SPIM

Thèse de Doctorat

école doctorale sciences pour l'ingénieur et microtechniques

UNIVERSITÉ DE FRANCHE-COMTÉ

Contribution à l'homogénéisation périodique d'un problème spectral et de l'équation d'onde

Thi Trang NGUYEN

SPIM

Thèse de Doctorat

école doctorale **sciences pour l'ingénieur et microtechniques**

UNIVERSITÉ DE FRANCHE-COMTÉ

N° | X | X | X |

THÈSE présentée par

THI TRANG NGUYEN

pour obtenir le

Grade de Docteur de
l'Université de Franche-Comté

Spécialité : **Sciences pour l'ingénieur**

Contribution à l'homogénéisation périodique d'un problème spectral et de l'équation d'onde

Unité de Recherche :
FEMTO-ST, Département Temps-Fréquence, Université de Franche-Comté

Soutenue publiquement le 3 Décembre 2014 devant le Jury composé de :

MORVAN OUISSE	Président	Professeur, ENSMM, Besançon
CARLOS CONCA	Rapporteur	Professeur, Université du Chili, Chili
SÉBASTIEN GUENNEAU	Rapporteur	Directeur de recherche CNRS, Institut Fresnel, Université d'Aix Marseille
JUAN CASADO-DÍAZ	Rapporteur	Professeur, Université de Séville, Espagne
MICHEL LENCZNER	Directeur de Thèse	Professeur, Université de Technologie Belfort-Montbéliard
MATTHIEU BRASSART	Co-Directeur de Thèse	Maître de conférences, Université de Franche-Comté

Département Temps-Fréquence
UFR ST Sciences et Techniques

École doctorale SPIM Besançon

Contributions to periodic homogenization of a spectral problem and of the wave equation

Dissertation

Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Ph.D. in Engineering Sciences

3 December 2014

Doctoral School of University of Franche-Comté,

By

Thi Trang NGUYEN

Doctoral Committee:

President :	Professor Morvan OUISSE
Reviewers :	Professor Carlos CONCA Director of research Sébastien GUENNEAU Professor Juan CASADO-DÍAZ
Examiners :	Professor Michel LENCZNER Assistant professor Matthieu BRASSART Professor Morvan OUISSE

INSTITUT FEMTO-ST

ABSTRACT

In this dissertation, we present the periodic homogenization of a spectral problem and the wave equation with periodic rapidly varying coefficients in a bounded domain. The asymptotic behavior is addressed based on a method of Bloch wave homogenization. It allows modeling both the low and high frequency waves. The low frequency part is well-known and it is not a new point here. In the opposite, the high frequency part of the model, which represents oscillations occurring at the microscopic and macroscopic scales, was not well understood. Especially, the boundary conditions of the high-frequency macroscopic equation established in [36] were not known prior to the commencement of thesis. The latter brings three main contributions. The first two contributions, are about the asymptotic behavior of the periodic homogenization of the spectral problem and wave equation in one-dimension. They are derived starting from a system of first order equation as in [36] but also from the usual second order equation. The two-scale models are only for high frequency waves in the case of the spectral problem and for both high and low frequencies for the wave equation. The high frequency models include a microscopic and a macroscopic part, both including boundary conditions, which for the latter is a novelty. Numerical simulation results are provided to corroborate the theory. The third contribution consists in an extension of the model for the spectral problem to a thin two-dimensional bounded strip $\Omega = (0, \alpha) \times (0, \varepsilon) \subset \mathbb{R}^2$. The homogenization result includes boundary layer effects occurring in the boundary conditions of the high-frequency macroscopic equation.

Keywords: Homogenization, Bloch waves, Bloch wave decomposition, Spectral problem, Wave equation, Two-scale transform, Two-scale convergence, Unfolding method, Boundary layers, Boundary layer two-scale transform, Macroscopic equation, Microscopic equation, Boundary conditions.

Résumé

Dans cette thèse, nous présentons des résultats d'homogénéisation périodique d'un problème spectral et de l'équation d'ondes avec des coefficients périodiques variant rapidement dans un domaine borné. Le comportement asymptotique est étudié en se basant sur une méthode d'homogénéisation par ondes de Bloch. Il permet de modéliser les ondes à basse et haute fréquences. La partie du modèle à basse fréquence est bien connu et n'est pas donc abordée dans ce travail. A contrario, la partie à haute fréquence du modèle, qui représente des oscillations aux échelles microscopiques et macroscopiques, est un problème laissé ouvert. En particulier, les conditions aux limites de l'équation macroscopique à hautes fréquences établies dans [36] n'étaient pas connues avant le début de la thèse. Ce dernier travail apporte trois contributions principales. Les deux premières contributions, portent sur le comportement asymptotique du problème d'homogénéisation périodique du problème spectral et de l'équation des ondes en une dimension. Elles sont dérivées soit à partir d'un système d'équation du premier ordre comme dans [36], soit à partir de l'équation du second ordre. Les modèles à deux échelles sont obtenus pour des ondes à haute fréquence seulement pour le problème spectral et pour les basses et hautes fréquences pour l'équation des ondes. Les modèles à haute fréquence comprennent à la fois une partie microscopique et une partie macroscopique, cette dernière incluant des conditions au bord, ce qui est une nouveauté. Des résultats de simulations numériques corroborent la théorie. La troisième contribution consiste en une extension du modèle du problème spectral posé dans une bande mince bidimensionnelle et bornée. Le résultat d'homogénéisation comprend des effets de couche limite qui se produisent dans les conditions aux limites de l'équation macroscopique à haute fréquence.

Mots-clés: Homogénéisation, Ondes de Bloch, Décomposition en ondes de Bloch, Problème spectral, Equation des ondes, Transformée à deux-échelles, Convergence à deux échelles, Méthode d'éclatement périodique, Couches limites, Transformation à deux échelles pour des couche limites.

Acknowledgment

First of all, I would like to deeply thank to my advisor, Prof. Michel LENCZNER, for his kindheartedly advising. With honor, I would like to say "Thank you, Prof. Michel LENCZNER, for all of your help, guidance, dedication, enthusiasm, patience and instructions. It was really lucky for me to work with you". Moreover, I owe a big thank to my co-advisor, Prof. Matthieu BRASSART, who was always ready to guide, help and encourage me throughout this thesis. I do appreciate all of your support, enthusiasm, advice, and all mathematical discussions in this thesis.

I would like to send my appreciation to all the members of Jury, Prof. Carlos CONCA, Prof. Sébastien GUENNEAU, Prof. Juan CASADO-DÍAZ and Prof. Morvan OUISSE for reviewing and examining my thesis.

I hereby express my gratitude to especially thank to all staff members at Institute of FEMTO-ST and all my friends at University of Franché-Comté for providing me with a lot of kind assistances during the time when I have worked at FEMTO-ST. In particular, I would like to sincerely thank to Dr. Philip Lutz, my PhD director, and Dr. Vincent GIORDANO for all of their generous support. I am very grateful to get the aid and kindness of Prof. Nicolas RATIER, Prof. Emmanuel BIGLER and Prof. Bernard DULMET. I am very thankful to the secretaries Mrs. Fabienne CORNU and Mrs. Sarah DJAOUTI from Time frequency department, Mrs. Isabelle GABET and Mrs. Sandrine FRANCHI of our Lab for helping me in work contract issues. I am also thankful to the secretaries of doctoral school of University of Franché-Comté for helping me with registration and preparation of my defense.

I also express my gratitude to thank Prof. Duc Trong DANG, Prof. Minh Duc DUONG and Prof. Pascal OMES for teaching me for the past several years, helping me to develop my background knowledge in mathematics and encouraging me to pursue my research career.

A huge thank is given to my colleagues, Hui HUI, Bin YANG, Raj Narayan DHARA, Youssef YAKOUBI, Huu Quan DO, Duy Duc NGUYEN, and Mohamed ABAIDI for their kindly help and instructions. A lot of thanks are sent to all of my friends living in Vietnam and France for their kindness and sharing in my work and my life.

Last but not least, a great thank goes to my parents who always wish me all the best in life, to my younger brother and two younger sisters who always encourage and help me whenever I need them. A special thank is given to my husband for his invaluable support, love, and understanding. He is always by my side to cheer me up and stands by me through the good times and bad. Many respectful thanks are expressed to my parents-in-law, uncle Duy Chinh LE, aunt Thi Minh Thanh NGUYEN, cousin Anh Tuan LE and all of my family.

One more time, I am thankful to everyone who has contributed to this dissertation.

To my parents
my younger sisters
my younger brother
my husband
my uncle and aunt

TABLE OF CONTENTS

List of Figures

List of Tables

Introduction

The homogenization theory was introduced in order to describe the behaviour of composite materials. Composite materials are characterized by both a microscopic and macroscopic scales describing heterogeneities and the global behaviour of the composite respectively, see Figure 1 as an example. The aim of homogenization is precisely to give macroscopic properties of the composite by taking into account properties of the microscopic structure. The name "homogenization" was introduced in 1974 by Babuska in [14] and it became an important subject in Mathematics. In the mathematical literature, the homogenization of physical systems with a periodic microstructure or periodic media is called "periodic homogenization". A vast literature exists where one distinguishes between stochastic and deterministic homogenization corresponding to stochastic and deterministic micro-structures, the later being mostly concerned with periodic homogenization. Nevertheless, there are also research works on non-periodic deterministic micro-structures as in [91], [93], [92], [34]. I recommend the introductory book by D. Cioranescu and P. Donato [44] which is a good start to study the homogenization theory of partial differential equations. I also recommend the books [99], [19], [63], [109] to understand, not only the homogenization theory, but also its motivation, historical development and larger view over various methods, see also two thesis works [108] and [55] for a brief history.

This thesis falls within the area of deterministic homogenization and its aim is to study the periodic homogenization of a spectral problem, at high frequency, and of the wave equation, simultaneously at high and low frequencies, in an open bounded domain $\Omega \subset \mathbb{R}^N$ with time-independent periodic coefficients. Our complete results are presented for a one-dimensional geometry and also for a thin two-dimensional strip, however a significant part of our results extend trivially to multi-dimensional cases. The model derivation method is based on the modulated-two-scale transform and the Bloch wave decomposition. I recall that the two-scale transform or periodic unfolding operator [77], [79], [78], [76], [45], [37] or [46], transforms a function of the variable in the physical space into a function of two variables, namely the macroscopic variable and the microscopic variables. This is how the concept of two scale-convergence turns out to be a usual notion of convergence of functions that can be weak or strong. The modulated-two-scale transform was defined in [36] by multiplying the usual two-scale transform by a family of oscillating exponential functions which effect is to yield a corresponding family of two-scale limits with all possible periodicities also refered as quasi-periodicities. We also use its counterpart defined from the two-scale convergence issued from [89], [90], [1], [2], [81].

The Bloch wave decomposition, also known as Floquet decomposition, was introduced in the original work of F. Bloch [29], and is well exposed in [111], [99] and [102].

A Bloch wave decomposition of a function, consists in an expansion over a family of the eigenfunctions solution to the spectral problem

$$\mathrm{div}_y\left(a\nabla_y\phi^k\right) = -\lambda^k\phi^k$$

posed in the reference cell $Y \subset \mathbb{R}^N$ equipped with k-quasi-periodic boundary conditions for some $k \in [-1/2, 1/2)^N$. We refer to [104], [99], [111] for an introduction to the Bloch waves in spectral analysis. Such a decomposition is used in the so-called Bloch wave homogenization method for spectral problems [7], [51], [8] and for elliptic problems [48], [50]. We notice that we call our approach with the same name "Bloch wave homogenization" even if the techniques differ in some aspects but we think that they yield similar results.

For a two-dimensional strip, a boundary corrector is required so that the asymptotic solution satisfies the nominal boundary condition. It is solution to a boundary layer problem posed in $\mathbb{R}^+ \times (0,1)$. Its solution might decreases exponentially with respect to the first variable. The derivation of this part of model is achieved by a two-scale transform dedicated to boundary layers that can be related to the two-scale convergence for boundary layers as in [9].

In all this work, the homogenization process starts with a very weak formulation of the spectral or wave equation. Applying our method, provides two-scale models including the expected high frequency parts but also a low frequency part for the wave equation. The latter is well known since it has been found by various authors, so our work focuses mainly on the high frequency part. It comprises so-called high-frequency microscopic and macroscopic equations, the first being a second order partial differential equation and the second a system of first order partial differential equations. In the strip case, the boundary layer problem is a second order partial differential equation.

The thesis includes three main contributions. In the first one, we consider the solution $(w^\varepsilon, \lambda^\varepsilon)$ of the spectral problem

$$-\partial_x\left(a^\varepsilon\partial_x w^\varepsilon\right) = \lambda^\varepsilon\rho^\varepsilon w^\varepsilon, \tag{1}$$

posed in a one-dimensional open bounded domain $\Omega \subset \mathbb{R}$, with Dirichlet or Neumann boundary conditions. An asymptotic analysis of this problem is carried out where $\varepsilon > 0$ is a parameter tending to zero and the coefficients are ε-periodic, namely $a^\varepsilon = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon = \rho\left(\frac{x}{\varepsilon}\right)$, $a(y)$ and $\rho(y)$ being 1-periodic in $\mathbb{R}$. Homogenization of spectral problems has been studied in various works providing the asymptotic behaviour of eigenvalues and eigenvectors. The low frequency part of the spectrum has been investigated in [69], [70], [110]. Then, many configurations have been analyzed, as [52] and [49] for a fluid-structure interaction, [21], [5] for neutron transport, [86], [98] for ρ which changes sign or [6] for the first high frequency eigenvalue and eigenvector for a one-dimensional non-self-adjoint problem with Neumann boundary conditions. Higher order of asymptotic of the eigenvalues have been studied in [106] and [101]. A survey on recent spectral problems encountered in mathematical physics is available in [71]. In an important contribution [8], G. Allaire and C. Conca studied the asymptotic behaviour of both the low and high frequency spectrum. In order to analyze the asymptotic behaviour of the high frequency eigenvalues, they used the Bloch wave homogenization method. They have shown that the limit of the set of renormalized eigenvalues $\varepsilon^2\lambda^\varepsilon$ is the union of the Bloch spectrum and the boundary layer spectrum, when ε goes to 0. However, the asymptotic behaviour of the corresponding eigenvectors was not addressed for a bounded domain Ω. This is the goal of this contribution. We only focuses on the Bloch spectrum of the high frequency part. By applying the Bloch wave homogenization method, the two-scale model is derived including both microscopic and macroscopic eigenmodes with boundary conditions. We derive the homogenization model from both the second order equation (2.1) and an equivalent first order system of equations. We observe that the two models are equivalent. The asymptotic behaviour of the eigenvalue λ^ε and corresponding eigenvector w^ε are provided.

In the second contribution, we establish a homogenized model for the wave equation,

$$\begin{aligned} &\rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x \left(a^\varepsilon \partial_x u^\varepsilon\right) = f^\varepsilon, \\ &u^\varepsilon\left(t=0,x\right) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon\left(t=0,x\right) = v_0^\varepsilon, \end{aligned} \tag{2}$$

posed in a finite time interval $I \subset \mathbb{R}^+$ and in a one-dimensional open bounded domain $\Omega \subset \mathbb{R}$ with Dirichlet boundary conditions. The asymptotic analysis is carried out under the same assumptions as for the spectral problem regarding ε and the coefficients. The homogenization of the wave equation has been studied in various works. The construction of homogenization and corrector results for the low frequency waves has been published in [33], [60]. These works were not taking into account fast time oscillations, so the models reflect only a part of the physical solution. Similar solutions have been derived for the case where the coefficients depend on the time variable t in [47], [38]. In [35] and [36], an asymptotic analysis of the solution $u^\varepsilon(t,x)$, that conserves time and space oscillations occurring both at low and high frequencies in a bounded domain, has been introduced. It is derived from a formulation of the wave equation as a first order system and uses a decomposition over Bloch modes. It extends the thesis work [65] achieved in one-dimension. By using the Bloch wave homogenization method, the resulting asymptotic model includes separated parts for low and high frequency waves respectively. The latter is comprised with a microscopic equation and with a first order macroscopic equation which boundary conditions are missing. A similar result has been obtained in [39], based on the second order formulation of the wave equation, which homogenized solution is periodic in space because it does not include a decomposition on Bloch modes. In the present contribution, we synthesize these ideas in a model, based on the second order formulation of the wave equation, using the Bloch wave decomposition of the solution and more importantly including boundary conditions. The main result of this contribution is the boundary conditions of the high frequency macroscopic model. However, the high frequency macroscopic model is also new since it differs from this in [36] derived from a first order system only. In addition, the proof has been simplified. Moreover, for the sake of comparison, the homogenization is also presented under the first order formulation as in [35] and [36], then boundary conditions for the one-dimensional model of these works have been announced. In conclusion, the physical solution u^ε is approximated by a sum of a low frequency term, the usual corrector in elliptic problems, using the solution of the cell problem, and a sum of Bloch waves being the corrector for the high frequency part. The same result is also established for the Neumann boundary conditions and also for a generalization of the wave equation taking into account a zero order term as well as first order time and space derivatives.

We quote that in both contributions, the models and proofs have been written in one-dimension but they extend trivially to multi-dimensional cases, except what refers to the high frequency macroscopic boundary conditions which remains an open question in higher dimension. Hence, to do a step towards the possibility of taking into account a multi-dimensional geometry, we address the case of a two-dimensional bounded strip. This yields the third contribution. Due to time limitation, only results on the spectral problem are reported, but we expect that they extend to the wave equation. We study the periodic homogenization of the spectral problem

$$-\text{div}\,(a^\varepsilon \nabla w^\varepsilon) = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon$$

posed in an open bounded strip $\Omega = \omega_1 \times (0, \varepsilon) \subset \mathbb{R}^2$ with $\omega_1 = (0, \alpha) \subset \mathbb{R}^+$, with the boundary conditions

$$w^\varepsilon = 0 \text{ on } \partial\omega_1 \times (0, \varepsilon) \text{ and } a^\varepsilon \nabla_x w^\varepsilon . n_x = 0 \text{ on } \omega_1 \times \{0, \varepsilon\},$$

with the same assumptions regarding ε and the coefficients excepted that the reference cell $Y \subset \mathbb{R}^2$. The results of this part are an extension of those obtained in the first one, and the main remaining difficulty consists in establishing the boundary conditions of the macroscopic equation. The model derivation method is still based on the Bloch wave homogenization method using the modulated-two-scale transform, however this tool is not enough. So, a boundary corrector is added, it is solution to a boundary layer problem which is an Helmholtz equation posed in $\mathbb{R}^+ \times (0, 1)$ and with a non-homogeneous boundary condition at left. Its solution is expected to decrease exponentially with respect to the first variable. The derivation of this part of model is achieved by a two-scale transform dedicated to boundary layers that can be related to the two-scale convergence for boundary layers as in [9]. The complete asymptotic behaviour of the eigenvalue λ^ε and corresponding eigenvectors w^ε including the boundary layer effects are provided. We observe that a similar problem was also investigated in [53] but posed in the unbounded domain $\Omega = \mathbb{R}^2$ and for $k \in \left\{0, \frac{1}{2}\right\}$ only. The derivation uses the asymptotic expansion technique and the macroscopic equation arises as a compatibility condition. Higher order equations are also derived. Related works [26] and [27] focus on the homogenization in a vicinity of a gap edge of the Bloch spectra. We recently have been aware of the paper [40] which provides the boundary condition for the high frequency macroscopic equation for the periodic case ($k = 0$).

In our viewpoint, the asymptotics of eigenvalues, eigenvectors and wave propagation are important problems. They have been widely studied in transport theory, reaction-diffusion equations and fluid dynamics, so we present more bibliographical references. For general results on the spectral problem, we refer to [18], [24], [41], [66], [69], [70], [84], [85], [86], [110] and the references therein. In a fixed domain, the homogenization of spectral problems with point-wise positive density function goes back to [69], [70]. In perforated domains, the first homogenization result is referred to [110]. Furthermore, many other authors have addressed similar problems connected with the homogenization of the wave equation for long-term approximation based on convergence methods or asymptotic expansions as [99], [32], [31], [42], [56], [57], [43], [58], [20], [11], [13] to take into account rapid spatial fluctuations. They are valid in the low frequency range only. Most of these results involve more than the two usual terms in the asymptotic expansion, so they involve higher order partial differential equations in addition to the usual second order macroscopic and microscopic equations, and thus additional regularity of the solutions is needed. Non linear cases have also been considered as for instance in [59] and the bibliography herein. Similar problems have been addressed with the perspective of effective coefficient derivation based on various approaches, see for instance the recent works [87] or [12]. They involve Bloch-mode analysis but also refer to a long history of works as the self-consistent schemes in dynamic homogenization by [103], [67] and [68] to cite only few. Other related problems have been studied on the asymptotic regime of the singularly perturbed wave equation for propagation in a periodic medium with volume mass $\varepsilon^2\rho^\varepsilon$ as in [10] or with a large potential as in [3]. Another work in [73] studied the very long time behaviour of waves in a strongly heterogeneous medium. In addition, other asymptotic results for the wave equation can be found in [105], [75], [97] and [61]. Another point of view refers to the midfrequency approach built upon the notions of effective energy density. One of such method was initiated in [22], [23] and has been pursued in the recent years by several authors including [64], [74] and [30]. Other numerical techniques have been developed in the recent years as [72] or those in the review paper [54].

We conclude this introduction by giving a few references to related works on boundary layers in homogenization. Also related to the homogenization of eigenvalue problem, boundary layer has been studied in many works such as [106], [82], [83], [96], [101]. In these publications, boundary layer equations are elliptic equation posed in the macroscopic domain Ω, and they yields correctors for the low frequency part. In [9] and [8], the boundary spectra is studied, it involved a spectral problem which solution is localized along the boundary. Moreover, we refer to [88], [62], [100] for studies of boundary layers for homogenization of highly oscillating solution of elliptic equations. These boundary layers are correctors to the formal the two-scale expansion. In addition, we refer to [19], [106], [28], [109] for other works and references about boundary layers.

This dissertation is organized as follows. Chapter 1 introduces the notations, definitions and properties which are used throughout the thesis. In Chapter 2, we present the homogenization of the spectral problem in one dimension. This corresponds to the published paper [95]. Chapter 3 addresses the homogenization of the one-dimensional wave equation. A first part is based on the second order formulation, which corresponding paper is in preparation. Its second part is based on the first order formulation and is to appear in the proceeding of the conference ENUMATH 2013 held in Lausanne. The results for the strip are presented in Chapter 4. We draw our conclusions in Chapter 5 with some remarks on future research work. Some mathematical proofs and additional material are presented in Appendix.

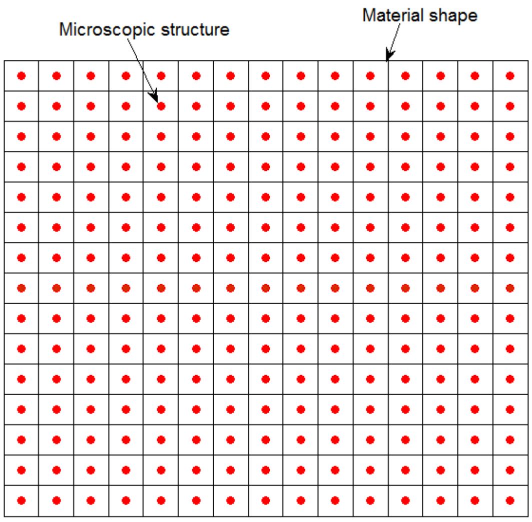

Figure 1: Composite material has a macroscopic shape and a microstructure. The ratio between the size of the microstructure and the size of the material is ε.

Chapter 1

Notations, assumptions and elementary properties

Contents

This chapter introduce the notations, definitions, elementary properties and assumptions which are used throughout the thesis.

1.1 Notations

For $N \in \mathbb{N}^*$ and an open bounded domain $\Omega \subset \mathbb{R}^N$, the functional space $L^2(\Omega)$ of square integrable functions is over $\mathbb{C}$. For m-dimensional complex-valued functions $u = (u_i)_i$ and $v = (v_i)_i$ of $L^2(\Omega)^m$, the dot product is denoted by $u.v := \sum_i u_i v_i$ and the hermitian inner product by

$$\int_\Omega u \cdot v \, dx = \int_\Omega u(x).\overline{v(x)} \, dx. \tag{1.1}$$

The notation $O(\varepsilon)$ refers to numbers or functions tending to zero when $\varepsilon \to 0$ in a sense made precise in each case, $\partial_x u = \frac{\partial u}{\partial x}$ is the $x-$derivative of the function u in one dimension and $[u]_{z=\alpha_1}^{z=\alpha_2}$ is the integration of a function u on the boundary $\partial X = \{\alpha_1, \alpha_2\}$ of an interval $X = (\alpha_1, \alpha_2) \subset \mathbb{R}$. The vectors n_x and n_y are the outer unit normals to the boundaries $\partial\Omega$ and ∂Y of Ω and Y. For the sake of convenience, we shall use the abbreviation "LF" and "HF" to refer to "low frequency" and "high frequency" respectively. Moreover, we introduce a characteristic function $\chi_0(k) = 1$ if $k = 0$ and $= 0$ otherwise.

In the following, we use the notations for Bloch wave decomposition defined in [36] where the dual cell or first Brillouin zone is $Y^* = [-1/2, 1/2)$ and the subset of the wave numbers used in the model is

$$L_K^* = \begin{cases} \{-\frac{K}{2K}, .., \frac{K}{2K} - \frac{1}{K}\} \subset L \text{ if } K \text{ is even,} \\ \{-\frac{K-1}{2K}, .., \frac{K-1}{2K}\} \subset L \text{ if } K \text{ is odd,} \end{cases} \tag{1.2}$$

for $K \in \mathbb{N}^*$. Note that $L_K^* \to Y^*$ when $K \to \infty$. The super-cell $Y_K = (0, K) \times (0, 1)^{N-1}$ is made of K cells translated from $Y = (0, 1)^N$. For $r \in \{1, ..., N\}$, the variable x is written as

$$x = (x_r, \widetilde{x_r}) \text{ with } \widetilde{x_r} = (x_1, ..., x_{r-1}, x_{r+1}, ..., x_N).$$

For any $k \in Y^*$ the space of square integrable $k-$quasi-periodic functions in x_r direction is

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}^N) \mid u(x_r + \ell, \widetilde{x_r}) = u(x)e^{2i\pi k\ell} \text{ a.e. in } \mathbb{R}^N \text{ for all } \ell \in \mathbb{Z}\},$$

or equivalently

$$L_k^2 = \{u \in L_{loc}^2(\mathbb{R}^N) \mid \exists v \in L_\sharp^2 \text{ such that } u(x) = v(x)e^{2i\pi k x_r} \text{ a.e. in } \mathbb{R}^N\},$$

where $L_\sharp^2$ is the traditional notation for L_k^2 in the periodic case i.e. when $k = 0$. Likewise, we set

$$H_k^2 := L_k^2 \cap H_{loc}^2\left(\mathbb{R}^N\right)$$

bearing in mind that the subscript $\sharp$ would be more appropriate in the periodic case $k = 0$. In addition, the operator $\varpi_k : L^2(Y) \to L_k^2$ denotes the $k-$quasi-periodic extension operator. Finally, we denote

$$I^k = \{-k, k\} \text{ if } k \in Y^* \backslash \left\{0, -\frac{1}{2}\right\} \text{ and } I^k = \{k\} \text{ otherwise.} \tag{1.3}$$

1.2 Bloch waves and two-scale transform

We distinguish between two cases: in one dimension and in a two-dimensional trip.

1.2.1 In one dimension

We consider $\Omega = (0, \alpha) \subset \mathbb{R}^+$ an interval, which boundary is denoted by $\partial\Omega$, and two functions $(a^\varepsilon, \rho^\varepsilon)$ assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \text{ and } \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right),$$

where $\rho \in L^\infty(\mathbb{R})$, $a \in W^{1,\infty}(\mathbb{R})$ are both Y-periodic where $Y = (0, 1)$. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions:

$$\rho^0 \leq \rho \leq \rho^1 \text{ and } a^0 \leq a \leq a^1,$$

for some given strictly positive ρ^0, ρ^1, a^0 and a^1.
Two-scale operators For the sake of notation simplicity, we denote $P^\varepsilon = -\partial_x(a^\varepsilon\partial_x)$ and $Q^\varepsilon = \rho^\varepsilon\partial_{tt}$. For a function $u(x, y)$ defined in $\Omega \times \mathbb{R}$ and a function $v(t, \tau)$ defined in $I \times \mathbb{R}$, we introduce,

$$P^0u = -\partial_x(a\partial_x u), \; P^1u = -\partial_x(a\partial_y u) - \partial_y(a\partial_x u) \text{ and } P^2u = -\partial_y(a\partial_y u), \tag{1.4}$$
$$Q^0v = \rho\partial_{tt}v, \; Q^1v = 2\rho\partial_t\partial_\tau v \text{ and } Q^2v = \rho\partial_{\tau\tau}v.$$

Bloch waves For a given $k \in Y^*$, the Bloch eigenelements (λ_n^k, ϕ_n^k) indexed by $n \in \mathbb{N}$ are solution to

$$\mathcal{P}(k) : -\partial_y \left(a\partial_y\phi_n^k\right) = \lambda_n^k\rho\phi_n^k \text{ in } Y \text{ with } \phi_n^k \in H_k^2(Y) \text{ and } \left\|\phi_n^k\right\|_{L^2(Y)} = 1, \tag{1.5}$$

where the eigenvalues λ_n^k constitute a non-negative increasing sequence. The zero eigenvalue only for $k = 0$ is denoted by λ_0^0. We state some properties of the Bloch eigenelements $\left(\lambda_n^k, \phi_n^k\right)$ solution to (1.5) which are useful in studying the HF-waves. For a given $k \in Y^*$, the operator $P_k^2 := -\partial_y \left(a\partial_y.\right) : D\left(P_k^2\right) \subset L_k^2(Y)/\operatorname{Ker}(P_k^2) \to L_k^2(Y)/\operatorname{Ker}(P_k^2)$ with dense domain is positive self-adjoint and with compact inverse, so its spectrum is made with an increasing sequence of positive real numbers tending to infinity. Moreover, the family $\left(\phi_n^k\right)_n$ constitutes an orthonormal basis of the space $L^2\left(Y\right)$ for the hermitian inner product. The only zero eigenvalue is λ_0^0 corresponding to a constant eigenvector, equal to one by normalization. Therefore, $\operatorname{Ker}\left(P_k^2\right) = \varnothing$ for all $k \in Y^*$ except for $k = 0$. This is the same for the case of a two-dimensional strip in Section 1.2.2.

