_{l=1,..,M} \in [0, 2\pi] \), and \( M \) weights \( (\omega_l)_{l=1,..,M} \). \( I_M(g) = \sum_{l=1}^{M} g(\theta_l) \omega_l \).

In the following equations, we state that the matrices \( k_R(\zeta) \) associated to the rational approximation with the numerator polynomial degrees \( R^N \) and the denominator polynomial degrees \( R^D \). So, for each \( z \in L^2(\Gamma)^{2N} \) and \( \zeta \in \mathbb{C} \), we introduce the 2\( N \)-dimensional vector field

\[
v^c = -i\zeta k_R(\zeta) (\zeta I - \Lambda)^{-1} z.
\]

Decomposing \( v^c \) into its real part \( v^c_1 \) and its imaginary part \( v^c_2 \), the couple \( (v^c_1, v^c_2) \) is solution of the system

\[
\begin{align*}
\zeta_1 v^c_1 - \zeta_2 v^c_2 - \Lambda v^c_1 &= Re \left(-i\zeta k_R(\zeta)\right) z, \\
\zeta_2 v^c_1 + \zeta_1 v^c_2 - \Lambda v^c_2 &= Im \left(-i\zeta k_R(\zeta)\right) z.
\end{align*}
\]

Thus, combining the rational approximation \( k_R \) and the quadrature formula yields an approximate realization \( k_{R,M}(\Lambda) \) of \( k(\Lambda) \),

\[
k_{R,M}(\Lambda) z = \frac{1}{2\pi} \sum_{l=1}^{M} v^c_1(\theta_l) \omega_l. \tag{14}
\]

This formula is central in the method, so it is the center of our attention in the simulations. A fundamental remark is that a "real-time" realization, \( k_{R,M}(\Lambda) z \), requires solving \( M \) systems like (13) corresponding to the \( M \) quadrature nodes \( \zeta(\theta_l) \). The matrices \( k_R(\zeta(\theta_l)) \) could be computed "off-line" once and for all, and stored in memory, so their determination would not penalize a rapid real-time computation. In total, the ultimate parameter responsible of accuracy in a real-time computation, apart from spatial discretization discussed in next Section, is \( M \) the number of quadrature points.

### 6. Circuit Implementation

To realize an optimal control by a set of distributed circuits, we introduce a spatial discretization and synthesis of Equation (13). The interval \( \Gamma \) is meshed with regularly spaced nodes separated by a distance \( h \), we introduce \( \Lambda^{-1} \) the finite difference discretization of \( \Lambda^{-1} \), associated with the clamping boundary condition. In practice, the discretization length \( h \) is chosen small compared to the distance between cantilevers. Then, \( z_h \) denoting the vector of nodal values of \( z \), for each \( \zeta \) we introduce \( (v^c_1,h, v^c_2,h) \), a discrete approximation of \( (v^c_1, v^c_2) \), solution of the discrete set of equations,

\[
\begin{align*}
\zeta_1 v^c_1,h - \zeta_2 v^c_2,h - \Lambda_h v^c_1,h &= Re \left(-i\zeta k_R(\zeta)\right) z_h, \\
\zeta_2 v^c_1,h + \zeta_1 v^c_2,h - \Lambda_h v^c_2,h &= Im \left(-i\zeta k_R(\zeta)\right) z_h.
\end{align*}
\]

Finally, an approximate optimal control, intended to be implemented in a set of spatially distributed actuators, could be estimated from the nodal values,

\[
k_{R,M,h}(\omega_l) z_h = \frac{1}{2\pi} \sum_{l=1}^{M} v^c_1(\theta_l) \omega_l,
\]

estimated at mesh nodes in the following. We propose a synthesis (15–16) by a distributed electronic circuit. The system is rewritten under the manageable form (17–18) and for the sake of simplicity, we use the notations \( \alpha = \ldots \)
We call $\rho_s$ the resistances between the potentials $u_1^{(k)}$ and $u_1^{(k+2)}$, and $\rho_2$ the resistances between the potentials $u_1^{(k-1)}$ and $u_1^{(k+1)}$. By applying the Kirchhoff Current Law (KCL) at node $u_1^{(k)}$, rearranging some terms and dividing by $h^4$, the equation of the cell $k$ can be written under the form:

$$\frac{1}{h^4}\left( -u_1^{(k-2)} + \frac{1}{\rho_1} u_1^{(k-1)} + 2 u_1^{(k)} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - \frac{1}{\rho_2} u_1^{(k+1)} - \frac{1}{\rho_4} u_1^{(k+2)} \right) = \frac{1}{h^2} v_1^{(k)}.$$ 