Notation 1 *For $k \neq 0$, $n \in M^k$, the conjugate $\overline{\phi_n^k}$ of ϕ_n^k is solution of $\mathcal{P}(-k)$. We choose the numbering of the eigenvectors ϕ_n^{-k} so that $\phi_n^{-k} = \overline{\phi_n^k}$ which implies that $\lambda_n^{-k} = \lambda_n^k$.*

Remark 2 *For each $k \in Y^*$, $n \in \mathbb{N}^*$, the second order differential equation (1.5) admits two independent solutions, which according to Notation 1, are ϕ_n^k and ϕ_n^{-k} when $k \notin \{0, -\frac{1}{2}\}$. So, the eigenvalues λ_n^k and λ_n^{-k} are both simple while in the other case the eigenvectors are or periodic or anti-periodic and the eigenvalues are or simple or double.*

The L^2-orthogonal projector onto ϕ_n^k is denoted by Π_n^k and the associated time scale is $\alpha_n^k = \frac{2\pi}{\sqrt{\lambda_n^k}}$, with $\alpha_0^0 = \infty$. Denote by M^k the set of the indices n of all Bloch eigenelements,

$$M^k = \mathbb{N} \text{ for } k = 0 \text{ and } M^k = \mathbb{N}^* \text{ for } k \neq 0. \tag{1.6}$$

The space-modulated-two-scale transform Let us assume from now on that the domain Ω is the union of a finite number of entire cells of size ε or equivalently that ε belongs to a subsequence of $\varepsilon_n = \frac{\alpha}{n}$ for $n \in \mathbb{N}^*$. The set of all cells of Ω is $C := \{\omega_\varepsilon = \varepsilon l + \varepsilon Y \mid l \in \mathbb{Z}, \varepsilon l + \varepsilon Y \subset \Omega\}$.

Definition 3 *For any $k \in Y^*$, the modulated-two-scale transform $S_k^\varepsilon : L^2\left(\Omega\right) \to L^2\left(\Omega \times Y\right)$ of a function $u \in L^2\left(\Omega\right)$ is defined by*

$$S_k^\varepsilon u\left(x, y\right) = \sum_{\omega_\varepsilon \in C_\varepsilon} u\left(\varepsilon l_{\omega_\varepsilon} + \varepsilon y\right) \chi_{\omega_\varepsilon}\left(x\right) e^{-2i\pi k l_{\omega_\varepsilon}}, \tag{1.7}$$

where $\varepsilon l_{\omega_\varepsilon}$ stands for the unique node in εL of ω_ε and $\chi_{\omega_\varepsilon}$ is the characteristic function of ω_ε.

From Definition 3 of the modulated-two-scale transform, the three following properties can be checked by using (1.7) and are admitted. For $k \in Y^*$ and two functions

$u, v \in L^2(\Omega)$

$$\|S_k^\varepsilon u\|^2_{L^2(\Omega\times Y)} = \int_{\Omega\times Y} |S_k^\varepsilon u|^2 \, dxdy = \sum_{\omega_\varepsilon \in C} \int_{\omega_\varepsilon} |u|^2 \, dx = \|u\|^2_{L^2(\Omega)}, \tag{1.8}$$

$$S_k^\varepsilon(uv) = S_0^\varepsilon(u) S_k^\varepsilon(v),$$

$$\text{and } S_k^\varepsilon(\partial_x u)(x,y) = \frac{1}{\varepsilon}\partial_y S_k^\varepsilon u(x,y) \text{ for } u \in H^1(\Omega). \tag{1.9}$$

The adjoint $S_k^{\varepsilon *} : L^2(\Omega\times Y) \to L^2(\Omega)$ of S_k^ε is defined by

$$\int_\Omega (S_k^{\varepsilon *} v)(x) \cdot w(x) \, dx = \int_{\Omega\times Y} v(x,y) \cdot (S_k^\varepsilon w)(x,y) \, dxdy, \tag{1.10}$$

for all $w \in L^2(\Omega)$ and $v \in L^2(\Omega\times Y)$. A direct computation, see [95], shows that the explicit expression of $S_k^{\varepsilon *} v$ is

$$(S_k^{\varepsilon *} v)(x) = \sum_{\omega_\varepsilon \in C} \varepsilon^{-1} \int_{\omega_\varepsilon} v\left(z, \frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dz \chi_{\omega_\varepsilon}(x) e^{2i\pi k l_{\omega_\varepsilon}}, \tag{1.11}$$

it maps regular functions in $\Omega \times Y$ to a piecewise-constant functions in Ω.

Remark 4 *Let $k \in Y^*$ and a bounded sequence u^ε in $L^2(\Omega)$ such that $S_k^\varepsilon u^\varepsilon$ converges to u^k in $L^2(\Omega\times Y)$ weakly when $\varepsilon \to 0$, then $S_{-k}^\varepsilon u^\varepsilon$ converges to some u^{-k} in $L^2(\Omega \times Y)$ weakly. Moreover, since $S_k^\varepsilon u^\varepsilon$ and $S_{-k}^\varepsilon u^\varepsilon$ are conjugate then u^k and u^{-k} are also conjugate.*

According to (1.11), $S_k^{\varepsilon *} v$ is not a regular function. For various reasons, we need a regular approximation of $S_k^{\varepsilon *} v$ that will denote by $\mathfrak{R}^k v$. The expression of $\mathfrak{R}^k v$ depends on the regularity of v with respect to its first variable. Prior to defining $\mathfrak{R}^k v$, it is required to extend $v(x,y)$ to $y \in \mathbb{R}$ by $k-$quasi-periodicity. Hence, we denote by $\mathfrak{R}^k$ the operator operating on functions $v(x,y)$ defined in $\Omega \times \mathbb{R}$ and $k-$quasi-periodic in y,

$$(\mathfrak{R}^k v)(x) = v(x, \frac{x}{\varepsilon}). \tag{1.12}$$

The next lemma shows that $\mathfrak{R}^k$ is an approximation of $S_k^{\varepsilon *}$ for $k-$quasi-periodic functions.

Lemma 5 *Let $v \in C^1(\Omega\times Y)$ be a $k-$quasi-periodic function in y then*

$$S_k^{\varepsilon *} v = \mathfrak{R}^k v + O(\varepsilon) \text{ in the } L^2(\Omega) \text{ weak sense.} \tag{1.13}$$

Moreover, for $v \in C^2(\Omega\times Y)$ a $k-$quasi-periodic function in y then

$$\mathfrak{R}^k v = S_k^{\varepsilon *}\left(v + \varepsilon\left(y - \frac{1}{2}\right)\partial_x v\right) + \varepsilon O(\varepsilon) \text{ in the } L^2(I\times\Omega) \text{ weak sense.} \tag{1.14}$$

We refer to Lemma 3 in [95] and to [80] for the proof, see also the proof of forthcoming Lemma 8 in Appendix when the time variables are dismissed. In the proof, we constantly use the following consequence.

Corollary 6 *Let $v \in C^1(\Omega \times Y)$ and $k-$quasi-periodic in y, for any bounded sequence u^ε in $L^2(\Omega)$ such that $S_k^\varepsilon u^\varepsilon$ converges to u in $L^2(\Omega \times Y)$ weakly when $\varepsilon \to 0$ then*

$$\int_\Omega u^\varepsilon \cdot \mathfrak{R}^k v \; dx \to \int_{\Omega \times Y} u \cdot v \; dxdy \quad \textit{when } \varepsilon \to 0.$$

Note that for $k = 0$, this corresponds to the definition of two-scale convergence in [1] and [89].

The time-two-scale transform A two-scale transform is then introduced for the time variable, let $\mathbb{Z}$ be as a canonical lattice and $\Lambda = (0,1)$ as a time unit cell, we set $D := \{\theta_\varepsilon = \varepsilon l + \varepsilon\Lambda \mid l \in \mathbb{Z}, \, \varepsilon l + \varepsilon\Lambda \subset I\}$ the family of all $\varepsilon\Lambda-$cells contained in I.

Definition 7 *The time two-scale transform $T^\varepsilon : L^2(I) \to L^2(I \times \Lambda)$ of the function $u \in L^2(I)$ is defined by*

$$T^\varepsilon u(t,\tau) := \sum_{\theta_\varepsilon \in D} u(\varepsilon l_{\theta_\varepsilon} + \varepsilon\tau)\, \chi_{\theta_\varepsilon}(t) \tag{1.15}$$

where $\varepsilon l_{\theta_\varepsilon} \in \varepsilon\mathbb{Z}$ stands for the left end point of θ_ε and $\chi_{\theta_\varepsilon}$ is the characteristic function of θ_ε.

Similarly, for $u \in L^2(I)$ and $v \in H^1(I)$, the two following properties can be checked by using (1.15),

$$\|T^\varepsilon u\|^2_{L^2(I\times\Lambda)} = \int_{I\times\Lambda} |T^\varepsilon u|^2 \, dtd\tau = \sum_{\theta_\varepsilon \in D} \int_{\theta_\varepsilon} |u|^2 \, dt = \|u\|^2_{L^2(I)} \tag{1.16}$$

$$\text{and } T^\varepsilon(\partial_t v)(t,\tau) = \frac{1}{\varepsilon}\partial_\tau (T^\varepsilon v)(t,\tau). \tag{1.17}$$

The adjoint $T^{\varepsilon *} : L^2(I \times \Lambda) \to L^2(I)$ of T^ε is defined by

$$\int_I (T^{\varepsilon *} v)(t) \cdot w(t) \; dt = \int_{I\times\Lambda} v(t,\tau) \cdot (T^\varepsilon w)(t,\tau) \; dtd\tau, \tag{1.18}$$

for all $w(t) \in L^2(I)$ and $v(t,\tau) \in L^2(I \times \Lambda)$. The explicit expression of $T^{\varepsilon *} v$ is

$$(T^{\varepsilon *} v)(t) = \sum_{\theta_\varepsilon \in D} \varepsilon^{-1} \int_{\theta_\varepsilon} v\left(z, \frac{t - \varepsilon l_{\theta_\varepsilon}}{\varepsilon}\right) dz \chi_{\theta_\varepsilon}(t), \tag{1.19}$$

it maps regular functions in $I \times \Lambda$ to a piecewise-constant functions in I.

The operator $\mathfrak{B}_n^k$, transforming two-scale functions $v(t,\tau,x,y)$ defined in $I \times \mathbb{R} \times \Omega \times \mathbb{R}$ by functions of the physical space-time variables, is then

$$(\mathfrak{B}_n^k v)(t,x) = v(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}). \tag{1.20}$$

Next Lemma presents the relation between $\mathfrak{B}_n^k v$ and $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v$ for a function v which is periodic in τ and $k-$quasi-periodic in y for any $n \in \mathbb{N}^*$.

Lemma 8 *For $k \in Y^*$ and $n \in \mathbb{N}^*$, let $v \in C^1(I \times \Lambda \times \Omega \times Y)$ be a periodic function in τ and $k-$quasi-periodic function in y, then*

$$\mathfrak{B}_n^k v = T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v + O(\varepsilon) \text{ in the } L^2(I \times \Omega) \text{ weak sense.} \tag{1.21}$$

Moreover, if $v \in C^2(I \times \Lambda \times \Omega \times Y)$ is a periodic function in τ and $k-$quasi-periodic function in y, then $\mathfrak{B}_n^k v$ can be approximated at the first order by

$$\mathfrak{B}_n^k v = T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\left(v + \varepsilon\alpha_n^k\left(\tau - \frac{1}{2}\right)\partial_t v + \varepsilon\left(y - \frac{1}{2}\right)\partial_x v\right) + \varepsilon O(\varepsilon) \tag{1.22}$$

in the $L^2(I \times \Omega)$ weak sense.

It would take long to present here the proof, based on Lemma 3 in [95] and [80], of Lemma 8 in details, thus it is postponed in Appendix. Moreover, for a function $u(x,y)$ defined in $\Omega \times \mathbb{R}$ and a function $v(t,\tau)$ defined in $I \times \mathbb{R}$, we observe that,

$$P^\varepsilon \mathfrak{R}^k u = \sum_{n=0}^{2} \varepsilon^{-n} \mathfrak{R}^k P^n u,\ P^\varepsilon\left(\mathfrak{B}_n^k u\right) = \mathfrak{B}_n^k\left(P^0 u + \varepsilon^{-1} P^1 u + \varepsilon^{-2} P^2 u\right), \tag{1.23}$$
$$\text{and } Q^\varepsilon \mathfrak{B}_n^k v = \mathfrak{B}_n^k\left(Q^0 v + \left(\varepsilon\alpha_n^k\right)^{-1} Q^1 v + \left(\varepsilon\alpha_n^k\right)^{-2} Q^2 v\right).$$

1.2.2 In a two-dimensional strip

We consider an open bounded domain $\Omega = \omega_1 \times \omega_2$ with $\omega_1 = (0,\alpha) \subset \mathbb{R}^+$ and $\omega_2 = (0,\varepsilon)$ with ends $\Gamma_{end} = \partial\omega_1 \times \omega_2$ and lateral boundary $\Gamma_{lat} = \omega_1 \times \partial\omega_2$. As usual in homogenization papers, $\varepsilon > 0$ denotes a small parameter intended to go to zero. A 2×2 matrix a^ε and a real function ρ^ε are assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \text{ and } \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right),$$

where $\rho \in L^\infty(\mathbb{R}^2)$ and $a \in W^{1,\infty}(\mathbb{R}^2)^{2\times 2}$ is symmetric. They are both $Y-$periodic with respect to the reference cell $Y \subset \mathbb{R}^2$. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions,

$$\rho^0 \le \rho \le \rho^1 \text{ and } a^0 \left\|\xi\right\|^2 \le \xi^T a \xi \le a^1 \left\|\xi\right\|^2 \text{ for all } \xi \in \mathbb{R}^2 \tag{1.24}$$

for some given strictly positive numbers ρ^0, ρ^1, a^0 and a^1. We note that variables x and y can be written as,

$$x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

For $Y_1 = Y_2 = (0,1)$, let us define the unit cell $Y = Y_1 \times Y_2 = (0,1)^2$ which, upon rescaling to size ε, becomes the period in Ω. The boundary ∂Y is decomposed into $\partial Y = \gamma_{end} \cup \gamma_{lat}$ where $\gamma_{end} = \partial Y_1 \times Y_2$ and $\gamma_{lat} = Y_1 \times \partial Y_2$. The cells $\left(\omega_{1\varepsilon}^j\right)_{j\in\mathbb{N}}$ and $\left(\omega_\varepsilon^j\right)_{j\in\mathbb{N}}$ of size ε, and their indices $\left(l_{\omega_\varepsilon^j}\right)_{j\in\mathbb{N}}$ are decomposed accordingly

$$\omega_{1\varepsilon}^j = \varepsilon(j + Y_1),\ \omega_\varepsilon^j = \omega_{1\varepsilon}^j \times \omega_2 \text{ and } l_{\omega_\varepsilon^j} = (j, 0) \text{ for } j \in \mathbb{N}.$$

With the same convention, the boundary layer cell is in direction y_1 and is defined as,

$$Y_\infty^+ = \mathbb{R}^+ \times Y_2.$$

The boundary ∂Y_∞^+ of Y_∞^+ is decomposed into $\gamma_{\infty,end}^+ = \{0\} \times Y_2$ and $\gamma_{\infty,lat}^+ = \mathbb{R}^+ \times \partial Y_2$.
Two-scale operators Similarly to the one-dimensional case, we denote $P^\varepsilon = -\operatorname{div}_x(a^\varepsilon \nabla_x .)$ and

$$P^0 = -\partial_{x_1}(a_{11}\partial_{x_1}.)\,,\ P^1 = -\partial_{x_1}(a_{1.}\nabla_y .) - \operatorname{div}_y(a_{.1}\partial_{x_1}.)\,,\ P^2 = -\operatorname{div}_y(a\nabla_y .)\,.$$

Bloch waves For $k \in Y^*$, the Bloch eigenelements (λ_n^k, ϕ_n^k) indexed by $n \in \mathbb{N}$ are solution to

$$\begin{gathered}\mathcal{P}(k) : -\operatorname{div}_y\left(a\nabla_y \phi_n^k\right) = \lambda_n^k \rho \phi_n^k \text{ in } Y \text{ with } \phi_n^k \in H^2(Y) \cap L^2\left(H_k^2(Y_1); Y_2\right) \\ \text{such that } a\nabla_y \phi_n^k . n_y = 0 \text{ on } \gamma_{lat} \text{ and } \left\|\phi_n^k\right\|_{L^2(Y)} = 1,\end{gathered} \tag{1.25}$$

where the eigenvalues λ_n^k constitute a non-negative increasing sequence. The zero eigenvalue only for $k=0$ is denoted by λ_0^0.
The modulated-two-scale transform In the statement of the results, the asymptotic behaviour of the solution is expressed by using the following definition of the modulated-two-scale transform. Let us assume from now on that Ω is the union of a finite number of entire cells of size ε or equivalently that the sequence ε is exactly $\varepsilon_n = \frac{\alpha}{n}$ for $n \in \mathbb{N}^*$. We set,

$$J = \left\{ j \in \mathbb{N} \text{ such that } \omega_\varepsilon^j \subset \Omega \right\},$$

then J is the set of indices of finite cells of size ε.

Definition 9 *For any $k \in Y^*$, the modulated-two-scale transform of the function $u \in L^2(\Omega)$, $S_k^\varepsilon : L^2(\Omega) \to L^2(\omega_1 \times Y)$, is defined by*

$$S_k^\varepsilon u(x_1, y) = \sum_{j \in J} u(\varepsilon j + \varepsilon y_1, \varepsilon y_2)\, \chi_{\omega_{1\varepsilon}^j}(x_1)\, e^{-2i\pi kj} \tag{1.26}$$

where $\chi_{\omega_\varepsilon^j}$ is the characteristic function on $\omega_{1\varepsilon}^j$.

From Definition 9 of the modulated-two-scale transform, the three following properties can be checked by using (1.26) and are admitted. For $u, v \in L^2(\Omega)$

$$\|S_k^\varepsilon u\|_{L^2(\omega_1\times Y)}^2 = \int_{\omega_1 \times Y} |S_k^\varepsilon u|^2 dx_1 dy = \sum_{j\in J} \int_{\omega_{1\varepsilon}^j} \left(\frac{1}{\varepsilon^2}\int_{\omega_\varepsilon^j} |u|^2\, dx\right) \chi_{\omega_{1\varepsilon}^j}(x_1) dx_1 = \frac{1}{\varepsilon}\|u\|_{L^2(\Omega)}^2\,, \tag{1.27}$$

$$\begin{gathered} S_k^\varepsilon(uv) = S_0^\varepsilon(u) S_k^\varepsilon(v), \\ \text{and } S_k^\varepsilon(\nabla_x u)(x_1, y) = \frac{1}{\varepsilon}\nabla_y(S_k^\varepsilon u)(x_1, y) \ \text{ for } u \in H^1(\Omega)\,. \end{gathered}$$

Then, the adjoint $S_k^{\varepsilon *} : L^2(\omega_1 \times Y) \to L^2(\Omega)$ of S_k^ε is defined by

$$\frac{1}{\varepsilon}\int_\Omega (S_k^{\varepsilon *} v)(x) \cdot w(x)\, dx = \int_{\omega_1 \times Y} v(x_1, y) \cdot (S_k^\varepsilon w)(x_1, y)\, dx_1 dy, \tag{1.28}$$

for all $w \in L^2(\Omega)$ and $v \in L^2(\omega_1 \times Y)$. A direct computation shows that the explicit expression of $S_k^{\varepsilon *} v$ is

$$\left(S_k^{\varepsilon *} v\right)(x) = \sum_{j \in J} \int_{\omega_{1\varepsilon}^j} \varepsilon^{-1} v\left(z, \frac{x - \varepsilon l_{\omega_\varepsilon^j}}{\varepsilon}\right) dz \, \chi_{\omega_\varepsilon^j}(x) e^{2i\pi k j}, \tag{1.29}$$

it maps regular functions in $\omega_1 \times Y$ to a piecewise-constant function in Ω. Moreover, the operator $\mathfrak{R}^k$, transforming two-scale functions $v(x_1, y)$ defined in $\omega_1 \times \mathbb{R}^2$ and $k-$quasi-periodic in y_1 by functions of the physical space variables, is then

$$(\mathfrak{R}^k v)(x) = v(x_1, \frac{x}{\varepsilon}). \tag{1.30}$$

The next Lemma shows that $\mathfrak{R}^k$ is an approximation of $S_k^{\varepsilon *}$ for $k-$quasi-periodic functions in y_1, it is a simple extension of Lemma 5 also of [80]. The proof is referred in Appendix.

Lemma 10 *Let $v \in C^1(\omega_1 \times Y)$ a $k-$quasi-periodic function in y_1 then*

$$S_k^{\varepsilon *} v = \mathfrak{R}^k v + O(\varepsilon) \text{ in the } L^2(\Omega) \text{ weak sense.} \tag{1.31}$$

Moreover, for $k \in Y^*$, the definition of the modulated-two-scale transform yield relations between $S_k^\varepsilon u^\varepsilon$ and $S_{-k}^\varepsilon u^\varepsilon$:

- $S_k^\varepsilon u^\varepsilon$ and $S_{-k}^\varepsilon u^\varepsilon$ are conjugate,
- if u^ε is a sequence such that $S_k^\varepsilon u^\varepsilon$ converges weakly to u^k in $L^2(\omega_1 \times Y)$ when $\varepsilon \to 0$, then $S_{-k}^\varepsilon u^\varepsilon$ converges weakly to u^{-k} in $L^2(\omega_1 \times Y)$ weakly; moreover u^k and u^{-k} are conjugate.

The boundary layer two-scale transform In order to study the oscillations of waves near the boundary, we introduce the boundary layer two-scale transform which will be defined by adapting the modulated-two-scale transform to the case boundary layers, that is, sequences of functions in Ω which concentrate near the boundary $\{0\} \times \omega_2$ and $\{\alpha\} \times \omega_2$. It is also based on the motivation of two-scale convergence for boundary layers in [9].

Definition 11 *For $\vartheta \in \{0, \alpha\}$, the boundary layer two-scale transform S_b^ϑ applies to functions $u(x) \in L^2(\Omega)$,*

$$S_b^\vartheta : L^2(\Omega) \to L^2\left(Y_\infty^+\right)$$

is a simple $\varepsilon^{-1}-$dilation and is defined by,

$$\left(S_b^0 u\right)(y) = u(\varepsilon y) \chi_{(0,\alpha/\varepsilon)}(y_1), \tag{1.32}$$

and

$$\left(S_b^\alpha u\right)(y) = u(-\varepsilon y_1 + \alpha, \varepsilon y_2) \chi_{(0,\alpha/\varepsilon)}(y_1). \tag{1.33}$$

For $u \in L^2(\Omega)$, the boundness property of $S_b^\vartheta u$ can be showed in the next lemma.

Lemma 12 *For $u \in L^2(\Omega)$ such that u is bounded in $L^2(\Omega)$, then*

$$\varepsilon^{-2} \int_\Omega |u|^2(x)\, dx = \int_{Y_\infty^+} \left|S_b^\vartheta u\right|^2(y)\, dy \text{ for } \vartheta \in \{0, \alpha\}. \tag{1.34}$$

Moreover, the adjoint $S_b^{\vartheta*}: L^2(Y_\infty^+) \to L^2(\Omega)$ of S_b^ϑ, is defined by

$$\frac{1}{\varepsilon}\int_\Omega \left(S_b^{\vartheta*}v\right)(x)\cdot w(x)\,dx = \varepsilon\int_{Y_\infty^+} v(y)\cdot\left(S_b^\vartheta w\right)(y)\,dy \text{ for any } \vartheta\in\{0,\alpha\} \tag{1.35}$$

for all $w(x)\in L^2(\Omega)$ and $v(y)\in L^2(Y_\infty^+)$. Furthermore, for a function $v(y)$ defined in Y_∞^+, the operators $\mathfrak{R}_b^0$ and $\mathfrak{R}_b^\alpha$, transforming the functions $v(y)$ defined in Y_∞^+ by functions of the physical variables, are introduced by

$$\mathfrak{R}_b^0(v)(x) = v\left(\frac{x}{\varepsilon}\right) \text{ and } \mathfrak{R}_b^\alpha(v)(x) = v\left(\frac{\alpha-x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right) \text{ for } x\in\Omega. \tag{1.36}$$

The next lemma presents the relation between $S_b^{\vartheta*}$ and $\mathfrak{R}_b^\vartheta$ for a function v defined in Y_∞^+.

Lemma 13 *For $v(y)\in C^1(Y_\infty^+)$ then*

$$S_b^{\vartheta*}v = \mathfrak{R}_b^\vartheta v \text{ in } L^2(\Omega) \text{ for any } \vartheta\in\{0,\alpha\}. \tag{1.37}$$

The proofs of Lemma 12 and Lemma 13 are postponed in Appendix. For the functions $v(x_1,y)$ and $w(y)$ defined respectively in $\omega_1\times\mathbb{R}^2$ and Y_∞^+, we observe that

$$P^\varepsilon\mathfrak{R}^k v = \sum_{n=0}^{2}\varepsilon^{-n}\mathfrak{R}^k P^n v \text{ and } P^\varepsilon\mathfrak{R}_b^\vartheta w = \varepsilon^{-2}\mathfrak{R}_b^\vartheta P^2 w. \tag{1.38}$$

Finally, from now on the notation P^2 is used instead of P_k^2 for all cases.

1.3 Assumption of sequence ε

The following condition on the sequence ε is made so that to be able to pass to the limit the boundary conditions at $x_1=\alpha$.

Assumption 14 *For each $k\in Y^*$, considering the decomposition in*

$$\frac{\alpha k}{\varepsilon} = h_\varepsilon^k + l_\varepsilon^k, \tag{1.39}$$

with the integer part $h_\varepsilon^k = \left[\frac{\alpha k}{\varepsilon}\right]$ and the rest $l_\varepsilon^k\in[0,1)$, we assume that the sequence ε belongs to a set $E_k\subset\mathbb{R}^{+*}$ such that the sequence l_ε^k is convergent

$$l_\varepsilon^k \to l^k \text{ when } \varepsilon\to 0. \tag{1.40}$$

For $k=0$, we observe that $h_\varepsilon^0=0$, $l_\varepsilon^0=0$ so $l^0=0$ and $E_0=\mathbb{R}^{+*}$.

Chapter 2

Homogenization of the spectral problem in one-dimension

Contents

Abstract. In this chapter, the asymptotic behavior of a one-dimensional spectral problem with periodic coefficient is addressed for HF-modes by a method of Bloch wave homogenization. The analysis leads to a spectral problem including both HF-microscopic and HF-macroscopic eigenmodes. Numerical simulation results are provided to corroborate the theory. This work has been published in [95].

2.1 Introduction

We consider the spectral problem

$$-\partial_x \left(a^\varepsilon \partial_x w^\varepsilon \right) = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \tag{2.1}$$

posed in an one-dimensional open bounded domain $\Omega \subset \mathbb{R}$ with Dirichlet boundary conditions. An asymptotic analysis of this problem is carried out where $\varepsilon > 0$ is a parameter tending to zero and the coefficients are ε-periodic, namely $a^\varepsilon = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon = \rho\left(\frac{x}{\varepsilon}\right)$ where $a(y)$ and $\rho(y)$ are 1-periodic in $\mathbb{R}$. In this chapter, we search eigenvalues λ^ε satisfying the expansion

$$\varepsilon^2\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \varepsilon O(\varepsilon). \tag{2.2}$$

It comes that λ^0 is equal to an eigenvalue λ_n^k solution of the Bloch wave spectral problem (1.5) for $n \in \mathbb{N}^*$ and $k \in Y^*$, also called the HF-microscopic equation in this work. To guarantee that Bloch waves are kept in the weak limit, we apply the modulated two-scale transform S_k^ε defined in (1.7). Passing to the limit in the weak formulation, it is shown that $\sum\limits_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ is weakly converging to two-scale modes

$$g_k(x,y) = \sum_{\sigma\in I^k}\sum_m u_m^\sigma(x)\,\phi_m^\sigma(y)$$

where the second sum runs over all modes ϕ_m^σ with the same eigenvalue λ_n^k. Here, the modes ϕ_m^σ are called Bloch modes. The factors $(u_m^\sigma)_m$ are solution of the HF-macroscopic system of first order differential equation,

$$\sum_m c(\sigma,n,m)\,\partial_x u_m^\sigma + \lambda^1 b(\sigma,n,m)\,u_m^\sigma = 0 \text{ in } \Omega \text{ for each } \sigma \in I^k, \tag{2.3}$$

which boundary conditions and the constant $c(\sigma,n,m)$ are depending on the involved Bloch modes and eigenvalues. The physical solution w^ε is then approximated by two-scale modes

$$w^\varepsilon(x) \approx \sum_{\sigma\in I^k}\sum_m u_m^\sigma(x)\,\phi_m^\sigma\left(\frac{x}{\varepsilon}\right). \tag{2.4}$$

These results are also established for Neumann boundary conditions.

In fact, the method introduced in this chapter is inspired from [36] dedicated to the wave equation, except that in the latter work the two-scale transforms $S_k^\varepsilon w^\varepsilon$ and $S_{-k}^\varepsilon w^\varepsilon$ were analyzed separately and the macroscopic boundary conditions were lacking. Moreover, the model derivation in [36] is starting from the wave equation written as a first order system. So, for the sake of comparison, we derive the homogenized spectral equation from a first order formulation. All presented results are straightforwardly extended to multiple space dimensions except the macroscopic boundary conditions satisfied by the modulation coefficient $u_m^\sigma(x)$ of the Bloch modes.

In addition, we report exploration results regarding approximations of physical eigenmodes by two-scale modes. First, for a given ε and each high frequency physical eigenelement $(\lambda^\varepsilon, w^\varepsilon)$, we show how to find quadruplets $\left(\lambda_n^k, \lambda_1, \phi_n^k, u_n^k\right)_{n,k}$ satisfying the approximations (2.2) and (2.4). This shows that each high frequency eigenelement can be approximated by a two-scale mode. Conversely, the high frequency physical eigenelements can be built from the two-scale eigenelements only. Namely, for a given Bloch mode $\left(\lambda_n^k, \phi_n^k\right)$, a macroscopic eigenelement $\left(\lambda^1, u_n^k\right)$ is minimizing the error on the physical equation (2.1) where w^ε and λ^ε are replaced by their approximations (2.2) and (2.4).

Remaining of this chapter is organized as follows. In Section 3.5.1 we state the physical spectral equation with Dirichlet boundary conditions. In Section 2.3 and 2.4, the model homogenization is derived based on the second order and first order formulations respectively. Finally, the numerical results are reported in the last section.

2.2 Statement of the problem

We consider $\Omega = (0, \alpha) \subset \mathbb{R}^+$ an interval, which boundary is denoted by $\partial\Omega$, and two functions $(a^\varepsilon, \rho^\varepsilon)$ assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \text{ and } \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right), \tag{2.5}$$

where $\rho \in L^\infty(\mathbb{R})$, $a \in W^{1,\infty}(\mathbb{R})$ are both Y-periodic where Y is an open interval. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions:

$$\rho^0 \leq \rho \leq \rho^1 \text{ and } a^0 \leq a \leq a^1, \tag{2.6}$$

for some given strictly positive ρ^0, ρ^1, a^0 and a^1.

With the operators $P^\varepsilon = -\partial_x(a^\varepsilon\partial_x\cdot)$, the spectral problem with Dirichlet boundary conditions is

$$P^\varepsilon w^\varepsilon = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \text{ in } \Omega \text{ and } w^\varepsilon = 0 \text{ on } \partial\Omega, \tag{2.7}$$

where as usual $\varepsilon > 0$ denotes a small parameter intended to go to zero.

The eigenvectors $w^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ are normalized by

$$\|w^\varepsilon\|_{L^2(\Omega)} = \left(\int_\Omega |w^\varepsilon|^2 \, dx\right)^{\frac{1}{2}} = 1, \tag{2.8}$$

and we search the eigenvalues such that

$$\varepsilon^2\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \varepsilon O(\varepsilon), \tag{2.9}$$

where λ^0 is a non negative real number and $O(\varepsilon)$ tends to zero with ε. The weak formulation of the spectral problem (2.7) is: find $w^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega a^\varepsilon \partial_x w^\varepsilon \partial_x v \, dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon v \, dx \quad \text{for all } v \in H_0^1(\Omega). \tag{2.10}$$

Since $\varepsilon^2\lambda^\varepsilon$ is bounded, it results the uniform bound

$$||\varepsilon\partial_x w^\varepsilon||_{L^2(\Omega)} \leq N_0. \tag{2.11}$$

2.3 Homogenization of the high-frequency eigenvalue problem

Before starting the homogenized results, for any $n, m \in \mathbb{N}^*$, we introduce the HF-macroscopic model coefficients

$$c(k,n,m) = \int_Y a\partial_y\phi_m^k \cdot \phi_n^k - \phi_m^k \cdot a\partial_y\phi_n^k \, dy \text{ and } b(k,n,m) = \int_Y \rho\phi_m^k \cdot \phi_n^k \, dy \tag{2.12}$$

and observe that the following properties hold,

$$c(k,n,m) = \overline{c(-k,n,m)},\ c(k,m,n) = -\overline{c(k,n,m)},\ c(k,n,m) = -c(-k,m,n)$$

and

$$b(k,n,m) = \overline{b(k,m,n)},\ b(k,n,m) = \overline{b(-k,m,n)},\ b(k,n,n) > 0.$$

In particular for $k = 0$, if the eigenvectors are chosen as real functions thus $c(0,n,n) = 0$. In the special case $\rho = 1$, $b(k,n,m) = 1$ for $n = m$ and $b(k,n,m) = 0$ otherwise. Here we study our problem with $k \in Y^* = \left(-\frac{1}{2}, \frac{1}{2}\right)$. The process is similar to $k = -\frac{1}{2}$ but the detail is not reported.

2.3.1 Main result

The HF-macroscopic equation is stated for each $k \in Y^*$ and each Bloch wave eigenvalue λ_n^k. For $k \neq 0$, we assume that $c(\sigma, n, n) \neq 0$ for each $\sigma \in I^k$, so it is stated as an eigenvalue problem

$$c(\sigma, n, n)\, \partial_x u_n^\sigma + \lambda^1 b(\sigma, n, n)\, u_n^\sigma = 0 \quad \text{in} \quad \Omega \tag{2.13}$$

for each σ, with the boundary conditions

$$\sum_{\sigma \in I^k} u_n^\sigma(x)\, \phi_n^\sigma(0)\, e^{sign(\sigma) 2i\pi \frac{l^k x}{\alpha}} = 0 \text{ on } x \in \partial\Omega, \tag{2.14}$$

where l^k is defined in (1.40). We observe that the first order operator $c(k, n, n) \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}$ of this system is self-adjoint on the domain

$$D^k = \left\{ (u_n, v_n) \in H^1(\Omega)^2 \text{ satisfying (2.14)} \right\}$$

so λ^1 is real.

For $k = 0$, assuming that λ_n^0 is a double eigenvalue corresponding to two eigenvectors ϕ_n^0 and ϕ_m^0, and that $c(0, n, m) \neq 0$, the HF-macroscopic system states

$$\sum_{q \in \{n,m\}} c(0, p, q)\, \partial_x u_q^0 + \lambda^1 b(0, p, q)\, u_q^0 = 0 \text{ in } \Omega \text{ for } p \in \{n, m\}, \tag{2.15}$$

with the boundary conditions

$$\sum_{q \in \{n,m\}} u_q^0(x)\, \phi_q^0(0) = 0 \text{ on } x \in \partial\Omega. \tag{2.16}$$

Again $\lambda^1 \in \mathbb{R}$ since $c(0, n, m) \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}$ is self-adjoint on

$$D^0 = \left\{ (u_n, u_m) \in H^1(\Omega)^2 \text{ satisfying (2.16)} \right\}.$$

Remark 15 *(i) If $c(k, n, n) = 0$ for $k \neq 0$ or $c(0, p, q) = 0$ for all p, q varying in $\{n, m\}$, the HF-macroscopic equations (2.13) or (2.15) are $\lambda^1 = 0$ or $u = (u_n^\sigma)_{n,\sigma} = 0$. If $\lambda^1 = 0$ then this model does not provide any equation for u_n^σ.*

(ii) For $k \neq 0$, if $\phi_m^k(0) = 0$ then $\phi_m^k(1) = 0$ and ϕ_m^k is a periodic solution that is a solution of $k = 0$. So, we consider always that $\phi_m^k(0) \neq 0$ for $k \neq 0$.

(iii) For $k = 0$, in case where $\phi_n(0) = \phi_m(0) = 0$ the boundary conditions of the HF-macroscopic equation vanishes.

Remark 16 *This work focuses on the Bloch spectrum. To avoid eigenmodes related to the boundary spectrum, according to Proposition 7.7 in [8], we shall assume that the weak limit of $S_k^\varepsilon w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ is not vanishing.*

The main Theorem states as follows.

Theorem 17 *For $k \in Y^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be solution of (2.7) then $\sum_{\sigma \in I^k} S_\sigma^\varepsilon w^\varepsilon$ is bounded in $L^2(\Omega; H^1(Y))$. For $\varepsilon \in E_k$, as in (1.39, 1.40), assuming that the weak limit of $S_k^\varepsilon w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ is non-vanishing and the renormalized sequence $\varepsilon^2 \lambda^\varepsilon$ satisfies the decomposition (2.9), there exists $n \in \mathbb{N}^*$ such that $\lambda^0 = \lambda_n^k$ with λ_n^k an eigenvalue of the Bloch wave spectrum and the limit g_k of any weakly converging extracted subsequence of $\sum_{\sigma \in I^k} S_\sigma^\varepsilon w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ can be decomposed on the Bloch modes*

$$\begin{aligned} g_k(x,y) &= \sum_{\sigma \in I^k} u_n^\sigma(x)\, \phi_n^\sigma(y) \text{ for } k \neq 0 \\ \text{and } g_0(x,y) &= \sum_{q \in \{n,m\}} u_q^0(x)\, \phi_q^0(y) \text{ otherwise.} \end{aligned} \tag{2.17}$$

Moreover, $u_m^\sigma \in H^1(\Omega)$ and $(u_m^\sigma)_{m,\sigma}$ are solutions of the HF-macroscopic equations (2.13, 2.14) and (2.15, 2.16). Finally, u_m^k and u_m^{-k} are conjugate.

Thus, it follows from (2.17) that the physical solution w^ε is approximated by two-scale modes

$$\begin{aligned} w^\varepsilon(x) &\approx \sum_{\sigma \in I^k} u_n^\sigma(x)\, \phi_n^\sigma\left(\frac{x}{\varepsilon}\right) \text{ for } k \neq 0 \\ \text{and } w^\varepsilon(x) &\approx \sum_{q \in \{n,m\}} u_q^0(x)\, \phi_q^0\left(\frac{x}{\varepsilon}\right) \text{ otherwise.} \end{aligned} \tag{2.18}$$

The boundary conditions (2.14) and (2.16) can be directly derived by replacing w^ε in the physical boundary condition by its approximations,

$$\begin{aligned} \sum_{\sigma \in I^k} u_n^\sigma(x)\, \phi_n^\sigma\left(\frac{x}{\varepsilon}\right) &= 0 \qquad \text{for } k \neq 0 \\ \text{and } \sum_{q \in \{n,m\}} u_q^0(x)\, \phi_q^0\left(\frac{x}{\varepsilon}\right) &= 0 \qquad \text{otherwise at } x \in \partial\Omega. \end{aligned} \tag{2.19}$$

For $k \neq 0$, they result from

$$\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) = \phi_n^\sigma(0)\, e^{2i\pi\sigma\frac{x}{\varepsilon}} = \phi_n^\sigma(0)\, e^{sign(\sigma)2i\pi x\frac{h_\varepsilon^k + l_\varepsilon^k}{\alpha}} = \phi_n^\sigma(0)\, e^{sign(\sigma)2i\pi x\frac{l_\varepsilon^k}{\alpha}} \text{ for } x \in \partial\Omega$$

and the assumption $l_\varepsilon^k \to l^k$. For $k = 0$, the conditions follow from the periodicity of ϕ_n^0. Furthermore, we observe that $g_k(x,0)$ and $g_k(x,1)$ are generally not vanishing except for $k = 0$.

Proposition 18 *For $k \in Y^*$, $n \in \mathbb{N}^*$, if the HF-macroscopic solution u_n^k is a non-vanishing constant, then any two-scale mode (2.18) is a physical eigenmode i.e. a solution to (2.7).*

Proof. For $k \in Y^*$, $n \in \mathbb{N}^*$, if the HF-macroscopic solution u_n^k is constant then $\lambda^1 = 0$ and $(u_m^\sigma)_{m,\sigma}$ are constant for all $\sigma \in I^k$ and $m \in \mathbb{N}^*$ such that $\lambda_m^\sigma = \lambda_n^\sigma$. Now, we consider $\rho = 1$ and the proof is similar for $\rho \neq 1$. Based on Remark 23 about the macroscopic solutions in Section 2.3.4, $\lambda^1 = 0$ is equivalent to $\ell = \frac{2k\alpha}{\varepsilon}$. From the $\sigma-$quasi-periodicity of ϕ_n^σ,

$$\phi_n^\sigma\left(\frac{\alpha}{\varepsilon}\right) = \phi_n^\sigma(0)\, e^{sign(\sigma)2i\pi k\frac{\alpha}{\varepsilon}} = \phi_n^\sigma(0)\, e^{sign(\sigma)i\pi\ell} = \pm\phi_n^\sigma(0),$$

then ϕ_n^σ is $\alpha-$periodic or $\alpha-$anti-periodic for $\sigma \in I^k$. Hence $\phi_n^\sigma\left(\frac{x}{\varepsilon}\right)$ is a solution of the equation

$$\begin{aligned} \partial_x\left(a\left(\frac{x}{\varepsilon}\right)\partial_x\phi_n^\sigma\left(\frac{x}{\varepsilon}\right)\right) &= -\frac{\lambda_n^\sigma}{\varepsilon^2}\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) \text{ in } \Omega \\ \text{and } \phi_n^\sigma\left(\frac{x}{\varepsilon}\right) &\text{ is } \alpha-\text{periodic or } \alpha-\text{anti-periodic,} \end{aligned} \tag{2.20}$$

and $u_m^\sigma\phi_m^\sigma\left(\frac{x}{\varepsilon}\right)$ is also a solution of (2.20). Denote by $w^\varepsilon := \sum_{\sigma\in I^k}\sum_m u_m^\sigma\phi_m^\sigma\left(\frac{x}{\varepsilon}\right)$ and observe that w^ε is a solution of the equation

$$\partial_x(a^\varepsilon\partial_x w^\varepsilon) = -\lambda^\varepsilon w^\varepsilon \text{ in } \Omega$$

with the boundary conditions

$$w^\varepsilon(0) = \sum_{\sigma\in I^k}\sum_m u_m^\sigma\phi_m^\sigma(0) = 0 \text{ and } w^\varepsilon(\alpha) = \sum_{\sigma\in I^k}\sum_m u_m^\sigma\phi_m^\sigma\left(\frac{x}{\varepsilon}\right) = \pm w^\varepsilon(0) = 0.$$

Finally, Proposition 18 is concluded. ■

Remark 19 *The converse is probably true, and is numerically studied in Section 2.5.2, i.e. for any* $(\lambda^\varepsilon, w^\varepsilon)$ *solution to (2.7), there exist* $k \in Y^*$, $n \in \mathbb{N}^*$ *and two complex numbers* ξ_1 *and* ξ_2 *such that* $\lambda^\varepsilon = \lambda_n^k/\varepsilon^2$ *and*

$$\begin{aligned} w^\varepsilon(x) &= \xi_1\phi_n^k\left(\frac{x}{\varepsilon}\right) + \xi_2\phi_n^{-k}\left(\frac{x}{\varepsilon}\right) \text{ if } k \neq 0 \\ \text{and } w^\varepsilon(x) &= \xi_1\phi_n^0\left(\frac{x}{\varepsilon}\right) + \xi_2\phi_m^0\left(\frac{x}{\varepsilon}\right) \text{ otherwise} \end{aligned} \tag{2.21}$$

for ξ_1, ξ_2 *two numbers such that the boundary conditions (2.16), respectively (2.14), are satisfied for* $k = 0$*, respectively for* $k \neq 0$*. In the later case* ξ_1 *and* ξ_2 *are conjugate.*

Remark 20 *(i) The case of non-constant coefficients* u_n^k *is used for approximations of the solution to the homogenized wave equation that may be derived from [36]. In such case* k *belongs to a finite subset* L_K^* *of* Y^* *made with values distant from* $1/K$ *and including* 0*. We cannot expect that there always exists a pair* (k, n) *such that* u_n^k *is a constant.*

(ii) The case of non-constant coefficients u_n^k *is also seen as a preparation to derive homogenized spectral problems in higher dimension where the boundary conditions constitute a more difficult problem and may require a more general solution than constant* u_n^k*.*

Proof. [Proof of Theorem 17] The proof is based on Lemma 21 in Section 2.3.2 and on the HF-macroscopic model derivation in Section 2.3.3. For a given $k \in Y^*$, let w^ε be solution of (2.7) which is bounded in $L^2(\Omega)$, the property (1.8) yields the uniform boundness of $\|S_\sigma^\varepsilon w^\varepsilon\|_{L^2(\Omega\times Y)}$ for any $\sigma \in I^k$. So there exist $w^\sigma \in L^2(\Omega \times Y)$ such that up the extraction of a subsequence $S^\varepsilon_\sigma w^\varepsilon \to w^\sigma$ in $L^2(\Omega \times Y)$ weakly. Furthermore, $\|S^\varepsilon_\sigma(\varepsilon \partial_x w^\varepsilon)\|_{L^2(\Omega\times Y)} = \|\partial_y S^\varepsilon_\sigma w^\varepsilon\|_{L^2(\Omega\times Y)}$ is uniformly bounded as $\|\varepsilon\partial_x w^\varepsilon\|_{L^2(\Omega)}$, hence

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y}\partial_y S^\varepsilon_\sigma w^\varepsilon \cdot v dxdy = \lim_{\varepsilon\to 0}\int_{\Omega\times Y} -S^\varepsilon_\sigma w^\varepsilon \cdot \partial_y v dxdy = -\int_{\Omega\times Y} w^\sigma \cdot \partial_y v dxdy$$

for all $v \in L^2(\Omega; H^1_0(Y))$. If $w^\sigma \in L^2(\Omega; H^1(Y))$ then

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y}\partial_y S^\varepsilon_\sigma w^\varepsilon \cdot v dxdy = \int_{\Omega\times Y}\partial_y w^\sigma \cdot v dxdy.$$

Therefore $S^\varepsilon_\sigma w^\varepsilon$ tends weakly to w^σ also in $L^2(\Omega; H^1(Y))$. Hence, $\sum_{\sigma\in I^k} S^\varepsilon_\sigma w^\varepsilon$ converges to

$$g_k(x,y) = \sum_{\sigma\in I^k} w^\sigma(x,y).$$

Using the decomposition (2.22) of w^σ in Lemma 21, for $(\phi^\sigma_p)_{\sigma,p}$ the Bloch wave eigenmodes corresponding to λ^0,

$$\begin{cases} g_k(x,y) = \sum_{\sigma\in I^k} u^\sigma_n(x)\,\phi^\sigma_n(y) \text{ for } k\neq 0, \\ g_0(x,y) = \sum_{p\in\{n,m\}} u^0_p(x)\,\phi^0_p(y) \text{ for } k=0. \end{cases}$$

Finally, $(u^\sigma_p)_{\sigma,p}$ is solution of the HF-macroscopic problem as proved in Section 2.3.3. ■

2.3.2 Modal decomposition on the Bloch modes

Lemma 21 *For $(\lambda^\varepsilon, w^\varepsilon)$ solution of (2.7) and satisfying (2.9), for a fixed $k \in Y^*$ there exists at least a subsequence of $S^\varepsilon_k w^\varepsilon$ converging weakly towards non-vanishing function w^k in $L^2(\Omega\times Y)$ when ε tends to zero. If $w^k \in L^2(\Omega; H^2(Y))$ then (λ^0, w^k) is solution of the Bloch wave equation (1.5) and w^k admits the modal decomposition,*

$$w^k(x,y) = \sum_m u^k_m(x)\,\phi^k_m(y) \text{ for } u^k_m \in L^2(\Omega) \tag{2.22}$$

where the sum is over all Bloch modes ϕ^k_m associated to λ^0. Moreover for $k\neq 0$ the two factors u^k_m and u^{-k}_m are conjugate.

Proof. The test functions of the weak formulation (2.10) are chosen as

$$v^\varepsilon := \mathfrak{R}^k v \in H^1_0(\Omega)\cap H^2(\Omega), \tag{2.23}$$

with

$$v \in H^1_0(\Omega; L^2_k(Y)) \cap L^2(\Omega; H^2_k(Y)) \cap H^2(\Omega; L^2_k(Y)). \tag{2.24}$$

Applying two integrations by parts and the boundary conditions satisfied by w^ε and by $\mathfrak{R}^k v$, it remains

$$\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon\rho^\varepsilon) v^\varepsilon \, dx = 0. \tag{2.25}$$

From (1.23) multiplied by ε^2 and (2.9),

$$\int_\Omega w^\varepsilon \cdot \mathfrak{R}^k((P^2-\lambda^0\rho)v)\ dx = O(\varepsilon).$$

Since $(P^2-\lambda^0\rho)v$ is $k-$quasi-periodic and $S_k^\varepsilon w^\varepsilon \to w^k$ in $L^2(\Omega\times Y)$ weakly, Corollary 6 allows to pass to the limit

$$\int_{\Omega\times Y} w^k\cdot(P^2-\lambda^0\rho)v\ dxdy = 0,$$

or equivalently

$$\int_{\Omega\times Y} w^k\cdot\partial_y\left(a\partial_y v\right) + w^k\cdot\lambda^0\rho v\ dxdy = 0. \tag{2.26}$$

Using the assumption $w^k\in L^2(\Omega;H^2(Y))$ and applying integrations by parts,

$$\int_{\Omega\times Y}\partial_y\left(a\partial_y w^k\right)\cdot v + w^k\cdot\lambda^0\rho v\ dxdy + \int_\Omega\left[aw^k\cdot\partial_y v - a\partial_y w^k\cdot v\right]_{y=0}^{y=1}dx = 0.$$

Then, choosing test functions $v\in L^2(\Omega;H_0^2(Y))$ comes the strong form

$$-\partial_y\left(a\partial_y w^k\right) = \lambda^0\rho w^k \text{ in } \Omega\times Y. \tag{2.27}$$

So, it remains

$$\int_\Omega\left[aw^k\cdot\partial_y v - a\partial_y w^k\cdot v\right]_{y=0}^{y=1}\ dx = 0$$

for general test functions (2.24), which implies that w^k and $\partial_y w^k$ are $k-$quasi-periodic in the variable y.

As we know that λ^0 is an eigenvalue λ_n^k of the Bloch wave spectrum, then w^k is a Bloch eigenvector and is decomposed as

$$w^k(x,y) = \sum_m u_m^k(x)\,\phi_m^k(y)\ \text{ with } u_m^k\in L^2(\Omega)$$

the sum being over all Bloch modes ϕ_m^k associated to λ^0 where $u_m^k(x) = \int_Y w^k(x,y)\cdot\phi_m^k(y)\,dy$. For $k\neq 0$, $\phi_m^k = \overline{\phi_m^{-k}}$ and from Definition 3 of modulated two-scale transform, $S_k^\varepsilon w^\varepsilon = \overline{S_{-k}^\varepsilon w^\varepsilon}$ thus u_m^k and u_m^{-k} are conjugate i.e. $u_m^k = \overline{u_m^{-k}}$. ■

2.3.3 Derivation of the high-frequency macroscopic equation

Before continuing with the derivation of the HF-macroscopic equation, we establish an auxiliary result for existence of special test functions. For $k\in Y^*\backslash\{0\}$, $n\in\mathbb{N}^*$ and $\sigma\in I^k$, we consider the two functions $\psi^k(x), \psi^{-k}(x)\in H^2(\Omega)$ such that

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi l^k\frac{x}{\alpha}} + \psi^{-k}(x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}} = 0 \text{ on } \partial\Omega \tag{2.28}$$

where l^k is defined in (1.40).

Lemma 22 *For $k\in Y^*\backslash\{0\}$, let $\varepsilon\in E_k$, there exist $\psi^{k,\varepsilon}$, $\psi^{-k,\varepsilon}\in H^2(\Omega)$ satisfying*

i) the boundary conditions

$$\psi^{k,\varepsilon}(x)\,\phi_n^k(0)\,e^{2i\pi k\frac{x}{\varepsilon}} + \psi^{-k,\varepsilon}(x)\,\phi_n^{-k}(0)\,e^{-2i\pi k\frac{x}{\varepsilon}} = 0 \text{ on } \partial\Omega, \tag{2.29}$$

ii) and the strong convergence

$$\psi^{\sigma,\varepsilon}\to\psi^\sigma \text{ in } H^2(\Omega) \text{ when } \varepsilon\to 0 \text{ for } \sigma\in I^k. \tag{2.30}$$

Proof. [Proof of Lemma 22] For any $\varepsilon \in E_k$ and let the two functions $\psi^k(x)$, $\psi^{-k}(x) \in H^2(\Omega)$ satisfying (2.28), we prove that the following choice satisfies the conditions,

$$\begin{aligned} \psi^{k,\varepsilon}(x) &= \psi^k(x) \in H^2(\Omega) \\ \text{and } \psi^{-k,\varepsilon}(x) &= \psi^{-k}(x) + \mu^\varepsilon(x) \text{ where } \mu^\varepsilon(x) \in H^2(\Omega) \end{aligned} \tag{2.31}$$

with

$$\mu^\varepsilon(x) = -\left(1 - e^{4i\pi\left(l_k^\varepsilon - l^k\right)}\right)\psi^{-k}(\alpha)\frac{x}{\alpha}$$

where l_ε^k and l^k is defined in (1.39) and (1.40).

i) Replacing (2.31) in (2.29), the boundary conditions are

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi k\frac{x}{\varepsilon}} + \left(\psi^{-k}(x) + \mu^\varepsilon(x)\right)\phi_n^{-k}(0)\,e^{-2i\pi k\frac{x}{\varepsilon}} = 0 \text{ on } \partial\Omega.$$

Using (1.39) and (1.40) with remarking that $e^{2i\pi h_k^\varepsilon\frac{x}{\alpha}} = 1$ at $x \in \partial\Omega$, so

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi l_k^\varepsilon\frac{x}{\alpha}} + \left(\psi^{-k}(x) + \mu^\varepsilon(x)\right)\phi_n^{-k}(0)\,e^{-2i\pi l_k^\varepsilon\frac{x}{\alpha}} = 0 \text{ on } \partial\Omega.$$

Or equivalently,

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} + \left(\psi^{-k}(x) + \mu^\varepsilon(x)\right)\phi_n^{-k}(0)\,e^{-2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} = 0$$

on $\partial\Omega$. Or,

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi l^k\frac{x}{\alpha}}e^{2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} + \left(\psi^{-k}(x) + \mu^\varepsilon(x)\right)\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} = 0$$

on $\partial\Omega$. From (2.28),

$$\psi^k(x)\,\phi_n^k(0)\,e^{2i\pi l^k\frac{x}{\alpha}} = -\psi^{-k}(x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}} \text{ on } \partial\Omega.$$

After replacement, the equation remains,

$$\begin{aligned} &\psi^{-k}(x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}\left(e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} - e^{2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}}\right) \\ &+\mu^\varepsilon(x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} = 0 \text{ on } \partial\Omega. \end{aligned}$$

This equation is satisfied with the above μ^ε.

ii) For $\sigma = k$, the strong convergence is true since $\psi^{k,\varepsilon}$ is independent on ε. For $\sigma = -k$, the strong convergence of $\mu^\varepsilon(x)$ in $H^2(\Omega)$ is trivial, i.e. $\mu^\varepsilon(x) \to 0$ in $H^2(\Omega)$ strongly when $\varepsilon \to 0$. Therefore, $\psi^{-k,\varepsilon} \to \psi^{-k}$ in $H^2(\Omega)$ strongly when $\varepsilon \to 0$. ■

In the HF-macroscopic model derivation, we distinguish between the two cases $k \neq 0$ and $k = 0$.