If one choose the negative potential $\rho_1 = -h^4 \rho_0$ and positive potential $\rho_2 = h^4 \rho_0 / 4$, then the potential at node $u_1^{(k)}$ is expressed as a function of its neighbor voltages as

$$\frac{1}{h^2} \left[ u_1^{(k-2)} - 4 u_1^{(k-1)} + 6 u_1^{(k)} - 4 u_1^{(k+1)} + u_1^{(k+2)} \right] = \rho_0 v_1^{(k)} ,$$

which is the stencil of the differential operation $\Lambda^{-1}$. Consequently, the whole electronic circuit composed of $N - 1$ cells computes the finite difference approximation $u_1^{(k)}$. The numerical value of $\rho_0$ only changes the magnitude of the voltages $u_1^{(k)}$. The values of the resistances inside a cell depend only on the circuit topology and are easily expressed as a function of $\rho_1$ or $\rho_2$, here the resistances of the cells can be taken as $r_1 = r_3 = r_4 = r_6 = \rho_1 / 4$ and $r_2 = r_5 = \rho_2 / 2$.

The VCSC (Voltage Controlled Current Source) $i_1^{(k)}$ of Figure 3 is controlled by the voltage $v_1^{(k)}$ through the equation $i_1^{(k)} = g v_1^{(k)}$. The four boundary cells are represented in Figure 4. The imposed values of the voltages correspond to the clamping boundary condition. Remark that the terminals denoted by a cross are not connected, so the resistances are linked by one side at them can be removed without changing the behavior of the circuits. They are saved to

$$\begin{align*}
\text{Fig. 2: Analog computation of } \Lambda_h v_1. \\
\text{Fig. 3: Five adjacent interior cells.}
\end{align*}$$
show clearly the real difference between interior cells and boundary cells.

6.2 Analog computation of equation (17)
The analog computation of Equation (17) can be made by an array of classical non inverting summing amplifiers of Figure 5. Notice that there is no current exchange between these circuits and PNR inputs and outputs, see buffers in Figure 3. Analysis of the circuit of Figure 5 leads to (19).

\[ v^{(k)} = \frac{R_0}{R_c} \alpha + \frac{R_2}{R_B} \beta + \frac{R_c}{R_2} g(\Lambda h v_1)^k + \frac{R_c}{R_2} g(\Lambda h v_2)^k, \]

where \( \frac{1}{R_c} = \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} + \frac{1}{R_d}. \)

6.3 Analog computation of equation (18)
In a very similar way, the analog computation of Equation 18 can be made by an array of classical difference summing amplifiers of Figure 6. Analysis of the circuit of Figure 6 leads to (20). With a proper choice of resistances, Figure 6 solve (18),

\[ v^{(k)} = \frac{R_0}{R_c} \beta - \frac{R_0}{R_c} \alpha + \frac{R_2}{R_B} \beta + \frac{R_c}{R_2} g(\Lambda h v_2)^k, \]

where \( \frac{1}{R_c} = \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} + \frac{1}{R_d} \) and \( \frac{1}{R_c} = \frac{1}{R_c} + \frac{1}{R_d} + \frac{1}{R_2}. \)
Numerical integrations have been performed with a standard trapezoidal quadrature rule. The relative error, \( E = \frac{\|k_{n,M} - k\|_{L^2(I)}}{\|k\|_{L^2(I)}} \), between the exact control function and final approximation are shown in Figure 7, for the number of nodes \( M \) varying from 5 to 20. It may be easily tuned without changing spatial complexity associated with the finite difference discretization of \( \Lambda^{-1} \).

We also present the ratio of the computation time of solving the whole system for varying number of nodes \( M \) to the reference computation time of solving the whole system for \( M = 20 \), see Figure 8.

8. Conclusion

In this paper, we have presented a semi-decentralized approximation of an optimal control operator applied to a two-scale model of microcantilever arrays. This model is discretized in \( y \)-direction projecting on a transformed basis of Chebyshev polynomials. It has been shown that the semi-decentralized optimal controller can be implemented by a set of distributed electronic circuits. Numerical simulations have been carried out to validate the method and study its performances. This method can be extended to other optimal control theories, such as \( LQG \) or \( H_\infty \).

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References