Case $k \neq 0$

We consider $\lambda^0 = \lambda_n^k$ and the two conjugate eigenvectors ϕ_n^k and ϕ_n^{-k} discussed in Notation 1. We restart from the very weak formulation (2.25) with the test function

$$v^\varepsilon(x) := \mathfrak{R}^k(v^{k,\varepsilon} + v^{-k,\varepsilon}) \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.32}$$

Furthermore, we pose $v^{\sigma,\varepsilon}(x,y) = \psi^{\sigma,\varepsilon}(x)\phi_n^\sigma(y)$ with $\psi^{\sigma,\varepsilon} \in H^2(\Omega)$ for $\sigma \in I^k$ and use the $\sigma-$quasi-periodicity of ϕ_n^σ, i.e. $\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) = \phi_n^\sigma(0)\,e^{2i\pi k\frac{x}{\varepsilon}}$ at any $x \in \partial\Omega$. So the boundary condition in (2.32) is equivalent to

$$\psi^{k,\varepsilon}(x)\,\phi_n^k(0)e^{2i\pi k\frac{x}{\varepsilon}} + \psi^{-k,\varepsilon}(x)\,\phi_n^{-k}(0)e^{-2i\pi k\frac{x}{\varepsilon}} = 0 \text{ at any } x \in \partial\Omega.$$

Applying the relation (1.39),

$$\psi^{k,\varepsilon}(x)\phi_n^k(0)e^{2i\pi x\frac{h_\varepsilon^k+l_\varepsilon^k}{\alpha}} + \psi^{-k,\varepsilon}(x)\phi_n^{-k}(0)e^{-2i\pi x\frac{h_\varepsilon^k+l_\varepsilon^k}{\alpha}} = 0.$$

Since $x\frac{h_\varepsilon^k}{\alpha} = 0$ at $x = 0$ and $x\frac{h_\varepsilon^k}{\alpha} = h_\varepsilon^k$ at $x = \alpha$ with $h_\varepsilon^k \in \mathbb{Z}$ then $e^{\pm 2i\pi x\frac{h_\varepsilon^k}{\alpha}} = 1$. From (1.40), $e^{\pm 2i\pi\frac{l_\varepsilon^k x}{\alpha}} \to e^{\pm 2i\pi\frac{l^k x}{\alpha}}$ when $\varepsilon \to 0$. Using Lemma 22, passing to the limit, the limit v^σ of the test function $v^{\sigma,\varepsilon}$ is

$$v^\sigma(x,y) = \psi^\sigma(x)\phi_n^\sigma(y)$$

and the boundary conditions of the test function are

$$\psi^k(x)\phi_n^k(0)e^{2i\pi\frac{l^k x}{\alpha}} + \psi^{-k}(x)\phi_n^{-k}(0)e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega. \tag{2.33}$$

From (1.23) multiplied by ε, (2.9) and $P^2 v^\sigma - \lambda^0\rho v^\sigma = 0$,

$$\sum_{\sigma\in I^k}\int_\Omega w^\varepsilon \cdot \mathfrak{R}^k(-P^1 v^{\sigma,\varepsilon} + \lambda^1\rho v^{\sigma,\varepsilon})\,dx = O(\varepsilon). \tag{2.34}$$

Extracting a subsequence of w^ε so that $S_k^\varepsilon w^\varepsilon$ and $S_{-k}^\varepsilon w^\varepsilon$ are converging to w^k and w^{-k} in $L^2(\Omega\times Y)$ weak, since $-P^1 v^{\sigma,\varepsilon} + \lambda^1\rho v^{\sigma,\varepsilon}$ is $\sigma-$quasi-periodic then Corollary 6 and Lemma 22 infer that

$$\sum_{\sigma\in I^k}\int_{\Omega\times Y} w^\sigma \cdot (-P^1 v^\sigma + \lambda^1\rho v^\sigma)\,dxdy = 0,$$

i.e.

$$\sum_{\sigma\in I^k}\int_{\Omega\times Y} w^\sigma \cdot \left(\partial_x(a\partial_y v^\sigma) + \partial_y(a\partial_x v^\sigma) + \lambda^1\rho v^\sigma\right)dxdy = 0.$$

This is the very weak form of the HF-macroscopic equation for all test functions $v^\sigma \in H^1(\Omega; H_k^1(Y))$, reached by density, satisfying (2.33). Now, we derive the strong formulation. We assume that $w^\sigma \in H^1(\Omega; L^2(Y))$, since $w^\sigma \in L^2(\Omega; H^1(Y))$ after two integrations by parts,

$$\begin{aligned}\sum_{\sigma\in I^k}\Bigg[\int_{\Omega\times Y} &\partial_y(a\partial_x w^\sigma)\cdot v^\sigma + \partial_x(a\partial_y w^\sigma)\cdot v^\sigma + \lambda^1\rho w^\sigma\cdot v^\sigma\,dxdy \\ &+ \int_Y \left[w^\sigma\cdot a\partial_y v^\sigma - a\partial_y w^\sigma\cdot v^\sigma\right]_{x=0}^{x=\alpha}\,dy \\ &+ \int_\Omega \left[w^\sigma\cdot a\partial_x v^\sigma - a\partial_x w^\sigma\cdot v^\sigma\right]_{y=0}^{y=1}\,dy\Bigg] = 0.\end{aligned}$$

From Lemma 21, w^σ is solution to the Bloch mode equation and is decomposed as

$$w^\sigma(x,y) = u^\sigma(x)\phi_n^\sigma(y). \tag{2.35}$$

After replacement,

$$\sum_\sigma \Big[\int_Y \partial_y (a\phi_n^\sigma) \cdot \phi_n^\sigma + a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy \int_\Omega \partial_x u^\sigma \cdot \psi^\sigma dx \tag{2.36}$$
$$+\lambda^1 \int_Y \rho \phi_n^\sigma \cdot \phi_n^\sigma \, dy \int_\Omega u^\sigma \cdot \psi^\sigma dx + \int_Y \phi_n^\sigma \cdot a\partial_y \phi_n^\sigma - a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy \left[u^\sigma \cdot \psi^\sigma \right]_{x=0}^{x=\alpha}$$
$$+ \left[\phi_n^\sigma \cdot a\phi_n^\sigma \right]_{y=0}^{y=1} \int_\Omega u^\sigma \cdot \partial_x \psi^\sigma - \partial_x u^\sigma \cdot \psi^\sigma \, dx \Big] = 0.$$

Let us recall that $b(.,.,.)$ and $c(.,.,.)$ have been defined in (2.12). For the sake of simplicity, we use $c(\sigma,n) := c(\sigma,n,n)$ and $b(\sigma,n) := b(\sigma,n,n)$ and observe that

$$\int_Y \partial_y (a\phi_n^\sigma) \cdot \phi_n^\sigma + a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy = c(\sigma,n),$$

which results from integrations by parts and from the $\sigma-$quasi-periodicity of ϕ_n^σ. So, using the σ-quasi-periodicity of ϕ_n^σ, (2.36) can be rewritten as

$$\sum_\sigma \left[\int_\Omega (c(\sigma,n)\partial_x u^\sigma + \lambda^1 b(\sigma,n)\, u^\sigma) \cdot \psi^\sigma \, dx - c(\sigma,n) \left[u^\sigma \cdot \psi^\sigma \right]_{x=0}^{x=\alpha} \right] = 0.$$

Choosing the test function $\psi^\sigma = 0$ on $\partial\Omega$, the boundary condition (2.33) is satisfied and by density of $H_0^1(\Omega)$ in $L^2(\Omega)$, the internal equation satisfied by u^σ follows,

$$c(\sigma,n)\partial_x u^\sigma + \lambda^1 b(\sigma,n)\, u^\sigma = 0 \text{ in } \Omega \text{ for each } \sigma. \tag{2.37}$$

Choosing general $\psi^\sigma \in H^1(\Omega)$ satisfying (2.33) yields the boundary conditions

$$\sum_\sigma c(\sigma,n) u^\sigma \overline{\psi^\sigma} = 0 \text{ on } \partial\Omega. \tag{2.38}$$

We introduce the matrices $C_1 = diag((c(\sigma,n))_\sigma)$, $C_2 = diag((b(\sigma,n))_\sigma)$ and the vectors $u = (u^\sigma)_\sigma$, $\psi = (\psi^\sigma)_\sigma$, $\varphi = \left(\phi_n^\sigma(0)\, e^{sign(\sigma)2i\pi \frac{l^k x}{\alpha}} \right)_\sigma$ with $\sigma \in I^k$, so that (2.33, 2.37, 2.38) can be written on the matrix form

$$C_1 \partial_x u + \lambda^1 C_2 u = 0 \text{ in } \Omega \ ,$$
$$\text{and } C_1 u(x).\overline{\psi}(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \overline{\varphi}(x,0).\overline{\psi}(x) = 0 \text{ on } \partial\Omega.$$

The boundary condition is equivalent to

$$C_1 u(x) \text{ is collinear with } \overline{\varphi}(x,0) \text{ i.e. } \det(C_1 u(x), \overline{\varphi}(x,0)) = 0.$$

Equivalently

$$\begin{cases} c(k,n)u^k(0)\,\overline{\phi^{-k}(0)} - c(-k,n)u^{-k}(0)\,\overline{\phi^k(0)} = 0, \\ c(k,n)u^k(\alpha)\,\overline{\phi^{-k}(0)\, e^{-2i\pi l^k}} - c(-k,n)u^{-k}(\alpha)\,\overline{\phi^k(0)\, e^{2i\pi l^k}} = 0. \end{cases}$$

Finally, since $c(k,n) = -c(-k,n)$ and $c(k,n)$ is assumed to do not vanish, the boundary conditions of HF-macroscopic equation (2.37) are

$$u^k(x)\,\phi_n^k(0)\, e^{2i\pi \frac{l^k x}{\alpha}} + u^{-k}(x)\,\phi_n^{-k}(0)\, e^{-2i\pi \frac{l^k x}{\alpha}} = 0 \text{ at } x \in \partial\Omega.$$

Case $k = 0$

In case $k = 0$, to avoid any confusion with λ^0, the upper indices $k = 0$ are removed. We denote by ϕ_n, ϕ_m the eigenvectors associated to $\lambda^0 = \lambda_n = \lambda_m$, solutions to $\mathcal{P}(0)$ in (1.5), and by $\sum_p$, $\sum_q$ the sums over p or q varying in $\{n, m\}$. We restart with a test function

$$v^\varepsilon(x) := \mathfrak{R}^0(\sum_p v_p) \in H_0^1(\Omega) \cap H^2(\Omega) \tag{2.39}$$

for the very weak formulation (2.34). We pose $v_p(x, y) = \psi_p(x)\phi_p(y)$ with $\psi_p(x) \in H^2(\Omega)$ for $p \in \{n, m\}$. Since ϕ_p is periodic thus $\phi_p(\frac{x}{\varepsilon}) = \phi_p(0)$ at $x \in \partial\Omega$ and the boundary condition in (2.39) is equivalent to

$$\sum_p \psi_p(x)\phi_p(0) = 0 \text{ at } x \in \partial\Omega.$$

By setting $c(p, q) := c(0, p, q)$ for $p, q \in \{n, m\}$, using the expression in Lemma 21 of the weak limit w^0 of $S_0^\varepsilon w^\varepsilon$,

$$w^0(x, y) = \sum_p u_p(x)\,\phi_p(y), \tag{2.40}$$

using the periodicity of $(\phi_p)_p$ and conducting the same calculations as for $k \neq 0$, we obtain

$$\sum_{p,q} \left[\int_\Omega (c(p, q)\partial_x u_q + \lambda^1 b(p, q)u_q) \cdot \psi_p \, dx - \left[c(p, q)u_q \cdot \psi_p\right]_{x=0}^{x=\alpha} \right] = 0.$$

With $u = (u_p)_p$, $\psi = (\psi_p)_p$, $\phi = (\phi_p)_p$ and $C_1 = (c(p, q))_{p,q}$, $C_2 = (b(p, q))_{p,q}$, the HF-macroscopic problem turns to be

$$C_1\partial_x u + \lambda^1 C_2 u = 0 \text{ in } \Omega, \tag{2.41}$$

with the boundary conditions

$$C_1 u(x).\psi(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \psi(x).\phi(0) = 0 \text{ on } \partial\Omega.$$

Equivalently, $C_1 u(x)$ is collinear to $\phi(0)$ on $\partial\Omega$ or

$$\det(C_1 u(x), \phi(0)) = 0 \text{ on } \partial\Omega. \tag{2.42}$$

But $c(p, p) = 0$, so (2.42) simplifies to

$$\begin{cases} c(n, m)\, u_m(0)\, \phi_m(0) - c(m, n)\, u_n(0)\, \phi_n(0) = 0, \\ c(n, m)\, u_m(\alpha)\, \phi_m(0) - c(m, n)\, u_n(\alpha)\, \phi_n(0) = 0. \end{cases}$$

Finally, since $c(n, m) = -c(m, n)$ and $c(n, m) \neq 0$, the boundary conditions are

$$u_n(x)\, \phi_n(0) + u_m(x)\, \phi_m(0) = 0 \text{ on } \partial\Omega.$$

2.3.4 Analytic solutions

For $k \in Y^*$ and $\rho = 1$, we solve the HF-macroscopic equations In Section 2.3.4. These solutions are used to validate the numerical results in the final Section. Moreover, in Section 2.3.4.0, the exact formulations of the two-scale eigenmodes are found for $\rho = 1$ and $a = 1$.

The case $\rho = 1$

For $k \neq 0$ and $b(n,n) = 1$, the exact solutions of the HF-macroscopic equation (2.13) are

$$u_n^\sigma(x) = d^\sigma e^{-\lambda^1 c(\sigma,n)^{-1}x} \text{ for each } \sigma \in I^k$$

where d^σ is any complex number. Applying the boundary condition (2.14) and assuming that $\phi_n^k(0) \neq 0$, the eigenvalue is

$$\lambda^1 = \frac{c(k,n)}{\alpha}\left(2i\pi l^k - i\ell\pi\right) \text{ for } \ell \in \mathbb{Z}. \tag{2.43}$$

Furthermore, $u_n^k = \overline{u_n^{-k}}$ and $\phi_n^k(0) = \overline{\phi_n^{-k}}(0)$ then $\mathrm{Re}\left(d^k\phi_n^k(0)\right) = 0$, or $d^k\phi_n^k(0) = i\delta$ for any $\delta \in \mathbb{R}$. Thus,

$$d^k = \frac{i\delta}{\phi_n^k(0)} \text{ and } d^{-k} = -\frac{i\delta}{\phi_n^{-k}(0)} \text{ for any } \delta \in \mathbb{R}.$$

For $k = 0$, using the equalities $c(n,n) = c(m,m) = 0$, $b(n,m) = b(m,n) = 0$ and $b(n,n) = b(m,m) = 1$, the HF-macroscopic equation (2.15) is rewritten

$$\begin{cases} c(n,m)\,\partial_x u_m^0 + \lambda^1 u_n^0 = 0 & \text{in} \quad \Omega, \\ c(m,n)\,\partial_x u_n^0 + \lambda^1 u_m^0 = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.44}$$

If $\lambda^1 = 0$, $\partial_x u_m^0 = 0$ and $\partial_x u_n^0 = 0$ in Ω, then u_m^0 and u_n^0 are independent on x, equivalently, u_m^0 and u_n^0 are complex numbers.

If $\lambda^1 \neq 0$, the first equation gives $u_n^0 = -\frac{c(n,m)\partial_x u_m^0}{\lambda^1}$ in Ω and since $c(n,m) = -c(m,n)$ then

$$\partial_{xx} u_m^0 = -\left(\frac{\lambda^1}{c(n,m)}\right)^2 u_m^0 \tag{2.45}$$

and

$$u_m^0(x) = d_1 \cos\left(\frac{\lambda^1}{c(n,m)}x\right) + d_2 \sin\left(\frac{\lambda^1}{c(n,m)}x\right)$$

for two constants for $d_1, d_2 \in \mathbb{C}$ and u_n^0 follows by its above expression. Applying the boundary condition (2.16), if $\phi_m^0(0) \neq 0$,

$$\lambda^1 = \frac{\ell\pi c(n,m)}{\alpha} \text{ for } \ell \in \mathbb{Z} \text{ and } d_1 = -\, d_2\frac{\phi_n^0(0)}{\phi_m^0(0)} \tag{2.46}$$

for any $\ell \in \mathbb{Z}$ and $d_2 \in \mathbb{C}$. If $\phi_m^0(0) = 0$ then $\phi_n^0(0) = 0$ or $u_n^0(x) = 0$ on $\partial\Omega$. In the case $\phi_n^0(0) = 0$, the HF-macroscopic equation is lacking of boundary conditions and their solutions are not unique, they depend on arbitrary coefficients d_1, d_2 and λ^1. When $u_n^0(x) = 0$ at $\partial\Omega$, there is an alternative, or u_n^0 is the trivial solution or

$$\det\begin{pmatrix} 0 & 1 \\ -\sin\left(\frac{\lambda_1}{c(n,m)}\alpha\right) & \cos\left(\frac{\lambda_1}{c(n,m)}\alpha\right) \end{pmatrix} = 0$$

and then $d_2 = 0$, $\lambda^1 = \frac{\ell\pi c(n,m)}{\alpha}$ for any $\ell \in \mathbb{Z}$ and $d_1 \in \mathbb{C}$.

Remark 23 *According to (3.44) and (2.46), $\lambda^1 = 0$ iff $\ell = 2l^k$ for $k \neq 0$ and iff $\ell = 0$ otherwise. So, in any case small values of $\lambda^{1,\ell}$ correspond to indices ℓ in a vicinity of $2l^k$ or to $\frac{2k\alpha}{\varepsilon}$ when $\varepsilon > 0$.*

The case $a=\rho=1$

We consider the spectral problem

$$-\partial^2_{yy}\phi^k = \lambda^k \phi^k \text{ in } Y$$

with the $k-$quasi-periodicity conditions.

For $k \neq 0$, for a mapping $m \mapsto n(m)$ from $\mathbb{Z}$ to $\mathbb{N}^*$ not detailed here, $\lambda^k_{n(m)} = 4\pi^2(m+k)^2$ and there are exactly two conjugated solutions $\phi^\sigma_{n(m)}(y) = e^{sign(\sigma)2i\pi(m+k)y}$ for any $m \in \mathbb{Z}$ and $\sigma \in I^k$. It follows that $c(\sigma, n(m)) = sign\,(\sigma)\,4i\pi(m+k)$, $b\,(\sigma, n(m)) = 1$ and $\lambda^1 = -\frac{4\pi^2}{\alpha}(2l^k - \ell)(m+k)$ for any $\ell \in \mathbb{Z}$, so

$$u^\sigma_{n(m)}(x) = d^\sigma e^{\frac{sign(\sigma)i\pi}{\alpha}(2l^k-\ell)x}$$

and the resulting two-scale eigenmode is

$$w^\sigma(x,y) = d^\sigma e^{\frac{sign(\sigma)i\pi}{\alpha}(2l^k-\ell)x} e^{sign(\sigma)2i\pi(n+k)y}.$$

For $k=0$, for each $\lambda^0_{n(m)} = (2\pi m)^2$ there are two eigenvectors $\phi_{n(m)}(y) = \cos(2\pi m y)$ and $\phi_{n(m)+1}(y) = \sin(2\pi m y)$ so

$$C_1 = 2m\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ C_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} \phi_{n(m)}(0) \\ \phi_{n(m)+1}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It implies that $\lambda^1 = \frac{4m\ell\pi^2}{\alpha}$ for any $\ell \in \mathbb{Z}$ and

$$u_{n(m)}(x) = d_0 \sin\left(\ell\pi\frac{x}{\alpha}\right) \text{ and } u_{n(m)+1}(x) = d_0\cos(\ell\pi\frac{x}{\alpha}),$$

then the two-scale eigenmode is

$$w(x,y) = d_0[\sin\left(\ell\pi\frac{x}{\alpha}\right)\cos(2\pi m y) + \cos(\ell\pi\frac{x}{\alpha})\sin(2\pi m y)] \text{ for } \ell, m \in \mathbb{Z}.$$

2.3.5 Neumann boundary conditions

We consider the spectral problem with Neumann boundary conditions

$$P^\varepsilon w^\varepsilon = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_x w^\varepsilon = 0 \quad \text{on} \quad \partial\Omega.$$

The process of homogenization and the results are similar to the case of Dirichlet boundary conditions. The HF-microscopic problem and the internal HF-macroscopic equation are unchanged while the boundary conditions of the latter are

$$\sum_{\sigma\in I^k}\sum_m u^\sigma_m(x)\,\partial_y\phi^\sigma_m(0)\,e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega$$

where the cases $k \neq 0$ and $k=0$ are not separated so a general notation is adopted for the sum over m and σ. Their derivation follows the same steps, so we only mention the boundary condition satisfied by the test functions. They are chosen to satisfy $\partial_x v^\varepsilon(x) = 0$ on $\partial\Omega$ or equivalently,

$$\sum_{\sigma\in I^k}\sum_m \partial_x\psi^{\sigma,\varepsilon}_m(x)\,\phi^\sigma_m\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\psi^{\sigma,\varepsilon}_m(x)\,\partial_y\phi^\sigma_m\left(\frac{x}{\varepsilon}\right) = 0 \text{ on } \partial\Omega.$$

Multiplying by ε,

$$\sum_{\sigma\in I^k}\sum_m \psi_m^{\sigma,\varepsilon}(x)\,\partial_y\phi_m^\sigma\left(\frac{x}{\varepsilon}\right)+O(\varepsilon)=0 \text{ on } \partial\Omega, \tag{2.47}$$

then using the $\sigma-$quasi-periodicity of ϕ_m^σ and passing to the limit

$$\sum_{\sigma\in I^k}\sum_m \psi_m^{\sigma}(x)\,\partial_y\phi_m^\sigma(0)\,e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}}=0 \text{ on } \partial\Omega.$$

2.4 Homogenization based on a first order formulation

In this section, the homogenized model is derived based on a first order formulation. The calculations are less detailed than in Section 2.3, only the main results and the proof principles are given.

2.4.1 Reformulation of the spectral problem and the main result

We start by setting

$$U^\varepsilon=\left(\frac{\sqrt{a^\varepsilon}\partial_x w^\varepsilon}{i\sqrt{\lambda^\varepsilon}},\sqrt{\rho^\varepsilon}w^\varepsilon\right),\ \mu^\varepsilon=\sqrt{\lambda^\varepsilon},$$

$$A^\varepsilon=\left(\begin{array}{cc}0 & \sqrt{a^\varepsilon}\partial_x\left(\frac{1}{\sqrt{\rho^\varepsilon}}.\right)\\ \frac{1}{\sqrt{\rho^\varepsilon}}\partial_x\left(\sqrt{a^\varepsilon}.\right) & 0\end{array}\right),\ n_{A^\varepsilon}=\frac{1}{\sqrt{\rho^\varepsilon}}\left(\begin{array}{cc}0 & \sqrt{a^\varepsilon}n_\Omega\\ \sqrt{a^\varepsilon}n_\Omega & 0\end{array}\right)$$

with the domain of the operator A^ε,

$$D(A^\varepsilon):=\left\{(\varphi,\phi)\in L^2(\Omega)\times L^2(\Omega)\,\middle|\,\sqrt{a^\varepsilon}\varphi\in H^1(\Omega),\phi\in H_0^1(\Omega)\right\}\subset L^2(\Omega)^2,$$

so that iA^ε is self-adjoint on $L^2(\Omega)^2$ as proved in [36]. The spectral equation (2.7) can be recasted as a first-order system

$$A^\varepsilon U^\varepsilon=i\mu^\varepsilon U^\varepsilon \text{ in } \Omega \text{ and } U_2^\varepsilon=0 \text{ on } \partial\Omega, \tag{2.48}$$

where U_2^ε is the second component of U^ε. We observe that $\|\sqrt{\rho^\varepsilon}w^\varepsilon\|_{L^2(\Omega)}\le\|\sqrt{\rho^\varepsilon}\|_{L^\infty(\Omega)}$ and that $\left\|\frac{\sqrt{a^\varepsilon}\partial_x w^\varepsilon}{i\sqrt{\lambda^\varepsilon}}\right\|_{L^2(\Omega)}\le M_0$ can be deduced from the weak formulation (2.10), therefore U^ε is uniformly bounded,

$$\|U^\varepsilon\|^2_{L^2(\Omega)}\le M_1. \tag{2.49}$$

We start our analysis from the system expressed in a distributional sense,

$$\int_\Omega U^\varepsilon\cdot(i\mu^\varepsilon-A^\varepsilon)\,\Psi\,dx=0, \tag{2.50}$$

for all admissible test functions $\Psi = (\varphi, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$. We choose $\mu_0 = \sqrt{\lambda^0}$ and $\mu_1 = \frac{\lambda^1}{2\mu_0}$, so μ^ε can be decomposed as

$$\mu^\varepsilon = \frac{\mu_0}{\varepsilon} + \mu_1 + O(\varepsilon). \tag{2.51}$$

The asymptotic spectral problem (1.5) is also restated as a first order system by setting

$$A_k := \begin{pmatrix} 0 & \sqrt{a}\partial_y\left(\frac{1}{\sqrt{\rho}}.\right) \\ \frac{1}{\sqrt{\rho}}\partial_y\left(\sqrt{a}.\right) & 0 \end{pmatrix} \text{ and } n_{A_k} = \frac{1}{\sqrt{\rho}}\begin{pmatrix} 0 & \sqrt{a}n_Y \\ \sqrt{a}n_Y & 0 \end{pmatrix}, \tag{2.52}$$

and

$$e_n^k := \begin{pmatrix} -i\frac{sign(n)}{\sqrt{\lambda_{|n|}^k}}\sqrt{a}\partial_y\left(\phi_{|n|}^k\right) \\ \sqrt{\rho}\phi_{|n|}^k \end{pmatrix} \text{ and } \mu_n^k = sign(n)\sqrt{\lambda_{|n|}^k} \text{ for all } n \in \mathbb{Z}^*. \tag{2.53}$$

As proved in [36], iA_k is self-adjoint on the domain

$$D(A_k) := \left\{ (\varphi, \phi) \in L^2(Y)^2 \,|\, \sqrt{a}\varphi \in H_k^1(Y), \frac{\phi}{\sqrt{\rho}} \in H_k^1(Y) \right\} \subset L^2(Y)^2.$$

The Bloch wave spectral problem $\mathcal{P}(k)$ is equivalent to finding pairs $\left(\mu_n^k, e_n^k\right)$ indexed by $n \in \mathbb{Z}^*$ solution to

$$\mathcal{Q}(k) : A_k e_n^k = i\mu_n^k e_n^k \text{ in } Y \text{ with } e_n^k \in H_k^1(Y)^2. \tag{2.54}$$

The corresponding weak formulation is

$$\int_Y e_n^k \cdot \left(A_k - i\mu_n^k\right)\Psi\, dy = 0 \text{ for all } \Psi \in D(A_k). \tag{2.55}$$

The relation between the operator A^ε and the scaled operator A_k is obtained by considering any regular vector $\psi = \psi(x, y)$ depending on both space scales,

$$A^\varepsilon\left(\psi\left(x, \frac{x}{\varepsilon}\right)\right) = \left(\left(\frac{1}{\varepsilon}A_k + B\right)\psi\right)\left(x, \frac{x}{\varepsilon}\right), \tag{2.56}$$

where the operator B is defined as the result of the formal substitution of $x-$derivatives by $y-$derivatives in A_k, i.e.

$$B := \begin{pmatrix} 0 & \sqrt{a}\partial_x\left(\frac{1}{\sqrt{\rho}}.\right) \\ \frac{1}{\sqrt{\rho}}\partial_x\left(\sqrt{a}.\right) & 0 \end{pmatrix}. \tag{2.57}$$

For any $n \in \mathbb{Z}^*$ and $k \in Y^*$, $M_n^k := \left\{ i \in \mathbb{Z}^* \mid \mu_i^k = \mu_n^k \right\}$ is the set of indices of eigenvectors related to the same eigenvalue μ_n^k. For all $k \in Y^* \setminus \{0\}$, since $\mu_n^k = \mu_n^{-k}$ then $M_n^k = M_n^{-k}$.

Remark 24 *From now on, we shall assume that the weak limit of $S_k^\varepsilon U^\varepsilon$ in $L^2(\Omega \times Y)$ is not vanishing to avoid eigenmodes related to the boundary spectrum (see Proposition 7.7 in [8]).*

Theorem 25 *For $k \in Y^*$, let $(\mu^\varepsilon, U^\varepsilon)$ be solution of (2.48) then $\sum_{\sigma \in I^k} S_\sigma^\varepsilon U^\varepsilon$ is bounded in $L^2(\Omega \times Y)$. For $\varepsilon \in E_k$, assuming that the renormalized sequence $\varepsilon\mu^\varepsilon$ satisfies the decomposition (2.51) with $\mu_0 = \mu_n^k$ an eigenvalue of the Bloch wave spectrum, any weak limit G_k of $\sum_{\sigma \in I^k} S_\sigma^\varepsilon U^\varepsilon$ in $L^2(\Omega \times Y)$ has the form*

$$G_k(x,y) = \sum_{\sigma \in I^k, m \in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma(y), \tag{2.58}$$

where $(u_m^\sigma)_{m,\sigma}$ are the solutions of the HF-macroscopic equations (2.13, 2.14) or (2.15, 2.16).

Therefore, the physical solution U^ε can be approximated by

$$U^\varepsilon(x) \approx \sum_{\sigma \in I^k, m \in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma\left(\frac{x}{\varepsilon}\right). \tag{2.59}$$

Proof. For a given $k \in Y^*$, let U^ε be solution of (2.48) which is bounded in $L^2(\Omega)$, the property (1.8) yields the boundness of $\|S_\sigma^\varepsilon U^\varepsilon\|_{L^2(\Omega\times Y)}$. So there exist $U^\sigma \in L^2(\Omega \times Y)^2$ such that, up the extraction of a subsequence, $S_\sigma^\varepsilon U^\varepsilon$ tends weakly to U^σ in $L^2(\Omega \times Y)^2$ and hence, $\sum_{\sigma \in I^k} S_\sigma^\varepsilon U^\varepsilon$ converges to $G_k(x,y) = \sum_{\sigma \in I^k} U^\sigma(x,y)$. Using the decomposition (2.60) of U^σ in the forthcoming Lemma 26,

$$G_k(x,y) = \sum_{\sigma \in I^k, m \in M_n^\sigma} u_m^\sigma(x)\, e_m^\sigma(y).$$

The HF-macroscopic problem solved by the coefficients $(u_m^\sigma)_{\sigma,m}$ is derived in Section 2.4.2. ■

2.4.2 Model derivation

Modal decomposition on the Bloch modes

Lemma 26 *Let a sequence $(\mu^\varepsilon, U^\varepsilon)$ be solution of (2.48) and satisfies (2.51) with $\mu_0 = \mu_n^k$ for given $n \in \mathbb{Z}^*$ and $k \in Y^*$, we extract a subsequence of ε, still denoted by ε, such that $S_k^\varepsilon U^\varepsilon$ converges weakly to U^k in $L^2(\Omega \times Y)^2$. If $U^k \in D(A_k)$ then (μ_n^k, U^k) is solution of the Bloch wave equation (2.54) and U^k admits the modal decomposition*

$$U^k(x,y) = \sum_{m \in M_n^k} u_m^k(x)\, e_m^k(y) \text{ with } u_m^k \in L^2(\Omega). \tag{2.60}$$

Proof. For each $k \in Y^*$, taking $\Psi(x,y) := \theta(x)\phi(y)$ with $\theta(x) \in C_c^\infty(\Omega)$ and $\phi(y) \in C^\infty(Y)^2$ $k-$quasi-periodic in y, considering $\mathfrak{R}^k\Psi$ as a test functions in (2.50), and using (2.56,2.51),

$$\int_\Omega U^\varepsilon \cdot \mathfrak{R}^k \left(i\frac{\mu_0}{\varepsilon} + i\mu_1 - \frac{A_k}{\varepsilon} - B \right) \Psi \, dx + O(\varepsilon) = 0.$$

Multiplying by ε

$$\int_\Omega U^\varepsilon \cdot \mathfrak{R}^k (i\mu_0 - A_k)\, \Psi \, dx + O(\varepsilon) = 0,$$

and passing to the limit thanks to Corollary 6,

$$\frac{1}{|Y|}\int_{\Omega\times Y} U^k\cdot\left(i\mu_0 - A_k\right)\Psi\ dxdy = 0$$

which is the weak formulation of the Bloch wave equations. If in addition $U^k \in D\left(A_k\right)$, integrating by parts yields

$$\frac{1}{|Y|}\int_{\Omega\times Y}\left(A_k - i\mu_0\right)U^k\cdot\Psi\ dxdy - \frac{1}{|Y|}\int_{\Omega}\left[U^k\cdot n_{A_k}\Psi\right]_{y=0}^{y=1}\ dx = 0 \tag{2.61}$$

providing in turn the strong formulation,

$$A_k U^k = i\mu_0 U^k \quad \text{in} \quad \Omega\times Y. \tag{2.62}$$

Since the product of a periodic function by a $k-$quasi-periodic function is $k-$quasi-periodic then $n_{A_k}\Psi$ is $k-$quasi-periodic in y. Therefore, U^k is $k-$quasi-periodic in y and finally is a Bloch eigenvector in y. By projection, it can be decomposed as

$$U^k\left(x,y\right) = \sum_{m\in M_n^k} u_m^k\left(x\right)e_m^k\left(y\right) \text{ with } u_m^k = \frac{1}{b\left(k,m,m\right)}\int_Y U^k\cdot e_m^k\ dy \in L^2\left(\Omega\right).$$

■

Derivation of the HF-macroscopic equation

The HF-macroscopic equation is stated for each $k\in Y^*$ and each eigenvalue μ_n^k of the Bloch wave spectral problem $\mathcal{Q}(k)$. We pose

$$\kappa\left(k,n,m\right) = \frac{-ic\left(k,n,m\right)}{2\mu_0} \text{ for } m\in M_n^k \tag{2.63}$$

where $c\left(k,n,m\right)$ is defined in (2.12) and notice that

$$\kappa\left(k,n,m\right) = -\kappa\left(-k,m,n\right),\ \kappa\left(k,n,m\right) = -\overline{\kappa\left(-k,n,m\right)},$$
$$\kappa\left(k,n,m\right) = -\overline{\kappa\left(k,m,n\right)},\ \text{and } \kappa\left(0,n,n\right) = 0.$$

For the sake of simplicity, we do the proof for $n\in\mathbb{Z}^{*+}$ only and denote by $\kappa\left(k,n\right) = \kappa\left(k,n,n\right)$ and $\kappa\left(n,m\right) = \kappa\left(0,n,m\right)$. For general n, the proof is the same but ϕ_n^k is replaced by $\phi_{|n|}^k$.

Case $k\neq 0$ The pairs $\left(\mu_n^k, e_n^k\right)$ and $\left(\mu_n^{-k}, e_n^{-k}\right)$ are the eigenmodes of the spectral equations $\mathcal{Q}(\pm k)$ in (2.54) corresponding to the eigenvalue $\mu_0 = \mu_n^k = \mu_n^{-k}$. We pose $\Psi^\varepsilon = \mathfrak{R}^k\left(\Psi^{k,\varepsilon} + \Psi^{-k,\varepsilon}\right) \in H^1(\Omega)\times H_0^1(\Omega)$ as a test function in the weak formulation (2.50), with each $\Psi^{\sigma,\varepsilon}\left(x,y\right) = \psi^{\sigma,\varepsilon}\left(x\right)e_n^\sigma(y)$ where $\psi^{\sigma,\varepsilon}\in H^1(\Omega)$ and satisfies the boundary conditions,

$$\sum_\sigma \psi^{\sigma,\varepsilon}\left(x\right)\phi_n^\sigma\left(\frac{x}{\varepsilon}\right) = 0 \text{ on } \partial\Omega.$$

Notice that this condition is related to the second component of Ψ^ε only. Proceeding as in Section 2.3.3 yields (2.33). Since $\left(i\mu_0 - A_\sigma\right)\Psi^\varepsilon = 0$ for all σ, applying (2.51, 2.56), then Equation (2.50) yields

$$\sum_\sigma\int_\Omega U^\varepsilon\cdot\mathfrak{R}^k\left(i\mu_1 - B\right)\Psi^{\sigma,\varepsilon}dx + O\left(\varepsilon\right) = 0. \tag{2.64}$$

But $(i\mu_1 - B)\,\Psi^{\sigma,\varepsilon}$ is $\sigma-$quasi-periodic so passing to the limit thanks to Corollary 6 and Lemma 22 where Ψ^σ is limit of $\Psi^{\sigma,\varepsilon}$,

$$\frac{1}{|Y|}\sum_\sigma \int_{\Omega\times Y} U^\sigma \cdot (i\mu_1 - B)\,\Psi^\sigma dxdy = 0. \tag{2.65}$$

From Lemma 26, U^σ is decomposed as

$$U^\sigma(x,y) = u_n^\sigma(x)\,e_n^\sigma(y)\,.$$

After replacement,

$$\sum_\sigma \int_\Omega \left(-i\mu_1 b(\sigma,n)\,u_n^\sigma\cdot\psi^\sigma + \kappa(\sigma,n)\,u_n^\sigma\cdot\partial_x\psi^\sigma\right)dx = 0$$

for all $\psi^\sigma \in H^1(\Omega)$ fulfilling (2.33). Moreover, if $u_n^\sigma \in H^1(\Omega)$ it satisfies the strong form of the internal equations

$$\kappa(\sigma,n)\,\partial_x u_n^\sigma - i\mu_1 b(\sigma,n)\,u_n^\sigma = 0 \text{ in } \Omega \text{ for all } \sigma\in I^k, \tag{2.66}$$

and the boundary conditions

$$\sum_\sigma \kappa(\sigma,n)\,u_n^\sigma\cdot\psi^\sigma = 0 \text{ on } \partial\Omega.$$

Following the same calculations as in Section 2.3.3, with the matrices $C_1 = diag(\kappa(\sigma,n))$, $C_2 = diag(b(\sigma,n))$ and the vectors $u = (u_n^\sigma)_\sigma\,, \psi = (\psi^\sigma)_\sigma\,, \varphi = \left(\phi^\sigma(0)\,e^{sign(\sigma)2i\pi x\frac{l^k}{\alpha}}\right)_\sigma$, (2.66) is written on the matrix form

$$C_1\partial_x u = i\mu_1 C_2 u \text{ in } \Omega\,,$$

with boundary condition

$$C_1 u(x)\,.\overline{\psi}(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \overline{\varphi}(x,0)\,.\overline{\psi}(x) = 0 \text{ on } \partial\Omega.$$

Equivalently, $Cu(x)$ is collinear with $\overline{\varphi}(x,0)$ yielding the boundary conditions

$$u_n^k(x)\,\phi_n^k(0)\,e^{2i\pi\frac{l^k x}{\alpha}} + u_n^{-k}(x)\,\phi_n^{-k}(0)\,e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \partial\Omega \tag{2.67}$$

after remarking that $\kappa(\sigma,n)\neq 0$. Finally, with (2.63) and $\lambda^1 = 2\mu_0\mu_1$ the HF-macroscopic problem (2.13, 2.14) is recovered.

Case $k=0$ We adopt the same simplifications of notations that for the case of $k=0$ in Section 2.3.3. Let e_n and e_m be the Bloch eigenmodes of $\mathcal{Q}(0)$ in (2.54) regarding the double eigenvalue $\mu_0 = \mu_n = \mu_m$. In this case $M_n^0 = \{n,m\}$. Taking $\Psi^\varepsilon = \sum_{p\in M_n^0} \mathfrak{R}^0(\Psi_p) \in H^1(\Omega)\times H_0^1(\Omega)$ as a test function with $\Psi_p(x,y) = \psi_p(x)\,e_p(y)$ and $\psi_p \in H^1(\Omega)$. Due to the periodicity of ϕ_p, the second component of Ψ^ε satisfies the boundary conditions

$$\sum_{p\in M_n^0} \psi_p(x)\,\phi_p(0) = 0 \text{ on } \partial\Omega. \tag{2.68}$$

Following similar calculations as for the case $k \neq 0$, the weak limit U^0 of $S_0^\varepsilon U^\varepsilon$ in $L^2(\Omega \times Y)^2$ is

$$U^0(x,y) = \sum_{p \in M_n^0} u_p(x)\, e_p(y)$$

and u_p is solution to the weak formulation

$$\sum_{q \in M_n^0} \int_\Omega -i\mu_1 b(p,q)\, u_q \cdot \psi_p + \kappa(p,q)\, u_q \cdot \partial_x \psi_p \, dx = 0$$

for all $\psi_p \in H^1(\Omega)$ with $p \in M_n^0$. If $u_q \in H^1(\Omega)$ it is a solution to the internal equations

$$\sum_{q \in M_n^0} \kappa(p,q)\, \partial_x u_q - i\mu_1\ b(p,q)\, u_q = 0 \text{ in } \Omega \text{ for } p \in M_n^0, \tag{2.69}$$

and to the boundary conditions

$$\left[\sum_{p,q \in M_n^0} \kappa(p,q)\, u_q \cdot \psi_p \right]_{x=0}^{x=\alpha} = 0.$$

Here, with $C_1 = (\ \kappa(p,q))_{p,q}$, $C_2 = (\ b(p,q))_{p,q}$, $u = (u_p)_p$, $\psi = \left(\psi_p\right)_p$, $\phi = \left(\phi_p\right)_p$,

$$C_1 \partial_x u = i\mu_1 C_2 u \text{ in } \Omega\,,$$

$$\text{and } C_1 u(x) . \overline{\psi}(x) = 0 \text{ on } \partial\Omega \text{ for all } \psi \text{ such that } \phi(0) . \overline{\psi}(x) = 0 \text{ on } \partial\Omega.$$

But $\kappa(p,p)$= 0, therefore

$$u_n(x)\, \phi_n(0) + u_m(x)\, \phi_m(0) = 0 \text{ on } \partial\Omega. \tag{2.70}$$

As for $k \neq 0$, these HF-macroscopic equations are equivalent to (2.15, 2.16).

2.5 Numerical simulations

We report simulations regarding comparisons of physical eigenmodes and their approximation by two-scale modes for $\rho = 1$. In Subsection 2.5.2, for each given high frequency physical eigenelement, a two-scale eigenelement realizing a good approximation is identified. This shows that the two-scale model can actually be used as an approximation of the complete high-frequency spectra. Conversely, Subsection 2.5.3 addresses the modeling problem i.e. it introduces a way to generate approximations of high-frequency spectra from the two-scale model only. Finally, in 2.5.4 the order of convergence with respect to ε is analyzed. The next section describes the main simulation parameters.

2.5.1 Simulation methods and conditions

Both, the physical spectral problem and the Bloch wave spectral problem are discretized by a quadratic finite element method. The number of elements are respectively denoted N_{phys} and N_{bloch}. The implementation of the $k-$quasi-periodic boundary condition is achieved by elimination of the last degree of freedom. More precisely, for

$n \in \{1, ..., 2N_{bloch}+1\}$ the node indices, ϕ_n a degree of freedom of ϕ a Bloch eigenmode and φ_n the corresponding quadratic Lagrange interpolation function,

$$\phi(y) \simeq \sum_{n=2}^{2N_{bloch}} \phi_n \varphi_n + \phi_1 \varphi_1 + \phi_{2N_{bloch}+1} \varphi_{2N_{bloch}+1}.$$

Using the relation $\phi(1) = e^{2i\pi k}\phi(0)$ and taking $\varphi_1 + e^{2i\pi k}\varphi_{2N_{bloch}+1}$ as the first base function allows to eliminate $\phi_{2N_{bloch}+1}$,

$$\phi(y) \simeq \sum_{n=2}^{2N_{bloch}} \phi_n \varphi_n + \phi_1 \left(\varphi_1 + e^{2i\pi k}\varphi_{2N_{bloch}+1}\right).$$

The sets of indices considered in the simulations of high frequency physical modes and Bloch modes are denoted by $\mathcal{J}^\varepsilon$ and J^k, the former being generally included in $(\alpha/2\varepsilon, N_{phys}/2)$. The Bloch modes are calculated for $k \geq 0$ only, and the other cases can be deduced by conjugation. For each Bloch eigenmode $\left(\lambda_n^k, \phi_n^k\right)$, the macroscopic solutions $\left(\lambda^{1,\ell}, u_{m,\ell}^k\right)_{m,\ell}$ are given in Section 2.3.4 with $\delta = 1$ and $d_2 = \phi_m^0(0)$ for any m such that $\lambda_m^k = \lambda_n^k$ and $\ell \in \mathbb{Z}$. In fact, according to Remark 23 the index ℓ should vary in $J_n^k = \left[\frac{2k}{\varepsilon}\right] + \{-r, ..., r\}$, for a small integer r, so that only the first macroscopic eigenmodes be taken into account. In the next discussions, we use the following notations for the two-scale approximations of the eigenvalues and eigenmodes exhibiting clearly their parameters ε, k, n and ℓ,

$$\gamma_{n,\ell}^{\varepsilon,k} := \lambda_n^k + \varepsilon\lambda^{1,\ell} \text{ and } \psi_{n,\ell}^{\varepsilon,k}(x) := \sum_{\sigma\in I^k}\sum_m u_{m,\ell}^\sigma(x)\,\phi_m^\sigma\left(\frac{x}{\varepsilon}\right) \text{ for } \ell \in J_n^k,\ n \in J^k. \quad (2.71)$$

In the simulations reported in Sections 2.5.2 and 2.5.3 only one physical problem is used, namely $\Omega = (0,1)$, $a^\varepsilon(x) = \sin(2\pi x/\varepsilon) + 2$, 50 cells (i.e. $\varepsilon = 1/50$), and $N_{phys} = 2,000$. Other number of cells are used in Section 2.5.4 for the convergence analysis. Consequently, the coefficient of the Bloch wave spectral problem is $a(y) = \sin(2\pi y)+2$. The set Y^* of positive wave numbers in Y^* is discretized by $L_{125}^{*+} = \{0, ..., 62/125\}$ with step $\Delta_k = 1/125$ and $N_{bloch} = 50$. The subset of macroscopic eigenvalues is restricted by $r = 15$.

The first ten graphs $(k \mapsto \lambda_n^k)_{n=1,...,10}$ of Bloch eigenvalues are described in Figure 2.1. The graphs are symmetric about the axis $k = 0$ which confirms that $\lambda_n^k = \lambda_n^{-k}$ as remarked in Notation 1. Moreover, all eigenvalues λ_n^k are simple for $k \neq 0$ and double for $k \in \left\{0, \pm\frac{1}{2}\right\}$.

2.5.2 Approximation of physical modes by two-scale modes

We discuss the approximation of a given solution $\left(\lambda_p^\varepsilon, w_p^\varepsilon\right)$ of Equation (2.7) for a given value of ε. From Remark 3.2.4 we expect to show numerically that there exists a suitable pair (k,n) such that the equality $\left(\lambda_p^\varepsilon, w_p^\varepsilon\right) = (\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ is exact with $(\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ defined in (2.71) and $\lambda^{1,\ell} = 0$. Moreover, in the perspective of Remark 20, k varies in L_{125}^{*+} only and approximations with $\lambda^{1,\ell} \neq 0$ are expected. Whatever if $\lambda^{1,\ell}$ vanishes or not, we expect to search approximations for both eigenvalues and eigenvectors which turns to be an multi-objective optimization problem that might be solved by a

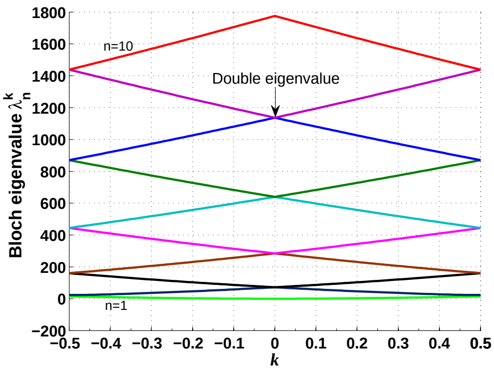

Figure 2.1: First ten eigenvalues of the Bloch wave spectral problem.

dedicated method. However, to reduce the computational cost, we propose an alternate approach consisting in minimizing the error on eigenvalues in the approximation (2.9),

$$er_{value}\left(k\right) = \min_{n\in\mathbb{N},\ \ell\in J_n^k} \left| \frac{\varepsilon^2\lambda_p^\varepsilon - \gamma_{n,\ell}^{\varepsilon,k}}{\varepsilon^2\lambda_p^\varepsilon} \right|, \tag{2.72}$$

for each $k \in L_{125}^{*+}$, and then in finding which one minimizes

$$er_{vector}\left(k\right) = \frac{\left\| w_p^\varepsilon - \psi_{n_k,\ell_k}^{\varepsilon,k} \right\|_{L^2(\Omega)}}{\left\| w_p^\varepsilon \right\|_{L^\infty(\Omega)}}$$

the error on eigenvectors in the approximation (2.18) where ℓ_k, n_k are the optimal arguments in (2.72). The optimal error on eigenvectors is then

$$er_{vector} = \min_{k\in L_{125}^{*+}} er_{vector}\left(k\right). \tag{2.73}$$

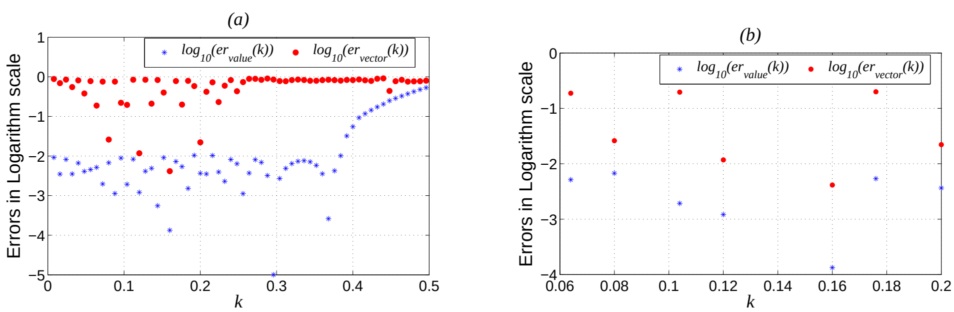

Figure 2.2: (a) Errors for $p = 85$ and $k \in L_{125}^{*+}$. (b) Errors for a selection of k s.t. $er_{vector}(k) \leq 0.2$.

Figure 2.2 (a) shows the distributions of errors $er_{value}(k)$ and $er_{vector}(k)$ in logarithmic scale for the index $p = 85$ of physical eigenmode with respect to k varying in L_{125}^{*+}. The minimal error is reached for $k = 0.16$, $n = 2$, $\ell = 17$, $\lambda_n^k = 51.1$ and $\lambda^{1,\ell} = 58.9$ yielding the errors $er_{value} = 10^{-4}$ and $er_{vector} = 4.10^{-3}$. Figure 2.2 (b)

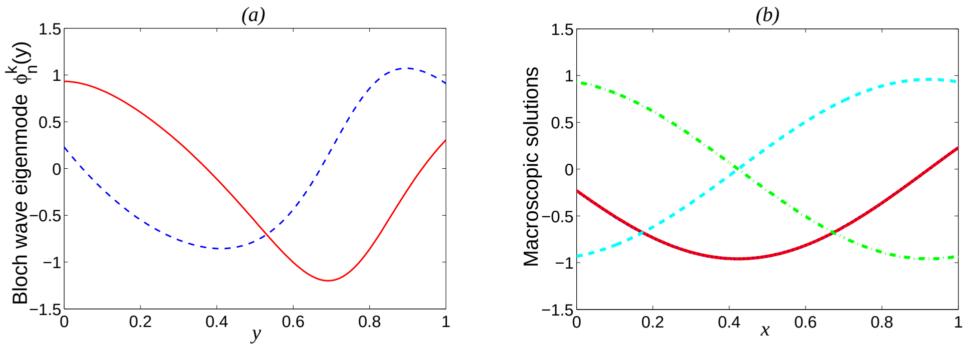

Figure 2.3: (a) Bloch wave solution ϕ_n^k. (b) Macroscopic solutions $u_{n,\ell}^k$ and $u_{n,\ell}^{-k}$.

focuses on values of k such that $er_{vector}(k) \leq 0.2$. In Figure 2.3 (a) the real (dashed line) and the imaginary (solid line) parts of the Bloch wave ϕ_n^k are shown when Figure 2.3 (b) presents the real (solid line) and the imaginary (dashed-dotted line) parts of $u_{n,\ell}^k$ and also the real (dotted line) and the imaginary (dashed line) parts of $u_{n,\ell}^{-k}$. In addition, the physical eigenmode w_p^ε and the relative error vector between w_p^ε and $\psi_{n,\ell}^{\varepsilon,k}$ are plotted in Figure 2.4 (a) and (b).

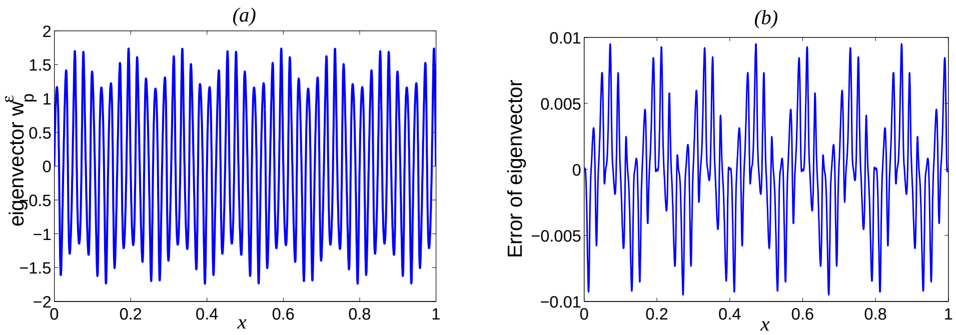

Figure 2.4: (a) Physical eigenmode w_p^ε. (b) Relative error between w_p^ε and $\psi_{n,\ell}^{\varepsilon,k}$.

After presenting a detailed study of the approximation of a given physical mode, i.e. for a single physical mode index p, we report approximation results for the list $\mathcal{J}_0^\varepsilon = \{40, ..., 150\} \setminus \{50\}$ of consecutive physical mode indices. The list starts at $p = 40$ corresponding to an intermediary mode between the low frequency modes approximated by the classical homogenized method and the high frequency modes considered in this chapter. The index $p = 50$ is excluded from the list since the corresponding eigenvector is evanescent, and as such corresponds to an element of the boundary spectrum. The previous optimization has been applied to each p yielding errors plotted in logarithm scale in Figure 2.5 (a). The error bounds are $er_{value} \leq 6.10^{-3}$ and $er_{vector} \leq 8.10^{-2}$.

Globally, the errors start by growing before to decrease except around $p = 100$ where they exhibit a peak that we do not explain. Figure 2.5 (b) reports the corresponding macroscopic eigenvalues $\lambda^{1,\ell}$. Some of them are close to pairs (k, n) such that $\lambda^{1,\ell}$ vanishes as discussed in Remark 3.2.4; their relative errors on eigenvalues are

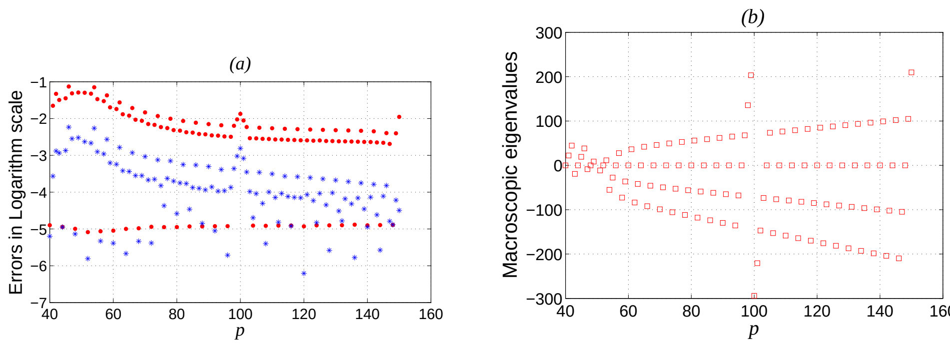

Figure 2.5: (a) Errors for p varying in $\mathcal{J}_0^\varepsilon$. (b) Macroscopic eigenvalues.

in the order of 10^{-5}. A way to answer the question in Remark 3.2.4 is to decrease the step Δ_k and see if all error decrease. A detailed presentation is made in Table 2.1 for two indices, namely $p = 66$ related to an eigenvalue in the beginning of the high frequency spectrum and $p = 102$ corresponding to one of the large errors. In both cases, the error diminishes as the step Δ_k is reduced from 8e-3 to 3e-3.

Table 2.1: Errors for $\Delta_k = 8.e - 3$ and $3e - 3$.

Δ_k	p	k	n	$\lambda^{1,\ell}$	er_{value}	er_{vector}
8.0e-3	66	2.16e-1	2	-92	1.2e-3	1.9e-2
3.0e-3	66	3.4e-1	2	21.7	9.0e-5	5.3e-3
8.0e-3	102	4.0e-2	3	-147	4.0e-4	5.8e-3
3.0e-3	102	1.5e-2	3	35.9	3.0e-5	1.4e-3

Figure 2.6 (a) is a global view of the errors in logarithm scale when $\Delta_k = 8.e - 3$ for $90 \leq p \leq 110$. It shows that for this k-step a large part of the errors on eigenvalues is in the range of 1.0e-5 i.e. almost the roundoff error. A measure of the error reduction is provided in Figure 2.6 (b) where the two ratios

$$E_{value} = \frac{er_{value}^{\Delta_k=3.e-3}}{er_{value}^{\Delta_k=8.e-3}} \text{ and } E_{vector} = \frac{er_{value}^{\Delta_k=3.e-3}}{er_{vector}^{\Delta_k=8.e-3}}$$

of error reduction are represented in logarithmic scale.

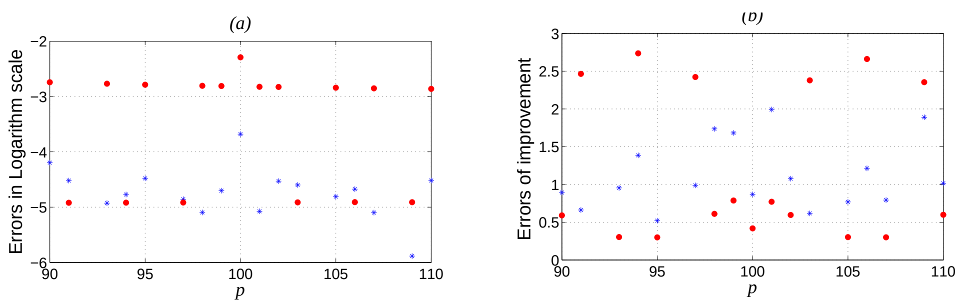

Figure 2.6: (a) Error of approximation for $\Delta_k = 3.0e-3$. (b) Ratios E_{value} and E_{vector} of error reduction.

2.5.3 The modeling problem

The modeling problem is reciprocal to the previous one. It consists in fixing a period ε as well as the parameters (k, n) of a Bloch mode and to search if there exists $\ell \in J_n^k$ such that $(\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ is close from a physical mode or in other words if it is almost a solution to the physical spectral problem i.e. if

$$\varepsilon^2 P^\varepsilon \psi_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} = O(\varepsilon) \text{ in } \Omega. \tag{2.74}$$

Posing for $\ell \in J_n^k$,

$$F_n^{\varepsilon,k}(\ell) = \frac{\left\| \varepsilon^2 P^\varepsilon \psi_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} \right\|_{L^2(\Omega)}}{\left\| \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} \right\|_{L^2(\Omega)}} \tag{2.75}$$

the modeling problem relies to the minimization problem $F_n^{\varepsilon,k}(\ell_0) = \min_{\ell \in J_n^k} F_n^{\varepsilon,k}(\ell)$. If the minimum is small enough, $(\gamma_{n,\ell_0}^{\varepsilon,k}, \psi_{n,\ell_0}^{\varepsilon,k})$ is close from a physical eigenelement and it is a solution to the modeling problem. A subsequent problem is to identify the corresponding physical eigenelement. This is done be minimizing the errors er_{value} and er_{vector} introduced in the previous section but considered as depending on the parameter $p \in \mathcal{J}^\varepsilon$ instead of k. Two illustrative examples are reported in Table 2.2, one yielding $\lambda^{1,\ell} = 0$ and the other $\lambda^{1,\ell} \neq 0$. The solution $\psi_{n,\ell}^{\varepsilon,k}$ and the relative error between $\psi_{n,\ell}^{\varepsilon,k}$ and w_p^ε are reported in Figures 2.7 (a) and (b).

Table 2.2: Results for the modeling problem.

k	n	λ_n^k	$F_n^{\varepsilon,k}(\ell)$	$\lambda^{1,\ell}$	p	er_{value}	er_{vector}
1.6e-1	2	5.11e1	8.9e-3	0	84	3.4e-5	2.1e-5
3.52e-1	2	3.14e1	4.5e-2	-8.55	65	1.5e-2	4.3e-3

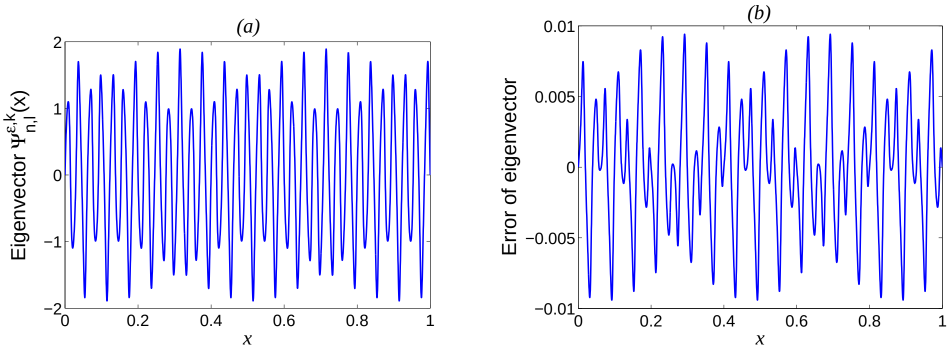

Figure 2.7: (a) Two-scale eigenmode $\psi_{n,\ell}^{\varepsilon,k}$. (b) Relative error vector between $\psi_{n,\ell}^{\varepsilon,k}$ and w_p^ε.

Additional results for $k = 3.52e-1$ with $n = \{1, ..., 15\}$ are reported in Figures 2.8 (a) and (b) showing $\lambda^{1,\ell}$ and $\gamma_{n,\ell}^k$ respectively.

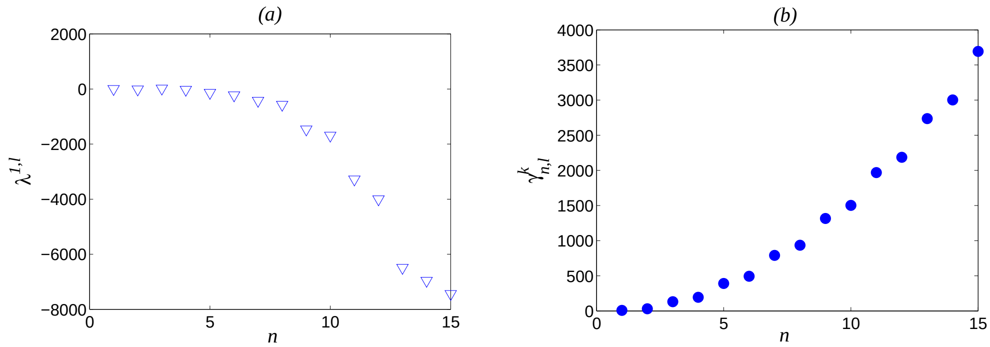

Figure 2.8: (a) $\lambda^{1,\ell}$ with respect to n. (b) $\gamma^k_{n,\ell}$ with respect to n.

2.5.4 Order of convergence

For a given pair k and $n \in J^k$, we investigate the order of convergence of the errors er_{value} and er_{vector} when the number of cells increases. To follow the convergence result, the sequence of periods ε is in fact a subsequence ε_h satisfying

$$\frac{1}{\varepsilon_h} = \frac{h+l}{k} \in \mathbb{N}^*$$

with $l \in [0,1)$ and for a sequence of $h \in \mathbb{N}^*$. Table 2.3 summarizes the results for $k = 0.3$, $l = 0.6$ and $h \in \{3, 9, 15, 21\}$.

Table 2.3: Errors for a decreasing subsequence ε_h.

h	ε_h	$er^{h,\ell}_{value}$	$er^{h,l}_{vector}$	p
3	8.3e-2	4.3e-2	6.3e-3	17
9	3.1e-2	1.6e-2	2.4e-3	45
15	1.9e-2	1.0e-2	1.5e-3	73
21	1.4e-2	7.0e-3	1.0e-3	101

To evaluate the decay rate of the errors, we pose $er^{h,\ell}_{value} = c_{value}\left(\varepsilon_h\right)^{q_{value}}$ and $er^{h,\ell}_{vector} = c_{vector}\left(\varepsilon_h\right)^{q_{vector}}$, so the decay rates satisfy

$$q_{value} = \frac{\log\left(er^{h,\ell}_{value}/er^{h',\ell}_{value}\right)}{\log\left(\varepsilon_h/\varepsilon_{h'}\right)} \text{ and } q_{vector} = \frac{\log\left(er^{h,\ell}_{vector}/er^{h',\ell}_{vector}\right)}{\log\left(\varepsilon_h/\varepsilon_{h'}\right)}.$$

Using successive results for h and h', yields

$$q_{value} = \{0.988,\ 0.995,\ 0.985\} \approx 1 \text{ and } q_{vector} = \{0.985,\ 0.993,\ 0.994\} \approx 1$$

with coefficients

$$c_{value} = \{0.504,\ 0.518,\ 0.497\} \approx 0.5 \text{ and } c_{vector} = \{0.0734,\ 0.0755,\ 0.0757\} \approx 0.07.$$

Chapter 3

Homogenization of the one-dimensional wave equation

Contents

Abstract In this chapter, we present a method for periodic homogenization of the one-dimensional wave equation in a bounded domain. It allows modelling both the low and high frequency waves. The high frequency model part includes oscillations occurring at the microscopic scale which amplitudes are governed by a well posed hyperbolic system of macroscopic equation. This model was already presented in [36] but for entire n-dimensional spaces, so the formulation of the boundary conditions were left as an open problem.

3.1 Introduction

We establish a homogenized model for the wave equation,

$$\begin{aligned} &\rho^{\varepsilon}\partial_{tt}u^{\varepsilon} - \partial_x\left(a^{\varepsilon}\partial_x u^{\varepsilon}\right) = f^{\varepsilon},\\ &u^{\varepsilon}\left(t=0,x\right) = u_0^{\varepsilon} \text{ and } \partial_t u^{\varepsilon}\left(t=0,x\right) = v_0^{\varepsilon}, \end{aligned} \tag{3.1}$$

posed in a finite time interval $I \subset \mathbb{R}^+$ and in an one-dimensional open bounded domain $\Omega \subset \mathbb{R}$ with Dirichlet boundary conditions. An asymptotic analysis of this problem is carried out where $\varepsilon > 0$ is a parameter tending to zero and the time-independent coefficients are $\varepsilon-$periodic, namely $a^{\varepsilon} = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^{\varepsilon} = \rho\left(\frac{x}{\varepsilon}\right)$ where $a\left(y\right)$ and $\rho\left(y\right)$ are $Y-$periodic with respect to a lattice of reference cell $Y \subset \mathbb{R}$.

In conclusion, the physical solution u^{ε} is approximated by a sum of a low frequency term u^0, the usual corrector in elliptic problems, using θ the solution of the cell problem, and a sum of Bloch waves being the corrector for the high frequency part,

$$u^{\varepsilon}\left(t,x\right) \approx u^0\left(t,x\right) + \varepsilon\theta\left(\frac{x}{\varepsilon}\right)\partial_x u^0\left(t,x\right) + \varepsilon\sum_k\sum_{n\in\mathbb{Z}^*} u_n^k\left(t,x\right)e^{sign(n)i\sqrt{\lambda^k_{|n|}}t/\varepsilon}\phi^k_{|n|}\left(\frac{x}{\varepsilon}\right).$$

The Bloch wave amplitudes $(u_n^k)_{n\in\mathbb{Z}^*}$ are solution of a first order system of differential equations constituting the high frequency macroscopic problem. In particular, for $k \in Y^*\backslash\left\{0,-\frac{1}{2}\right\}$ and for each n, the HF-macroscopic model has the form

$$\begin{aligned} b\left(k,n,n\right)\partial_t u_n^k + c\left(k,n,n\right)\partial_x u_n^k &= F_n^k\\ b\left(-k,n,n\right)\partial_t u_n^{\sigma} + c\left(-k,n,n\right)\partial_x u_n^{-k} &= F_n^{-k} \text{ in } I\times\Omega, \end{aligned} \tag{3.2}$$

with some initial conditions, and boundary conditions on the form

$$u_n^k\left(t,x\right)\phi^k_{|n|}\left(0\right)e^{2i\pi\frac{l^k x}{\alpha}} + u_n^{-k}\left(t,x\right)\phi^{-k}_{|n|}\left(0\right)e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } I\times\partial\Omega. \tag{3.3}$$

We observe that the two partial differential equations in (3.2) are not coupled, the coupling being due to the boundary conditions only. For $k \in \left\{0,-\frac{1}{2}\right\}$, in the case of double eigenvalue $\lambda^k_{|n'|} = \lambda^k_{|n|}$, the model is also a pair of equations indexed by $q \in \{n,n'\}$,

$$\sum_{p\in\{n,n'\}} b\left(k,p,q\right)\partial_t u_p^k + c\left(k,p,q\right)\partial_x u_p^k = F_q^k \text{ in } I\times\Omega, \tag{3.4}$$

with some initial conditions, and for $k=0$ with the boundary conditions

$$u_n^0\left(t,x\right)\phi^0_{|n|}\left(0\right) + u^0_{n'}\left(t,x\right)\phi^0_{|n'|}\left(0\right) = 0 \text{ on } I\times\partial\Omega, \tag{3.5}$$

and otherwise for $k = -\frac{1}{2}$,

$$\begin{aligned} &\left(c\left(k,n,n\right)\phi^k_{|n'|}\left(0\right) - c\left(k,n',n\right)\phi^k_{|n|}\left(0\right)\right)u_n^k(t,x)\\ &+\left(c\left(k,n,n'\right)\phi^k_{|n'|}\left(0\right) - c\left(k,n',n'\right)\phi^k_{|n|}\left(0\right)\right)u^k_{n'}\left(t,x\right) = 0 \text{ on } I\times\partial\Omega. \end{aligned} \tag{3.6}$$

The main contribution of this work is the boundary conditions of the HF-macroscopic model. However, the HF-macroscopic model is also new since it differs from this in [36] derived from a first order system. Moreover, the proof has been simplified. We quote that all models and proofs have been written in one-dimension but they extend trivially to the general case, except what refers to the HF-macroscopic boundary conditions which remains an open question in higher dimension.

The same result is also established for the Neumann boundary conditions and also for a generalization of the wave equation taking into account a zero order term as well as first order time and space derivatives. Moreover, the homogenization is also presented under the first order formulation as in [35] and [36], then boundary conditions for the one-dimensional model of these works have been announced.

This chapter is organized as follows. Section 3.2 is devoted to the statement of the model and the main results. Section 3.3 includes the model derivation. These results are then established for Neumann boundary conditions and for a generalization of the wave equation in Section 3.4. The homogenization is presented under the first order formulation in Section 3.5. Finally, numerical examples are provided for the first order formulation in the last section.

3.2 Statement of the results for the wave equation

We consider $I = (0,T) \subset \mathbb{R}^+$ a finite time interval and $\Omega = (0,\alpha) \subset \mathbb{R}^+$ a space interval, whose boundary is denoted by $\partial\Omega$. As usual in homogenization papers, $\varepsilon > 0$ denotes a small parameter intended to go to zero. Two functions $(a^\varepsilon, \rho^\varepsilon)$ are assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \text{ and } \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right), \tag{3.7}$$

where $\rho \in L^\infty(\mathbb{R})$ and $a \in W^{2,\infty}(\mathbb{R})$ are both $Y-$periodic with respect to the reference cell $Y = (0,1)$. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions,

$$0 < \rho^0 \leq \rho \leq \rho^1 \text{ and } 0 < a^0 \leq a \leq a^1, \tag{3.8}$$

for some given strictly positive numbers ρ^0, ρ^1, a^0 and a^1. In addition, $a \in W^{1,\infty}(\mathbb{R})$ is applied for the model based on the first order formulation in Section 3.5. We consider $u^\varepsilon(t,x)$ solution to the weak form of the wave equation with the source term $f^\varepsilon \in L^2(I \times \Omega)$, initial conditions $u_0^\varepsilon \in H^1(\Omega)$, $v_0^\varepsilon \in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions,

$$\begin{array}{l} \rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x\left(a^\varepsilon \partial_x u^\varepsilon\right) = f^\varepsilon \text{ in } I \times \Omega, \\ u^\varepsilon(t=0,x) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon \text{ in } \Omega, \\ u^\varepsilon = 0 \text{ on } I \times \partial\Omega. \end{array} \tag{3.9}$$

Assuming that the data are bounded,

$$\|v_0^\varepsilon\|_{L^2(\Omega)} + \|u_0^\varepsilon\|_{H^1(\Omega)} + \|f^\varepsilon\|_{L^2(I\times\Omega)} \leq c_0, \tag{3.10}$$

the uniform bound

$$\|\partial_t u^\varepsilon\|_{L^2(I\times\Omega)}, \|\partial_x u^\varepsilon\|_{L^2(I\times\Omega)} \leq c_1 \tag{3.11}$$

holds, see e.g. Theorem 3 in [36].

Remark 27 *The optimal regularity of the coefficient and of the solution is not in the focus on this work, so the regularity imposed to the coefficient a is not optimal. We refer to the recent work [40] which derives part of our results with $a \in L^\infty(Y)$ only, by being more careful on the manner to conduct the derivations. We do not see any obstacle to get our results with the same regularity.*

3.2.1 Assumptions

In the statement of the results, the assumptions on the data are expressed using the following definitions of two-scale transform when the first order approximation of the solution uses the operator $\mathfrak{B}_n^k$ defined in (1.20). The reason is that the latter yields approximations satisfying the periodicity or quasi-periodicity conditions, without further transformation. According to definition (1.20) of the operator $\mathfrak{B}_n^k$, it allows for the following definition of a generalization of the two-scale convergence of [89], [90] and [1] of a sequence $(u^\varepsilon)_\varepsilon$ defined in $I \times \Omega$ to a limit $u_n^{0,k}$ defined in $I \times \Lambda \times \Omega \times Y$ by

$$\int_{I\times\Omega} u^\varepsilon(t,x) \cdot (\mathfrak{B}_n^k \varphi)(t,x)\ dtdx \tag{3.12}$$
$$= \int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k}(t,\tau,x,y) \cdot \varphi(t,\tau,x,y)\ dtd\tau dxdy + O(\varepsilon)$$

for any $\varphi \in C^1(I \times \Lambda \times \Omega \times Y)$ being k-quasi-periodic in y and periodic in τ. We shall say that $u_n^{0,k}$ is a (n,k)-*mode two-scale* approximation of u^ε, since it relies to the Bloch mode ϕ_n^k, and denote it by

$$u^\varepsilon =^{WTS(n,k)} u_n^{0,k} + O(\varepsilon). \tag{3.13}$$

Similarly we define the first order (n,k)-mode wave-two-scale approximation $u_n^{0,k}(t,\tau,x,y) + \varepsilon u_n^{1,k}(t,\tau,x,y)$ of u^ε by

$$\int_{I\times\Omega} u^\varepsilon(t,x) \cdot \left(\mathfrak{B}_n^k \varphi\right)(t,x)\ dtdx \tag{3.14}$$
$$= \int_{I\times\Lambda\times\Omega\times Y} \left(u_n^{0,k}(t,\tau,x,y) + \varepsilon u_n^{1,k}(t,\tau,x,y)\right) \cdot \varphi(t,\tau,x,y)\ dtd\tau dxdy + \varepsilon O(\varepsilon)$$

for any $\varphi \in C^2\left(\Lambda \times Y; C^2(I \times \Omega) \cap C_c^0(I \times \Omega)\right)$ being periodic in τ that we denote (3.14)

$$u^\varepsilon =^{WTS(n,k)} u_n^{0,k} + \varepsilon u_n^{1,k} + \varepsilon O(\varepsilon). \tag{3.15}$$

We also require the so-called *wave-two-scale* approximation of u^ε towards u^0 by

$$\int_{I\times\Omega} u^\varepsilon(t,x) \cdot \sum_{k\in L_K^*} \sum_{n\in M^k} \left(\mathfrak{B}_n^k \Pi_n^k \varphi\right)(t,x)\ dtdx \tag{3.16}$$
$$= \int_{I\times\Lambda\times\Omega\times Y_K} u^0(t,\tau,x,y) \cdot \varphi(t,\tau,x,y)\ dtd\tau dxdy + O(\varepsilon)$$

for any $\varphi \in C^1(I \times \Lambda \times \Omega \times Y)$ being periodic in τ and denote it by

$$u^\varepsilon =^{WTS} u^0 + O(\varepsilon).$$

Finally, the first order wave-two-scale approximation of u^ε by $u^0(t,\tau,x,y) + \varepsilon u^1(t,\tau,x,y)$ satisfies

$$\int_{I\times\Omega} u^\varepsilon(t,x) \cdot \sum_{k\in L_K^*} \sum_{n\in M^k} \left(\mathfrak{B}_n^k \Pi_n^k \varphi\right)(t,x)\ dtdx \tag{3.17}$$
$$= \int_{I\times\Lambda\times\Omega\times Y} \left(u^0(t,\tau,x,y) + \varepsilon u^1(t,\tau,x,y)\right) \cdot \varphi(t,\tau,x,y)\ dtd\tau dxdy + \varepsilon O(\varepsilon)$$

for any $\varphi \in \mathcal{C}^2(\Lambda \times Y; C^2(I \times \Omega) \cap C_c^0(I \times \Omega))$ being periodic in τ, and is denoted by

$$u^\varepsilon =^{WTS} u^0 + \varepsilon u^1 + \varepsilon O(\varepsilon).$$

Remark 28 *Instead an assumption like (3.15), we could carry on the proof with the usual ansatz*

$$u^\varepsilon(t,x) = u^0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^2\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon^2 O(\varepsilon) \tag{3.18}$$

or to use a usual weak convergence approach. In this work, we adopt an intermediary method based on a solution expansion, to simplify the derivation, but expressed as an approximation in a weak sense keeping the essential idea lying in the convergence proof. Another reason for this choice is that we expect to use the model derivation in the context of automatic model derivation as in [25] which requires only computational steps and forbid abstract reasoning.

We already have assumed that f^ε, u_0^ε and v_0^ε are bounded in $L^2(I \times \Omega)$, $H^1(\Omega)$ and $L^2(\Omega)$ respectively, so according to the two-scale convergence theory in [77], [80] or [46] and the boundness property of S_0^ε, there exist $f^0 \in L^2(I \times \Omega \times Y)$, $\widehat{h^0} \in L^2(\Omega)$ and $g^0 \in L^2(\Omega \times Y)$ such that, up to a subsequence ε, the data converge weakly according to

$$f^0 = \lim_{\varepsilon \to 0} S_0^\varepsilon f^\varepsilon \in L^2(I \times \Omega \times Y), \ \widehat{h^0} = \lim_{\varepsilon \to 0} S_0^\varepsilon u_0^\varepsilon \in L^2(\Omega), \tag{3.19}$$

$$g^0 = \lim_{\varepsilon \to 0} S_0^\varepsilon v_0^\varepsilon \in L^2(\Omega \times Y),$$

$$\text{and the averages } \widehat{f^0} = \int_Y f^0 dy \in L^2(I \times \Omega), \ \widehat{g^0} = \frac{1}{\widehat{\rho}} \int_Y g^0 \cdot \rho \, dy \in L^2(\Omega). \tag{3.20}$$

These assumptions are the same as in [60], [33]. Moreover, in order to describe the HF-homogenized model, we assume that the additional weak convergences hold

$$g_n^k = \lim_{\varepsilon \to 0} \int_Y \frac{sign(n)\, i}{\sqrt{\lambda_n^k}} \partial_x S_k^\varepsilon u_0^\varepsilon \cdot \left(a \partial_y \phi_{|n|}^k + \partial_y \left(a \phi_{|n|}^k\right)\right) + S_k^\varepsilon v_0^\varepsilon \cdot \rho \phi_{|n|}^k dy \in L^2(\Omega), \tag{3.21}$$

$$\text{and } F_n^k = \lim_{\varepsilon \to 0} \int_{\Lambda \times Y} T^{\varepsilon \alpha_n^k} S_k^\varepsilon f^\varepsilon \cdot e^{sign(n) 2 i \pi \tau} \phi_{|n|}^k d\tau dy \in L^2(I \times \Omega),$$

for any $n \in \mathbb{Z}^*$. Finally, for each $k \in Y^*$ and $n \in M^k$, the sequence u^ε admits a first order (n,k)-mode two-scale approximation (3.15).

3.2.2 The model

In order to describe the LF-homogenized model, let us introduce the usual homogenized coefficients,

$$\widehat{a} = \int_Y a(1 + \partial_y \theta)(1 + \partial_y \theta) dy \text{ and } \widehat{\rho} = \int_Y \rho dy \tag{3.22}$$

where θ is a solution of the cell problem

$$\partial_y (a(\partial_y \theta + 1)) = 0 \text{ in } Y \text{ and } \theta \text{ is } Y - \text{periodic}. \tag{3.23}$$

Thus, the LF-homogenized equation states as in [36], [33],

$$\begin{gathered}\widehat{\rho}\partial_{tt}u^0 - \partial_x\left(\widehat{a}\partial_x u^0\right) = \widehat{f^0} \text{ in } I\times\Omega, \\ u^0(t=0) = \widehat{h^0} \text{ and } \partial_t u^0(t=0) = \widehat{g^0} \text{ in } \Omega, \\ u^0 = 0 \text{ on } I\times\partial\Omega.\end{gathered} \tag{3.24}$$

For $p, q \in \mathbb{Z}^*$ such that $\lambda^k_{|p|} = \lambda^k_{|q|}$, we introduce the coefficients

$$\begin{gathered}c(k,p,q) = \int_Y \phi^k_{|p|}\cdot a\partial_y\phi^k_{|q|} - a\partial_y\phi^k_{|p|}\cdot\phi^k_{|q|}dy \\ \text{and } b(k,p,q) = sign(p)\,2i\sqrt{\lambda^k_{|p|}}\int_Y \rho\phi^k_{|p|}\cdot\phi^k_{|q|}dy\end{gathered} \tag{3.25}$$

and observe that,

$$c(k,p,q) = \overline{c(-k,p,q)},\ c(k,q,p) = -\overline{c(k,p,q)},\ c(k,p,q) = -c(-k,q,p)\,.$$

In particular, $c(0,p,p) = 0$. Before to state the main result, the HF-macroscopic model is stated in all possible cases of k and of multiplicity of the Bloch eigenvalues.

A. $k \notin \left\{0, -\frac{1}{2}\right\}$ and

$$c(k,n,n) \neq 0 \text{ and } \phi^k_{|n|}(0) \neq 0 \text{ for all } n \in \mathbb{Z}^*. \tag{3.26}$$

The solutions of the HF-macroscopic model are the family of pairs $(u^k_n, u^{-k}_n)_{n\in\mathbb{Z}^*}$ solution to the system of equations where $\sigma \in \{-k,k\}$,

$$b(\sigma,n,n)\,\partial_t u^\sigma_n + c(\sigma,n,n)\,\partial_x u^\sigma_n = F^\sigma_n \text{ in } I\times\Omega \tag{3.27}$$

with the initial condition

$$b(\sigma,n,n)\,u^\sigma_n(t=0) = g^\sigma_n \text{ in } \Omega, \tag{3.28}$$

and the boundary condition,

$$u^k_n(t,x)\,\phi^k_{|n|}(0)\,e^{2i\pi\frac{l^k x}{\alpha}} + u^{-k}_n(t,x)\,\phi^{-k}_{|n|}(0)\,e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } I\times\partial\Omega. \tag{3.29}$$

We observe that the couple of partial differential equations (3.27) are not coupled, the coupling being due to the boundary conditions only.
If $c(k,n,n) = 0$ then whatever the value of $\phi^k_{|n|}(0)$, the HF-macroscopic equation (3.27) is replaced by

$$b(\sigma,n,n)\,\partial_t u^\sigma_n = F^\sigma_n \text{ in } I\times\Omega \text{ for } \sigma\in\{-k,k\} \tag{3.30}$$

which does not require any boundary condition.
If $\phi^k_{|n|}(0) = 0$ then whatever the value of $c(k,n,n)$, the k-quasi-periodicity implies that $\phi^k_{|n|}(1) = 0$ which says that $\phi^k_{|n|}$ is periodic. This case is covered by the case $k = 0$ and can be ignored when $k \neq 0$.

B. $k \in \left\{0, -\frac{1}{2}\right\}$, each eigenvalue λ^k_n is double, and (3.26), (3.31)

$$c(0,n,n') \neq 0 \text{ and } \phi^k_{|n|}(0) \neq 0 \text{ or } \phi^k_{|n'|}(0) \neq 0 \tag{3.31}$$

where n' is the index of the second eigenvalue $\lambda^k_{|n'|} = \lambda^k_{|n|}$. Each pair of the family $(u^k_n, u^k_{n'})_{n\in\mathbb{Z}^*}$ is solution to the system of first order boundary value problems where $q \in \{n, n'\}$,

$$\sum_{p\in\{n,n'\}} b\,(k,p,q)\,\partial_t u^k_p + c\,(k,p,q)\,\partial_x u^k_p = F^k_q \text{ in } I\times\Omega, \tag{3.32}$$

with initial condition

$$\sum_{p\in\{n,n'\}} b\,(k,p,q)\,u^k_p\,(t=0) = g^k_q \text{ in } \Omega. \tag{3.33}$$

The boundary condition is for $k=0$,

$$-\sum_{p\in\{n,n'\}} u^0_p\,(t,x)\,\phi^0_{|p|}\,(0) = 0 \text{ on } I\times\partial\Omega. \tag{3.34}$$

and for $k=-\frac{1}{2}$,

$$\begin{aligned}&\left(c\,(k,n,n)\,\phi^k_{|n'|}\,(0) - c\,(k,n',n)\,\phi^k_{|n|}\,(0)\right) u^k_n(t,x) \\ &+\left(c\,(k,n,n')\,\phi^k_{|n'|}\,(0) - c\,(k,n',n')\,\phi^k_{|n|}\,(0)\right) u^k_{n'}\,(t,x) = 0 \text{ on } I\times\partial\Omega.\end{aligned} \tag{3.35}$$

For $k=0$, if $c\,(0,n,n')=0$, whatever the values of $\phi^0_{|n|}\,(0)$ and $\phi^0_{|n'|}\,(0)$, then the HF-macroscopic model (3.32) is replaced by

$$\sum_{p\in\{n,n'\}} b\,(k,p,q)\,\partial_t u^0_p = F^0_q \text{ in } I\times\Omega \quad \text{for } q\in\{n,n'\} \tag{3.36}$$

and the boundary condition (3.34) does not apply.
Still for $k=0$, if $\phi^0_{|n|}(0) = \phi^0_{|n'|}(0) = 0$, whatever the values of $c\,(0,n,n')$, the boundary condition (3.34) does not apply.
Finally, for $k\in\left\{0,-\frac{1}{2}\right\}$ but if the eigenvalue $\lambda^k_{|n|}$ is simple, then the condition (3.31) does not apply and for $k=0$ the HF-macroscopic equation (3.32) is replaced by

$$b\,(0,n,n)\,\partial_t u^0_n = F^0_n \text{ in } I\times\Omega \text{ with } b\,(0,n,n)\,u^0_n\,(t=0) = g^0_n \text{ in } \Omega \tag{3.37}$$

without boundary condition, when for $k=-\frac{1}{2}$,

$$\begin{gathered} b\,(k,n,n)\,\partial_t u^k_n + c\,(k,n,n)\,\partial_x u^k_n = F^k_n \text{ in } I\times\Omega, \\ \text{with } b\,(k,n,n)\,u^k_n\,(t=0) = g^k_n \text{ in } \Omega,\end{gathered} \tag{3.38}$$

and without boundary condition if $c\,(k,n,n)=0$ whatever the values of $\phi^k_{|n|}\,(0)$ or $u^k_n = 0$ on $I\times\partial\Omega$ if $\phi^k_{|n|}\,(0)=0$ whatever the values of $c\,(k,n,n)$.

3.2.3 Approximation result

Theorem 29 *For any fixed $K\in\mathbb{N}^*$ and any bounded data as in (3.10), let u^ε be solution of the weak formulation of the wave equation (3.9) satisfying the uniform bound (3.11) and the assumption (3.15), then there exists u^0 in $H^1(I\times\Omega)$ and a family $(u^k_n)_{k\in L^*_K,n\in\mathbb{Z}^*}$ in $L^2(I\times\Omega)$ such that*

$$\begin{aligned} u^\varepsilon\,(t,\tau,x,y) &\overset{WTS}{=} u^0\,(t,x) + \varepsilon\theta\,(y)\,\partial_x u^0\,(t,x) \\ &+\varepsilon \sum_{k\in L^*_K,n\in\mathbb{Z}^*} u^k_n\,(t,x)\,e^{sign(n)2i\pi\tau}\phi^k_{|n|}\,(y) + \varepsilon O(\varepsilon).\end{aligned} \tag{3.39}$$

Moreover, if (3.19) satisfies then u^0 is the solution of the weak formulation of the LF-homogenized equation (3.24).

Finally, if $\varepsilon \in E_{1/K}$ and if for any $k \in L_K^*$ (3.21) is fulfilled and $u_n^k \in H^1(I \times \Omega)$ then the latter is the solution of the HF-macroscopic model (3.27)-(3.30), (3.32)-(3.38).

As a consequence, the two-scale structure of the solution u^ε including the correctors is

$$u^\varepsilon(t,x) \approx u^0(t,x) + \varepsilon\theta\left(\frac{x}{\varepsilon}\right)\partial_x u^0(t,x) + \varepsilon \sum_{k\in L_K^*, n\in\mathbb{Z}^*} u_n^k(t,x)\, e^{sign(n)i\sqrt{\lambda_{|n|}^k}t/\varepsilon}\phi_{|n|}^k\left(\frac{x}{\varepsilon}\right). \tag{3.40}$$

Remark 30 *To improve the asymptotic expansion (3.40) of u^ε near the boundary of Ω, we usually introduce a boundary layer term to compensate the lack of zero boundary condition of the LF-term $\theta\left(\frac{x}{\varepsilon}\right)\partial_x u^0(t,x)$, see e.g. [17], [16], [15], [107], [82], [4], [62] for elliptic problems, but the same equation holds for the wave equation. Furthermore, the boundary conditions (3.29) and (3.34) of the HF-model can be built independently. They can be derived directly by retaining only the eigenmodes corresponding to a same eigenvalue in the HF-decomposition of (3.40), by using the condition $u^\varepsilon = 0$ at $\partial\Omega$ and by simplifying the time-dependent term:*

$$\begin{aligned}
\sum_{\sigma\in\{k,-k\}} u_n^\sigma(t,x)\,\phi_{|n|}^\sigma\left(\frac{x}{\varepsilon}\right) &= 0 \text{ for } k \notin \left\{0, -\frac{1}{2}\right\},\\
\sum_{p\in\{n,n'\}} u_p^k(t,x)\,\phi_{|p|}^k\left(\frac{x}{\varepsilon}\right) &= 0 \text{ for } k = \left\{-\frac{1}{2}, 0\right\}.
\end{aligned}$$

Using the equalities

$$\phi_{|m|}^\sigma\left(\frac{x}{\varepsilon}\right) = \phi_{|m|}^\sigma(0)\, e^{2i\pi\sigma\frac{x}{\varepsilon}} = \phi_{|m|}^\sigma(0)\, e^{sign(\sigma)2i\pi x\frac{h_\varepsilon^k + l_\varepsilon^k}{\alpha}} = \phi_{|m|}^\sigma(0)\, e^{sign(\sigma)2i\pi x\frac{l_\varepsilon^k}{\alpha}}$$

for $x \in \partial\Omega$, the convergence $l_\varepsilon^k \to l^k$ as in Assumption 14 for $k \neq 0$, and the periodicity of $\phi_{|m|}^k$ for $k = 0$ yields the HF-macroscopic boundary condition of the model.

Remark 31 *The solution (3.39) is $\Lambda-$periodic in τ but the physical solution (3.40) is $\varepsilon\alpha_n^k$-periodic which is due to the choice of the time-scaling in $\mathfrak{B}_n^k$.*

Remark 32 *Assumptions (3.12) and (3.14) can be replaced by approximations*

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,\tau,x,y) + O(\varepsilon)$$

and

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,\tau,x,y) + \varepsilon\overline{u}_n^{1,k}(t,\tau,x,y) + \varepsilon O(\varepsilon)$$

in $L^2(I \times \Lambda \times \Omega \times Y)$ weakly as proved in Lemma 33.

3.2.4 Analytic solutions for the homogeneous equation ($f^\varepsilon = 0$)

For $k \in Y^*$, $n \in \mathbb{Z}^*$ and $\rho = 1$, we solve the HF-macroscopic equation with non degenerate boundary conditions. We solve the HF-macroscopic equations by distinguishing two cases $k \notin \left\{0, -\frac{1}{2}\right\}$ and $k = 0$. Here we solve the HF-macroscopic equation under the matrix form (3.116) as in proof of Lemma 36 in Subsection 3.3.1. For the present, we consider $n > 0$, then another case is similar.

Case $k \notin \left\{0, -\frac{1}{2}\right\}$

For each $\sigma \in I^k$, we observe that

$$\begin{aligned} c(\sigma, n, n) &= \int_Y \phi_n^\sigma a \partial_y \overline{\phi_n^\sigma} - a \partial_y \phi_n^\sigma \overline{\phi_n^\sigma}\, dy \\ &= \int_Y \phi_n^\sigma a \partial_y \overline{\phi_n^\sigma} - \overline{a \partial_y \overline{\phi_n^\sigma} \phi_n^\sigma}\, dy \\ &= 2i \operatorname{Im}\left(\int_Y \phi_n^\sigma a \partial_y \overline{\phi_n^\sigma}\, dy\right), \end{aligned}$$

and introduce the matrices $C = diag\left(c(\sigma, n, n)\right)_\sigma$, $B = diag\left(b(\sigma, n, n)\right)_\sigma$, hence, the operator $C\partial_x(.)$ with domain

$$D = \left\{w \in L^2(\Omega)^2 \text{ such that } w \cdot \phi = 0 \text{ at } x \in \partial\Omega\right\}$$

is self-adjoint on $L^2(\Omega)^2$. Thus the solution U of (3.116) can be decomposed by

$$U(t, x) = \sum_{l \in \mathbb{N}^*} r_l(t) V_l(x)$$

where $\left(\lambda_l^1, V_l\right)$ are solution of the eigenvalue problem

$$C\partial_x V_l + \lambda_l^1 B V_l = 0 \text{ in } \Omega, \tag{3.41}$$

and r_l are solution of the equation

$$\partial_t r_l + \lambda_l^1 r_l = 0 \text{ in } I. \tag{3.42}$$

We pose $V_l = \begin{pmatrix} v_l^1 \\ v_l^2 \end{pmatrix}$ and from the assumption $c(k, n, n) \neq 0$, then the equation (3.41) is equivalent to

$$\left\{\begin{array}{c} \partial_x v_l^1 + \lambda_l^1 b(k, n, n)/c(k, n, n)\, v_l^1 = 0 \\ \partial_x v_l^2 + \lambda_l^1 b(-k, n, n)/c(-k, n, n)\, v_l^2 = 0. \end{array}\right. \tag{3.43}$$

The exact solutions of the equations (3.42) and (3.43) are,

$$r_l(t) = r_l(0)\, e^{-\lambda_l^1 t},$$
$$v_l^1(x) = v_l^1(0)\, e^{-\left[\lambda_l^1 b(k,n,n)/c(k,n,n)\right]x} \text{ and } v_l^2(x) = v_l^2(0)\, e^{-\left[\lambda_l^1 b(-k,n,n)/c(-k,n,n)\right]x}.$$

The boundary condition (3.29) is equivalent to,

$$v_l^1(0)\, \phi_{|n|}^k(0) + v_l^2(0)\, \phi_{|n|}^{-k}(0) = 0$$

and $v_l^1(0)\, e^{-\lambda_l^1 \alpha b(k,n,n)/c(k,n,n) + 2i\pi l^k} \phi_{|n|}^k(0) + v_l^2(0)\, e^{-\lambda_l^1 \alpha b(-k,n,n)/c(-k,n,n) - 2i\pi l^k} \phi_{|n|}^{-k}(0) = 0.$

Since $b(k,n,n) = b(-k,n,n)$, $c(k,n,n) = -c(-k,n,n)$ and from the first boundary condition, $-v_l^1(0)\,\phi_{|n|}^k(0) = v_l^2(0)\,\phi_{|n|}^{-k}(0)$, so

$$e^{-\lambda_l^1 \alpha b(k,n,n)/c(k,n,n)+2i\pi l^k} = e^{\lambda_l^1 \alpha b(k,n,n)/c(k,n,n)-2i\pi l^k} \text{ or } e^{-2\lambda_l^1 \alpha b(k,n,n)/c(k,n,n)+4i\pi l^k} = 1.$$

Therefore, the eigenvalues of (3.41) are

$$\lambda_l^1 = \frac{c(k,n,n)}{\alpha b(k,n,n)}\left(2i\pi l^k - il\pi\right) \text{ for } l \in \mathbb{Z}. \tag{3.44}$$

Furthermore, $-v_l^1(0)\,\phi_{|n|}^k(0) = v_l^2(0)\,\phi_{|n|}^{-k}(0)$ then $\frac{v_l^1(0)}{v_l^2(0)} = -\frac{\phi_{|n|}^{-k}(0)}{\phi_{|n|}^k(0)}$. Thus,

$$v_l^1(0) = -\frac{\phi_{|n|}^{-k}(0)}{\phi_{|n|}^k(0)} v_l^2(0) \text{ for any } v_l^2(0) \in \mathbb{C}.$$

Using the orthogonality of the eigenvectors V_l, the initial condition is equivalent to,

$$r_l(0)\int_\Omega v_l^1 \cdot v_l^1 + v_l^2 \cdot v_l^2 dx = \frac{1}{b(k,n,n)}\int_\Omega g_n^k \cdot v_l^1 + g_n^{-k} \cdot v_l^2\, dx.$$

Finally,

$$r_l(0) = \frac{1}{b(k,n,n)}\frac{\int_\Omega g_n^k \cdot v_l^1 + g_n^{-k} \cdot v_l^2\, dx}{\int_\Omega v_l^1 \cdot v_l^1 + v_l^2 \cdot v_l^2 dx} \text{ or } r_l(0) = \frac{1}{b(k,n,n)}\frac{\int_\Omega g_n^k \cdot v_l^1 + g_n^{-k} \cdot v_l^2\, dx}{\|V_l\|_{L^2(\Omega)}^2}.$$

Case $k = 0$

In this case, $\lambda_n^0 = \lambda_m^0$ denote the double eigenvalue and $b(0,n,m) = sign(n)\, 2i\sqrt{\lambda_n^0}$ if $n = m$ and $= 0$ otherwise. By posing $C = (c(0,p,q))_{p,q}$ and $B = (b(0,p,q))_{p,q}$, we know that $iC\partial_x(.)$ with domain D is self-adjoint on $L^2(\Omega)^2$. Thus, the solution U of (3.116) can be decomposed by

$$U(t,x) = \sum_{l\in\mathbb{N}^*} r_l(t)\, V_l(x)$$

where $\left(\lambda_l^1, V_l\right)$ are solution of the eigenvalue problem

$$iC\partial_x V_l + \lambda_l^1 iBV_l = 0 \text{ in } \Omega, \tag{3.45}$$

and r_l are solution of the equation

$$\partial_t r_l + \lambda_l^1 r_l = 0 \text{ in } I. \tag{3.46}$$

We pose $V_l = \begin{pmatrix} v_l^1 \\ v_l^2 \end{pmatrix}$ and remark that $b(0,n,n) = b(0,m,m)$, $c(0,n,m) = -c(0,m,n)$, the equation (3.41) is equivalent to

$$c(0,n,m)\,\partial_x v_l^2 + \lambda_l^1 b(0,n,n)\, v_l^1 = 0 \text{ and } -c(0,n,m)\,\partial_x v_l^1 + \lambda_l^1 b(0,n,n)\, v_l^2 = 0.$$

From the first equation $v_l^1 = -\frac{c(0,n,m)\partial_x v_l^2(x)}{\lambda_l^1 b(0,n,n)}$, the second equation becomes,

$$\partial_{xx} v_l^2 = -\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)}\right)^2 v_l^2.$$

Thus, the exact solutions are,

$$v_l^2(x) = d_1 \cos\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} x\right) + d_2 \sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} x\right)$$

for all d_1, d_2 are complex numbers. Therefore,

$$v_l^1(x) = \frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \left(-d_1 \sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} x\right) + d_2 \cos\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} x\right)\right).$$

Applying the boundary condition (3.29),

$$\frac{\lambda_l^1 b(0,n,n)\, d_2}{c(0,n,m)} \phi_n^0(0) + d_1 \phi_m^0(0) = 0$$

$$\text{and } \frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \left(-d_1 \sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right) + d_2 \cos\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right)\right) \phi_n^0(0)$$
$$+ d_1 \cos\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right) + d_2 \sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right) \phi_m^0(0) = 0.$$

According to the first condition, the second condition remains

$$\sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right) \left(\frac{-d_1 \phi_n^0(0)\, \lambda_l^1 b(0,n,n)}{c(0,n,m)} + d_2 \phi_m^0(0)\right) = 0.$$

Hence,

$$\sin\left(\frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha\right) = 0 \text{ or } \frac{\lambda_l^1 b(0,n,n)}{c(0,n,m)} \alpha = \pi + l\pi \text{ or } \lambda_l^1 = \frac{c(0,n,m)(\pi + l\pi)}{\alpha b(0,n,n)} \text{ for } l \in \mathbb{Z}$$

and

$$d_1 = -\frac{(\pi + l\pi)\, \phi_n^0(0)\, d_2}{\alpha \phi_m^0(0)} \text{ for any } d_2 \in \mathbb{C}, l \in \mathbb{Z}.$$

Moreover, the exact solution of (3.46) is,

$$r_l(t) = r_l(0)\, e^{-\lambda_l^1 t}$$

Using the orthogonality of the eigenvector V_l, the initial condition is equivalent to,

$$r_l(0) \int_\Omega v_l^1 \cdot v_l^1 + v_l^2 \cdot v_l^2 dx = \frac{1}{b(0,n,n)} \int_\Omega g_n^0 \cdot v_l^1 + g_m^0 \cdot v_l^2\, dx.$$

Finally,

$$r_l^1(0) = \frac{1}{b(0,n,n)} \frac{\int_\Omega g_n^0 \cdot v_l^1 + g_m^0 \cdot v_l^2\, dx}{\int_\Omega v_l^1 \cdot v_l^1 + v_l^2 \cdot v_l^2 dx} \text{ or } r_l^1(0) = \frac{1}{b(0,n,n)} \frac{\int_\Omega g_n^0 \cdot v_l^1 + g_m^0 \cdot v_l^2\, dx}{\|V_l\|_{L^2(\Omega)}^2}.$$

3.3 Model derivation

According to Remark 31, for each (k,n), a two-scale transform in time is defined from the time cells

$$D := \left\{\theta_\varepsilon = \varepsilon\alpha_n^k l + \varepsilon\alpha_n^k \Lambda \mid l \in \mathbb{Z}, \varepsilon\alpha_n^k l + \varepsilon\alpha_n^k \Lambda \subset I\right\}$$

together with a scaling of the time variable $t \mapsto \frac{t}{\varepsilon\alpha_n^k}$. This yields a microscopic time variable τ always belonging to Λ. This plays an important role in the derivation of forthcoming Lemma 35 and justifies the use of the operator $T^{\varepsilon\alpha_n^k}$ instead of T^ε with the consistent convention $T^{\varepsilon\alpha_n^k} := 1$ when $\alpha_n^k = \infty$.

The decomposition of the time-space-two-scale function $T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon$ is provided in the next lemma, which justify Remak 32.

Lemma 33 *For any $k \in Y^*$, $n \in M^k$, for a sequence u^ε uniformly bounded in $L^2(I \times \Omega)$ satisfying (3.13) then*

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,\tau,x,y) + O(\varepsilon) \tag{3.47}$$

in $L^2(I \times \Lambda \times \Omega \times Y)$ weakly. Moreover, if a sequence u^ε uniformly bounded in $L^2(I \times \Omega)$ satisfies (3.15) then

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,\tau,x,y) + \varepsilon\overline{u}_n^{1,k}(t,\tau,x,y) + \varepsilon O(\varepsilon) \tag{3.48}$$

in $L^2(I \times \Lambda \times \Omega \times Y)$ weakly with the relation

$$\overline{u}_n^{1,k}(t,\tau,x,y) := \varepsilon\left(y - \frac{1}{2}\right)\partial_x u_n^{0,k}(t,x) + \varepsilon\alpha_n^k\left(\tau - \frac{1}{2}\right)\partial_t u_n^{0,k}(t,x) + \varepsilon u_n^{1,k}(t,\tau,x,y) \tag{3.49}$$

provided that $u_n^{0,k}$ is sufficiently regular.

Proof. [Proof of Lemma 33] For any $k \in Y^*$, $n \in M^k$, and a sequence u^ε uniformly bounded in $L^2(I\times\Omega)$, let $\varphi \in C^1(I \times \Lambda \times \Omega \times Y)$ a periodic function in τ and k-quasi-periodic function in y, according to (1.13) and (1.18) the definitions of the operators $T^{\varepsilon\alpha_n^k *}$ and $S_k^{\varepsilon *}$,

$$\int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon \cdot \varphi dt d\tau dx dy = \int_{I\times\Omega} u^\varepsilon \cdot T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\varphi dt dx.$$

Using the relation (1.21) between $\mathfrak{B}_n^k$ and $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}$,

$$= \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k \varphi dt dx + O(\varepsilon).$$

From (3.13), $u^\varepsilon =^{TSW(n,k)} u_n^{0,k} + O(\varepsilon)$,

$$= \int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k} \cdot \varphi dt d\tau dx dy + O(\varepsilon).$$

Therefore, the decomposition (3.47) follows.

Moreover, for $\varphi \in C^2(\Lambda \times Y; C^2(I\times\Omega) \cap C_c^0(I\times\Omega))$ a periodic function in τ and $k-$quasi-periodic function in y, according to (1.22) the first order approximation between $\mathfrak{B}_n^k$ and $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}$, we get

$$\int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k \varphi dt dx \tag{3.50}$$
$$= \int_{I\times\Omega} u^\varepsilon \cdot T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\left(\varphi + \varepsilon\alpha_n^k\left(\tau - \frac{1}{2}\right)\partial_t\varphi + \varepsilon\left(y-\frac{1}{2}\right)\partial_x\varphi\right) dtdx$$
$$= \int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon \cdot \left(\varphi + \varepsilon\alpha_n^k\left(\tau - \frac{1}{2}\right)\partial_t\varphi + \varepsilon\left(y-\frac{1}{2}\right)\partial_x\varphi\right) dtd\tau dxdy + \varepsilon O(\varepsilon).$$

From (3.47), we can decompose $T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon$ as

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,\tau,x,y) + \varepsilon\overline{u}_n^{1,k}(t,\tau,x,y) + \varepsilon O(\varepsilon)$$

in $L^2(I\times\Lambda\times\Omega\times Y)$ weakly. Furthermore, from (3.15), $u^\varepsilon =^{TSW(n,k)} u_n^{0,k} + \varepsilon u_n^{1,k} + \varepsilon O(\varepsilon)$, thus (3.50) yields

$$\int_{I\times\Lambda\times\Omega\times Y} \left(u_n^{0,k} + \varepsilon u_n^{1,k}\right)\cdot\varphi dtd\tau dxdy$$
$$= \int_{I\times\Lambda\times\Omega\times Y} \left(u_n^{0,k} + \varepsilon \overline{u}_n^{1,k}\right)\cdot\left(\varphi + \varepsilon\alpha_n^k\left(\tau - \frac{1}{2}\right)\partial_t\varphi + \varepsilon\left(y-\frac{1}{2}\right)\partial_x\varphi\right) dtd\tau dxdy + \varepsilon O(\varepsilon).$$

Assuming that $u_n^{0,k} \in H^1(I\times\Omega)$, taking the integration by parts and applying the conditions of φ on ∂I and $\partial\Omega$,

$$\int_{I\times\Lambda\times\Omega\times Y} \left(u_n^{0,k} + \varepsilon u_n^{1,k}\right)\cdot\varphi dtd\tau dxdy$$
$$= \int_{I\times\Lambda\times\Omega\times Y} \left(u_n^{0,k} + \varepsilon\overline{u}_n^{1,k} - \varepsilon\alpha_n^k\left(\tau-\frac{1}{2}\right)\partial_t u_n^{0,k} - \varepsilon\left(y-\frac{1}{2}\right)\partial_x u_n^{0,k}\right)\cdot\varphi dtd\tau dxdy + \varepsilon O(\varepsilon).$$

Or equivalently,

$$\int_{I\times\Lambda\times\Omega\times Y} \left(\varepsilon\overline{u}_n^{1,k} - \varepsilon\alpha_n^k\left(\tau-\frac{1}{2}\right)\partial_t u_n^{0,k} - \varepsilon\left(y-\frac{1}{2}\right)\partial_x u_n^{0,k} - \varepsilon u_n^{1,k}\right)\cdot\varphi dtd\tau dxdy = \varepsilon O(\varepsilon).$$

Finally, the decomposition (3.48) and the relation (3.49) follow. ■

For any $k \in Y^*$ and for each $n \in M^k$, let $u^\varepsilon \in L^2(I\times\Omega)$ satisfying the uniform bound (3.11), then the time-space-two-scale functions $T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon$ are bounded in $L^2(I\times\Lambda\times\Omega\times Y)$. According to (1.9) and (1.17),

$$T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_x u^\varepsilon = \frac{1}{\varepsilon}\partial_y\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right) \text{ and } T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_t u^\varepsilon = \frac{1}{\varepsilon\alpha_n^k}\partial_\tau\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)$$

where $\left\|T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_x u^\varepsilon\right\|_{L^2(I\times\Lambda\times\Omega\times Y)}$ and $\left\|T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_t u^\varepsilon\right\|_{L^2(I\times\Lambda\times\Omega\times Y)}$ are bounded thanks to (3.11) and the boundness of the two-scale operators S_k^ε and $T^{\varepsilon\alpha_n^k}$. Hence,

$$\begin{aligned} \left\|\partial_y\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)\right\|_{L^2(I\times\Lambda\times\Omega\times Y)} &= \varepsilon\left\|T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_x u^\varepsilon\right\|_{L^2(I\times\Lambda\times\Omega\times Y)} \\ \text{and } \left\|\partial_\tau\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)\right\|_{L^2(I\times\Lambda\times\Omega\times Y)} &= \varepsilon\alpha_n^k\left\|T^{\varepsilon\alpha_n^k} S_k^\varepsilon \partial_t u^\varepsilon\right\|_{L^2(I\times\Lambda\times\Omega\times Y)} \end{aligned} \tag{3.51}$$

tend to 0 when ε goes to 0. Thus, $u_n^{0,k}$ is independent on (τ, y), $u_n^{0,k} := u_n^{0,k}(t,x)$. Therefore, the decomposition (3.48) reads,

$$\left(T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon\right)(t,\tau,x,y) = u_n^{0,k}(t,x) + \varepsilon\overline{u}_n^{1,k}(t,\tau,x,y) + \varepsilon O(\varepsilon) \tag{3.52}$$

in the $L^2(I \times \Lambda \times \Omega \times Y)$ weak sense. In particular, for $k = 0$ and for any $n \in \mathbb{N}$, we know that $u_n^{0,0}$ is independent on n, see e.g. [46]. Then, there exists $u^0(t,x)$ such that

$$\lim_{\varepsilon\to 0} T^{\varepsilon\alpha_n^0} S_0^\varepsilon u^\varepsilon = u^0 \text{ in } L^2(I \times \Lambda \times \Omega \times Y) \text{ weakly} \tag{3.53}$$

for all $n \in M^0$.

Finally, for $k = 0$ and $n = 0$, $T^{\varepsilon\alpha_0^0} \equiv 1$ so the first order $(0,0)$-mode wave-two-scale approximation of u^ε is independent on τ. Thus, $u_0^{1,0}(t,x,y) = u_0^{1,0}(t,\tau,x,y)$ and (3.14) is rewritten by

$$\int_{I\times\Omega} u^\varepsilon(t,x) \cdot \left(\mathfrak{R}^0\varphi\right)(t,x)\ dtdx \tag{3.54}$$
$$= \int_{I\times\Omega\times Y} \left(u^0(t,x) + \varepsilon u_0^{1,0}(t,x,y)\right) \cdot \varphi(t,x,y)\ dtdxdy + \varepsilon O(\varepsilon)$$

for any $\varphi \in C^2(I \times Y; C^2(\Omega) \cap C_c^0(\Omega))$.

In order to prove the main result, we introduce some preliminary homogenized results including their proofs in Section 3.3.1 . Then, Theorem 29 is proved in Section 3.3.2.

3.3.1 Preliminary homogenization results and their proofs

Before to state the preliminary homogenized results, for $k \in Y^*$ and $n \in \mathbb{N}^*$, we pose

$$M_{n,\pm}^k = \{\pm n, \pm n'\} \text{ such that } \lambda_{n'}^k = \lambda_n^k \text{ for } k \in \{0, -\frac{1}{2}\} \tag{3.55}$$
$$\text{and } M_{n,\pm}^k = \{\pm n\} \text{ otherwise,}$$

$$M_{n,int}^k = \left\{ m \in \mathbb{Z}^* \text{ such that} \sqrt{\frac{\lambda_{|m|}^k}{\lambda_n^k}} \in \mathbb{N}^* \right\}, \tag{3.56}$$

In order to apply the assumptions (3.13), (3.15) of (n,k)-mode two-scale approximation of the wave solution u^ε, the proofs are always restarted with the very weak form of the wave equation (3.9).

The next lemma states the LF-part of the model from the (n,k)-modal two-scale approximations. Doing so, we recover the model of [33]. This was already done in [36] but in a different form since the calculation are done on a first order formulation of the wave equation.

Lemma 34 *For $k \in Y^*$, $n \in M^k$ and any bounded data as in (3.10), let u^ε be solution of the weak formulation of the wave equation (3.9) satisfying the uniform bound (3.11) and the assumption (3.14). Then,*

$$u_n^{0,k} = \chi_0(k)\, u^0, \tag{3.57}$$

and u^0 is the unique solution of the LF-homogenized model (3.24) and

$$u_0^{1,0}(t,x,y) = \partial_x u^0(t,x)\, \theta(y). \tag{3.58}$$

Proof. [Proof of Lemma 34] The proof is carried out in three steps. First, we prove that $u_n^{0,k} = 0$ if $k \neq 0$. Second, the two-scale model involving u^0 and $u_0^{1,0}$ is established. Then, the model (3.24) is derived thanks to (3.58).

i) For any $k \in Y^*$ and for each $n \in M^k$, we restart with the weak formulation of the wave equation (3.9) by choosing w decomposed as

$$w(t,\tau,x,y) = w_0(t,x) + \varepsilon w_1(t,\tau,x,y), \tag{3.59}$$

$$\begin{gathered}\text{with } w_0 \in C^\infty(I\times\Omega)\cap L^2\left(I;H_0^1(\Omega)\right) \text{ and } w_1 \in C^\infty(I\times\Lambda\times\Omega\times Y) \\ \cap L^2\left(I\times\Lambda\times\Omega;H_k^1(Y)\right)\cap L^2\left(I\times\Lambda;H_0^1\left(\Omega;L^2(Y)\right)\right)\cap L^2(L^2\left(I;H_\#^1(\Lambda)\right);\Omega\times Y) \\ \text{such that } w_0(t=T) = w_1(t=T) = 0 \text{ and } \partial_t w_0(t=T) = \partial_t w_1(t=T) = 0.\end{gathered} \tag{3.60}$$

Choosing $w^\varepsilon = \mathfrak{B}_n^k w$ as a test function,

$$w^\varepsilon \in H^2(I\times\Omega)\cap L^2\left(I;H_0^1(\Omega)\right),\ w^\varepsilon(t=T) = 0 \text{ and } \partial_t w^\varepsilon(t=T) = 0. \tag{3.61}$$

Applying two integrations by parts and the boundary conditions satisfied by u^ε and by $\mathfrak{B}_n^k w$, it remains,

$$\begin{aligned}&\int_{I\times\Omega} u^\varepsilon\cdot\left(Q^\varepsilon\left(\mathfrak{B}_n^k w\right)+P^\varepsilon\left(\mathfrak{B}_n^k w\right)\right) - f^\varepsilon\cdot\mathfrak{B}_n^k w\,dtdx \\ &+\int_\Omega u_0^\varepsilon\cdot\rho^\varepsilon\partial_t\left(\mathfrak{B}_n^k w\right)(t=0) - v_0^\varepsilon\cdot\rho^\varepsilon\mathfrak{B}_n^k w\,(t=0)\ dx = 0.\end{aligned} \tag{3.62}$$

According to (1.23),

$$\begin{aligned}&\int_{I\times\Omega} u^\varepsilon\cdot\mathfrak{B}_n^k\left(\sum_{l=0}^{2}\left(\left(\varepsilon\alpha_n^k\right)^{-l}Q^l w+\varepsilon^{-l}P^l w\right)\right) - f^\varepsilon\cdot\mathfrak{B}_n^k w\,dtdx \\ &+\int_\Omega u_0^\varepsilon\cdot\mathfrak{B}_n^k\rho\left(\partial_t w+\frac{1}{\varepsilon\alpha_n^k}\partial_\tau w\right)(t=0) - v_0^\varepsilon\cdot\mathfrak{B}_n^k\rho w\,(t=0)\ dx = 0.\end{aligned} \tag{3.63}$$

Moreover, from (1.4), (3.59), $\partial_\tau w_0 = \partial_y w_0 = 0$ and

$$\begin{aligned}&\sum_{l=0}^{2}\left(\left(\varepsilon\alpha_n^k\right)^{-l}Q^l w+\varepsilon^{-l}P^l w\right) = \rho\left(\partial_{tt}w_0+\frac{2}{\alpha_n^k}\partial_{t\tau}w_1+\frac{1}{\varepsilon\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1\right) \\ &-\left(\partial_x\left(a\partial_x w_0\right)+\partial_x\left(a\partial_y w_1\right)+\frac{1}{\varepsilon}\partial_y\left(a\partial_x w_0\right)+\partial_y\left(a\partial_x w_1\right)+\frac{1}{\varepsilon}\partial_y\left(a\partial_y w_1\right)\right),\end{aligned} \tag{3.64}$$

so, Equation (3.63) reads,

$$\begin{aligned}&\int_{I\times\Omega}[u^\varepsilon\cdot\mathfrak{B}_n^k[\rho(\partial_{tt}w_0+\frac{2}{\alpha_n^k}\partial_{t\tau}w_1+\frac{1}{\varepsilon\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1)-\partial_x\left(a\partial_x w_0\right)-\partial_x\left(a\partial_y w_1\right) \\ &\qquad -\frac{1}{\varepsilon}\partial_y\left(a\partial_x w_0\right)-\partial_y\left(a\partial_x w_1\right)-\frac{1}{\varepsilon}\partial_y\left(a\partial_y w_1\right)] - f^\varepsilon\cdot\mathfrak{B}_n^k w_0\,]dtdx \\ &+\int_\Omega\left(u_0^\varepsilon\cdot\mathfrak{B}_n^k\left(\rho\left(\partial_t w_0+\frac{1}{\alpha_n^k}\partial_\tau w_1\right)\right) - v_0^\varepsilon\cdot\mathfrak{B}_n^k\left(\rho w_0\right)\right)(t=0,\tau=0)\ dxdy = 0.\end{aligned} \tag{3.65}$$

Multiplying by ε, then using (3.12) and passing to the limit,

$$\int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k}\cdot\left(\rho\frac{1}{(\alpha_n^k)^2}\partial_{\tau\tau}w_1-\partial_y\left(a\partial_x w_0\right)-\partial_y\left(a\partial_y w_1\right)\right)dtd\tau dxdy=0. \tag{3.66}$$

Since $u_n^{0,k}$ is independent on (τ,y), w_1 is periodic in τ and a is periodic in y, so

$$\int_\Lambda u_n^{0,k}\cdot\rho\frac{1}{(\alpha_n^k)^2}\partial_{\tau\tau}w_1 d\tau=0 \text{ and } \int_Y u_n^{0,k}\cdot\partial_y\left(a\partial_x w_0\right)dy=0.$$

In equation (3.66), it remains,

$$\int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k}\cdot\partial_y\left(a\partial_y w_1\right)\,dtd\tau dxdy=0. \tag{3.67}$$

Or equivalently,

$$\int_{I\times\Omega} u_n^{0,k}\cdot\left(\int_\Lambda\left[a\partial_y w_1\right]_{y=0}^{y=1}d\tau\right)dtdx=0 \text{ for all } w_1 \text{ satisfying (3.60).}$$

Therefore w_1 is a periodic function in y or $u_n^{0,k}=0$ in $I\times\Omega$. It means that

$$u_n^{0,k}=0 \text{ in } I\times\Omega \text{ if } k\neq 0 \text{ or } u_n^{0,k}=\chi_0(k)\,u^0$$

where u^0 is introduced in (3.53).

ii) We restart with the very weak formulation (3.62) by choosing w decomposed as

$$w(t,x,y)=w_0(t,x)+\varepsilon w_{1,0}(t,x,y) \tag{3.68}$$

with

$$\begin{gathered} w_0\in C^\infty(I\times\Omega)\cap L^2\left(I;H_0^1(\Omega)\right) \text{ such that } w_0(t=T)=\partial_t w_0(t=T)=0 \\ \text{and } w_{1,0}\in C^\infty\left(I\times Y;C^\infty(\Omega)\cap H_0^1(\Omega)\right)\cap L^2\left(I\times\Omega;H_\#^1(Y)\right) \\ \text{such that } w_{1,0}(t=T)=\partial_t w_{1,0}(t=T)=0. \end{gathered} \tag{3.69}$$

Choosing $w^\varepsilon=\mathfrak{R}^0 w$ as a test function,

$$\begin{gathered} w^\varepsilon=\mathfrak{R}^0 w\in H^2(I\times\Omega)\cap L^2\left(I;H_0^1(\Omega)\right) \\ \text{such that } w^\varepsilon(t=T)=0 \text{ and } \partial_t w^\varepsilon(t=T)=0. \end{gathered} \tag{3.70}$$

The very weak formulation (3.62) yields,

$$\begin{aligned} &\int_{I\times\Omega} u^\varepsilon\cdot\left(Q^\varepsilon\left(\mathfrak{R}^0 w\right)+P^\varepsilon\left(\mathfrak{R}^0 w\right)\right)-f^\varepsilon\cdot\mathfrak{R}^0 w\,dtdx \\ +&\int_\Omega u_0^\varepsilon\cdot\rho^\varepsilon\partial_t\left(\mathfrak{R}^0 w\right)(t=0)-v_0^\varepsilon\cdot\rho^\varepsilon\mathfrak{R}^0 w\,(t=0)\,dx=0. \end{aligned}$$

From (1.23), (1.4), (3.68) and $\partial_\tau w=0$,

$$\begin{aligned} &\int_{I\times\Omega}[u^\varepsilon\cdot\mathfrak{R}^0[\rho\partial_{tt}w_0-\partial_x\left(a\partial_x w_0\right)-\partial_x\left(a\partial_y w_{1,0}\right)-\frac{1}{\varepsilon}\partial_y\left(a\partial_x w_0\right)-\partial_y\left(a\partial_x w_{1,0}\right) \quad (3.71)\\ &-\frac{1}{\varepsilon}\partial_y\left(a\partial_y w_{1,0}\right)]-f^\varepsilon\cdot\mathfrak{R}^0 w_0]dtdx+\int_\Omega u_0^\varepsilon\cdot\mathfrak{R}^0\rho\partial_t w_0(t=0)-v_0^\varepsilon\cdot\mathfrak{R}^0\rho w_0(t=0)\,dx=0. \end{aligned}$$

Choosing $w_0 = 0$, multiplying by ε and using (3.54), Equation (3.71) becomes

$$\int_{I\times\Omega\times Y} \left(u^0 + \varepsilon u_0^{1,0}\right) \cdot \left(-\varepsilon\partial_x \left(a\partial_y w_{1,0}\right) - \varepsilon\partial_y \left(a\partial_x w_{1,0}\right) - \partial_y \left(a\partial_y w_{1,0}\right)\right) dtdxdy = \varepsilon O\left(\varepsilon\right).$$

Observing that,

$$\int_Y u^0 \cdot \partial_y \left(a\partial_y w_{1,0}\right) dy = 0 \text{ and } \int_Y u^0 \cdot \partial_y \left(a\partial_x w_{1,0}\right) dy = 0,$$

hence, the equation yields,

$$\int_{I\times\Omega\times Y} -\varepsilon u^0 \cdot \partial_x \left(a\partial_y w_{1,0}\right) - \varepsilon u_0^{1,0} \cdot \partial_y \left(a\partial_y w_{1,0}\right) dtdxdy = \varepsilon O\left(\varepsilon\right).$$

Dividing by ε and passing to the limit,

$$\int_{I\times\Omega\times Y} u^0 \cdot \partial_x \left(a\partial_y w_{1,0}\right) + u_0^{1,0} \cdot \partial_y \left(a\partial_y w_{1,0}\right) \, dtdxdy = 0. \tag{3.72}$$

Assuming that $u^0 \in L^2\left(I; H^1\left(\Omega\right)\right)$, $u_0^{1,0} \in L^2\left(I \times \Omega; H^2\left(Y\right)\right)$, taking integrations by parts, using the boundary conditions $w_{1,0} = 0$ at $x \in \partial\Omega$ and the periodicity of $w_{1,0}$ in y, Equation (3.72) yields,

$$\int_{I\times\Omega\times Y} \left(\partial_y \left(a\partial_x u^0\right) + \partial_y \left(a\partial_y u_0^{1,0}\right)\right) \cdot w_{1,0} dtdxdy$$
$$+ \int_{I\times\Omega} \left[u_0^{1,0} \cdot a\partial_y w_{1,0} - a\partial_y u_0^{1,0} \cdot w_{1,0}\right]_{y=0}^{y=1} dtdx = 0.$$

By choosing the test function such that $w_{1,0}\left(t, x, .\right) \in H_0^2\left(Y\right)$ for all $(t, x) \in I \times \Omega$, the internal equation is stated as,

$$\partial_y \left(a\partial_x u^0\right) + \partial_y \left(a\partial_y u_0^{1,0}\right) = 0 \text{ in } I \times \Omega \times Y. \tag{3.73}$$

Thus, the boundary term remains,

$$\int_{I\times\Omega} \left[u_0^{1,0} \cdot a\partial_y w_{1,0} - a\partial_y u_0^{1,0} \cdot w_{1,0}\right]_{y=0}^{y=1} dtdx = 0.$$

Since $w_{1,0}$ is periodic in y so $\partial_y w_{1,0}$ is periodic in y. Therefore, $u_0^{1,0}$ and $\partial_y u_0^{1,0}$ are also periodic in y.

Moreover, by choosing $w_{1,0} = 0$ and multiplying by ε then Equation (3.71) is equivalent to

$$\int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{R}^0 \left[\varepsilon\rho\partial_{tt} w_0 - \varepsilon\partial_x \left(a\partial_x w_0\right) - \partial_y \left(a\partial_x w_0\right)\right] - \varepsilon f^\varepsilon \cdot \mathfrak{R}^0 w_0 dtdx$$
$$+ \int_\Omega \varepsilon u_0^\varepsilon \cdot \mathfrak{R}^0 \left(\rho\partial_t w_0\right)\left(t = 0\right) - \varepsilon v_0^\varepsilon \cdot \mathfrak{R}^0 \left(\rho w_0\right)\left(t = 0\right) \, dx = 0.$$

Using (3.54) and the data (3.19),

$$\int_{I\times\Omega\times Y} \left(u^0 + \varepsilon u_0^{1,0}\right) \cdot \left[\varepsilon\rho\partial_{tt} w_0 - \varepsilon\partial_x \left(a\partial_x w_0\right) - \partial_y \left(a\partial_x w_0\right)\right] - \varepsilon f^0 \cdot w_0 dtdxdy$$
$$+ \int_{\Omega\times Y} \varepsilon\widehat{h^0} \cdot \rho\partial_t w_0 \left(t = 0\right) - \varepsilon g^0 \cdot \rho w_0 \left(t = 0\right) \, dxdy = \varepsilon O\left(\varepsilon\right).$$

Remarking that $\int_Y u^0 \cdot \partial_y (a\partial_x w_0)\, dy = 0$, dividing by ε and passing to the limit,

$$\int_{I\times\Omega\times Y} u^0 \cdot \rho\partial_{tt} w_0 - u^0 \cdot \partial_x (a\partial_x w_0) - u_0^{1,0} \cdot \partial_y (a\partial_x w_0) - f^0 \cdot w_0 dtdxdy$$
$$+ \int_{\Omega\times Y} \widehat{h^0} \cdot \rho\partial_t w_0 (t=0) - g^0 \cdot \rho w_0 (t=0)\, dxdy = 0.$$

Assuming that $u^0 \in H^2(I\times\Omega)$ and $u_0^{1,0} \in L^2(I; H^1(\Omega\times Y))$, taking integrations by parts, using $w_0 \in L^2(I; H_0^1(\Omega))$ with $w_0(t=T) = 0$ and $\partial_t w_0 (t=T) = 0$, and periodicity of $u_0^{1,0}$, it remains,

$$\int_{I\times\Omega} ((\int_Y \rho dy)\partial_{tt} u^0 - \partial_x((\int_Y a dy)\partial_x u^0) - \partial_x(\int_Y a\partial_y u_0^{1,0} dy) - \int_Y f^0 dy) \cdot w_0 dtdx$$
$$+ \int_Y \rho dy \int_\Omega -u^0 \cdot \partial_t w_0 (t=0) + \partial_t u^0 \cdot w_0 (t=0)\, dx - \int_I \left[\left(\int_Y a dy\right) u^0 \cdot \partial_x w_0\right]_{x=0}^{x=\alpha} dt$$
$$+ \int_\Omega \left(\int_Y \rho dy\right) \widehat{h^0} \cdot \partial_t w_0 (t=0) - \left(\int_Y g^0 \cdot \rho dy\right) \cdot w_0 (t=0)\, dx = 0. \tag{3.74}$$

Choosing test functions $w_0 \in H_0^1(I\times\Omega)$, then the strong form comes

$$\left(\int_Y \rho dy\right) \partial_{tt} u^0 - \partial_x \left(\left(\int_Y a dy\right) \partial_x u^0\right) - \partial_x \left(\int_Y a\partial_y u_0^{1,0} dy\right) = \widehat{f_0} \text{ in } I\times\Omega, \tag{3.75}$$

So, Equation (3.74) remains,

$$\left(\int_Y \rho dy\right) \int_\Omega -u^0 \cdot \partial_t w_0 (t=0) + \partial_t u^0 \cdot w_0 (t=0)\, dx - \int_Y a dy \int_I \left[u^0 \cdot \partial_x w_0\right]_{x=0}^{x=\alpha} dt$$
$$+ \int_\Omega \left(\left(\int_Y \rho dy\right) \widehat{h^0} \cdot \partial_t w_0 - \left(\int_Y g^0 \cdot \rho dy\right) \cdot w_0\right) (t=0)\, dx = 0.$$

According to (3.20), the initial conditions are,

$$u^0 (t=0) = \widehat{h^0} \text{ and } \partial_t u^0 = \widehat{g^0} \text{ in } \Omega$$

and the boundary conditions are

$$u^0 = 0 \text{ on } I\times\partial\Omega.$$

iii) From (3.73), $u_0^{1,0}$ can be decomposed as

$$u_0^{1,0}(t,x,y) = \partial_x u^0 (t,x)\, \theta(y) \text{ with } \theta \in H_\#^2(Y). \tag{3.76}$$

After replacement, Equation (3.73) is equivalent to

$$(\partial_y (a\partial_y\theta) + \partial_y a)\, \partial_x u^0 = 0 \text{ in } I\times\Omega\times Y.$$

Without loss of generality, we consider $\partial_x u^0 \neq 0$. Therefore, θ is a solution of the cell equation (3.23). In addition, since (3.76) and

$$\int_Y a(1+\partial_y\theta)\, dy = \int_Y a(1+\partial_y\theta)\cdot(1+\partial_y\theta)\, dy - \int_Y a(1+\partial_y\theta)\cdot\partial_y\theta dy$$
$$= \int_Y a(1+\partial_y\theta)\cdot(1+\partial_y\theta)\, dy - \int_Y \partial_y (a(1+\partial_y\theta))\cdot\theta dy + [a(1+\partial_y\theta)\cdot\theta]_{y=0}^{y=1}$$
$$= \int_Y a(1+\partial_y\theta)\cdot(1+\partial_y\theta)\, dy = \widehat{a},$$

so,

$$\partial_x\left(\left(\int_Y a dy\right)\partial_x u^0\right)+\partial_x\left(\int_Y a\partial_y u_0^{1,0}dy\right)=\partial_x\left(\widehat{a}\partial_x u^0\right).$$

Thus, Equation (3.75) is equivalent to,

$$\widehat{\rho}\partial_{tt}u^0-\partial_x\left(\widehat{a}\partial_x u^0\right)=\widehat{f}^0 \text{ in } I\times\Omega.$$

■

This result shows that the LF-waves are related to $u_0^{0,0}$ only which is therefore not belonging to the HF-model. Therefore, the HF-waves are searched for $k\in Y^*$ and $n\in\mathbb{N}^*$. Let us define

$$\widehat{u}_n^{1,k}(t,\tau,x,y):=u_n^{1,k}(t,\tau,x,y)-\chi_0(k)\,u_0^{1,0}(t,x,y), \tag{3.77}$$

thus,

$$u_n^{1,k}(t,\tau,x,y)=\chi_0(k)\,u_0^{1,0}(t,x,y)+\widehat{u}_n^{1,k}(t,\tau,x,y). \tag{3.78}$$

Lemma 35 *For $k\in Y^*$, $n\in\mathbb{N}^*$ and any bounded data as in (3.10), let u^ε be the solution of the weak formulation of the wave equation (3.9) satisfying the uniform bound (3.11) and the assumption (3.14). Then $\widehat{u}_n^{1,k}(t,\tau,x,y)$ is solution of the HF-microscopic equation*

$$\left(\alpha_n^k\right)^{-2}\rho\partial_{\tau\tau}\widehat{u}_n^{1,k}-\partial_y\left(a\partial_y\widehat{u}_n^{1,k}\right)=0 \text{ in } I\times\Lambda\times\Omega\times Y \tag{3.79}$$
$$\text{where } \widehat{u}_n^{1,k} \text{ is periodic in } \tau \text{ and } k-\text{quasi-periodic in } y$$

in the very weak sense (i.e., it is solution of the very weak formulation (3.83)). Moreover, if $\widehat{u}_n^{1,k}\in L^2(I\times\Omega;H^2(\Lambda\times Y))$ then $\widehat{u}_n^{1,k}$ is solution of the HF-microscopic equation (3.79) and admits the modal decomposition,

$$\widehat{u}_n^{1,k}(t,\tau,x,y)=\sum_{m\in M_{n,int}^k}u_m^k(t,x)\,e^{sign(m)2i\pi\sqrt{\frac{\lambda_{|m|}^k}{\lambda_n^k}}\tau}\phi_{|m|}^k(y) \tag{3.80}$$

with $u_m^k(t,x)\in L^2(I\times\Omega)$.

Proof. [Proof of Lemma 35] For a given $k\in Y^*$ and $n\in\mathbb{N}^*$, we restart with the very weak formulation (3.62) in the proof of Lemma 34 by choosing test functions as in (3.59), (3.60), (3.61) but such that $w_0=0$ in $I\times\Omega$ and

$$w_1\in C^\infty(I\times\Omega\times\Lambda\times Y)\cap L^2\left(\Lambda\times Y;H_0^1(I\times\Omega)\right).$$

Multiplying by ε, Equation (3.65) becomes,

$$\int_{I\times\Omega}u^\varepsilon\cdot\mathfrak{B}_n^k[\rho(\varepsilon\frac{2}{\alpha_n^k}\partial_{t\tau}w_1+\frac{1}{(\alpha_n^k)^2}\partial_{\tau\tau}w_1)-\varepsilon\partial_x\left(a\partial_y w_1\right) \tag{3.81}$$
$$-\varepsilon\partial_y\left(a\partial_x w_1\right)-\partial_y\left(a\partial_y w_1\right)]\,dtdx=0.$$

Using (3.14) with $u_n^{0,k}=\chi_0(k)\,u^0$ as in Lemma 34 and remarking that,

$$\int_\Lambda\chi_0(k)\,u^0\cdot\partial_{t\tau}w_1d\tau=0,\ \int_\Lambda\chi_0(k)\,u^0\cdot\partial_{\tau\tau}w_1d\tau=0 \tag{3.82}$$
$$\text{and } \int_Y\chi_0(k)\,u^0\cdot\left(\partial_y\left(a\partial_y w_1\right)+\varepsilon\partial_y\left(a\partial_x w_1\right)\right)dy=0,$$

then dividing by ε and passing to the limit, Equation (3.81) yields,

$$\int_{I\times\Lambda\times\Omega\times Y} u_n^{1,k}\cdot\left(\frac{1}{\left(\alpha_n^k\right)^2}\rho\partial_{\tau\tau}w_1-\partial_y\left(a\partial_y w_1\right)\right)-\chi_0\left(k\right)u^0\cdot\partial_x\left(a\partial_y w_1\right)dtd\tau dxdy=0.$$

Using the decomposition (3.78) of $u_n^{1,k}$,

$$\begin{aligned}\int_{I\times\Lambda\times\Omega\times Y}&\left(\chi_0\left(k\right)u_0^{1,0}+\widehat{u}_n^{1,k}\right)\cdot\left(\frac{1}{\left(\alpha_n^k\right)^2}\rho\partial_{\tau\tau}w_1-\partial_y\left(a\partial_y w_1\right)\right)\\&-\chi_0\left(k\right)u^0\cdot\partial_x\left(a\partial_y w_1\right)dtd\tau dxdy=0.\end{aligned}$$

From (3.72), and because that $u_0^{1,0}$ is independent on τ and w_1 is periodic in τ,

$$\int_{I\times\Omega\times Y}u^0\cdot\partial_x\left(a\partial_y\int_\Lambda w_1d\tau\right)+u_0^{1,0}\cdot\partial_y\left(a\partial_y\int_\Lambda w_1d\tau\right)dtdxdy=0$$
$$\text{and}\int_\Lambda u_0^{1,0}\cdot\frac{1}{\left(\alpha_n^0\right)^2}\rho\partial_{\tau\tau}w_1d\tau=0.$$

Thus, $\widehat{u}_n^{1,k}$ is a solution of the very weak formulation

$$\int_{I\times\Lambda\times\Omega\times Y}\widehat{u}_n^{1,k}\cdot\left(\frac{1}{\left(\alpha_n^k\right)^2}\rho\partial_{\tau\tau}w_1-\partial_y\left(a\partial_y w_1\right)\right)dtd\tau dxdy=0.\qquad(3.83)$$

In addition, assuming that $\widehat{u}_n^{1,k}\in L^2\left(I\times\Omega;H^2\left(\Lambda\times Y\right)\right)$ and applying integrations by parts,

$$\begin{aligned}&\int_{I\times\Lambda\times\Omega\times Y}\left(\frac{1}{\left(\alpha_n^k\right)^2}\rho\partial_{\tau\tau}\widehat{u}_n^{1,k}-\partial_y\left(a\partial_y\widehat{u}_n^{1,k}\right)\right)\cdot w_1\,dtd\tau dxdy\qquad(3.84)\\&+\frac{1}{\left(\alpha_n^k\right)^2}\int_{I\times\Omega\times Y}\left[-\rho\widehat{u}_n^{1,k}\cdot\partial_\tau w_1+\rho\partial_\tau\widehat{u}_n^{1,k}\cdot w_1\right]_{\tau=0}^{\tau=1}dtdxdy\\&+\int_{I\times\Lambda\times\Omega}\left[\widehat{u}_n^{1,k}\cdot a\partial_y w_1-a\partial_y\widehat{u}_n^{1,k}\cdot w_1\right]_{y=0}^{y=1}dtd\tau dx=0.\end{aligned}$$

By choosing test function $w_1\in L^2\left(I\times\Omega\times Y;C_c^\infty\left(\Lambda\right)\right)\cap L^2\left(I\times\Lambda\times Y;C_c^\infty\left(Y\right)\right)$, the HF-microscopic equation associated to a value α_n^k is stated as,

$$\left(\alpha_n^k\right)^{-2}\rho\partial_{\tau\tau}\widehat{u}_n^{1,k}-\partial_y\left(a\partial_y\widehat{u}_n^{1,k}\right)=0\text{ in }I\times\Lambda\times\Omega\times Y.\qquad(3.85)$$

In Equation (3.84), it remains,

$$\begin{aligned}&\frac{1}{\left(\alpha_n^k\right)^2}\int_{I\times\Omega\times Y}\left[-\rho\widehat{u}_n^{1,k}\cdot\partial_\tau w_1+\rho\partial_\tau\widehat{u}_n^{1,k}\cdot w_1\right]_{\tau=0}^{\tau=1}dtdxdy\\&+\int_{I\times\Lambda\times\Omega}\left[\widehat{u}_n^{1,k}\cdot a\partial_y w_1-a\partial_y\widehat{u}_n^{1,k}\cdot w_1\right]_{y=0}^{y=1}dtd\tau dx=0.\end{aligned}$$

Since w_1 is periodic in τ and $k-$quasi-periodic in y, $\widehat{u}_n^{1,k}$and $\partial_\tau\widehat{u}_n^{1,k}$ are periodic in τ and $\widehat{u}_n^{1,k}$ and $\partial_y\widehat{u}_n^{1,k}$ are $k-$quasi-periodic in y.

Next, we notice that $\widehat{u}_n^{1,k}$ can be decomposed as

$$\widehat{u}_n^{1,k}\left(t,\tau,x,y\right)=\sum_{m\in\mathbb{N}^*}v_m^k\left(t,\tau,x\right)\phi_m^k\left(y\right)\qquad(3.86)$$

where $v_m^k \in L^2(I \times \Lambda \times \Omega)$ and ϕ_m^k is a solution of (1.5). Replacing (3.86) in (3.85) and since $\rho > 0$,

$$\sum_{m \in \mathbb{N}^*} \left(\left(\alpha_n^k\right)^{-2} \partial_{\tau\tau} v_m^k + \lambda_m^k v_m^k \right) \phi_m^k = 0. \tag{3.87}$$

For m and $m' \in \mathbb{N}^*$, applying the orthogonality in $L^2(Y)$ of ϕ_m^k and $\phi_{m'}^k$ to the equation (3.87) with $\int_Y \phi_m^k \cdot \phi_{m'}^k dy = 0$ if $m' \neq m$ and $\int_Y \phi_m^k \cdot \phi_{m'}^k dy = 1$ if $m' = m$, so

$$\left(\alpha_n^k\right)^{-2} \partial_{\tau\tau} v_m^k(t,\tau,x) + \lambda_m^k v_m^k(t,\tau,x) = 0 \text{ in } I \times \Lambda \times \Omega \text{ for all } m \in \mathbb{N}^*.$$

Since $\widehat{u}_n^{1,k}$ is periodic in τ, so v_m^k is also periodic in τ for any $m \in \mathbb{N}^*$. It implies that

$$\sqrt{\lambda_m^k}\alpha_n^k = 2\pi\sqrt{\frac{\lambda_m^k}{\lambda_n^k}} = 2\pi l \text{ for any } l \in \mathbb{N}^*. \tag{3.88}$$

For a given $m \in \mathbb{N}^*$ satisfying (3.88), v_m^k can be decomposed as

$$v_m^k(t,\tau,x) = u_m^k(t,x)\, e^{2i\pi\sqrt{\frac{\lambda_m^k}{\lambda_n^k}}\tau} + u_{-m}^k(t,x)\, e^{-2i\pi\sqrt{\frac{\lambda_m^k}{\lambda_n^k}}\tau}$$

where $\left(u_m^k, u_{-m}^k\right) \in L^2(I \times \Omega)^2$. Finally,

$$\widehat{u}_n^{1,k}(t,\tau,x,y) = \sum_{m \in M_{n,int}^k} u_m^k(t,x)\, e^{sign(m)2i\pi\sqrt{\frac{\lambda_{|m|}^k}{\lambda_n^k}}\tau} \phi_{|m|}^k(y)$$

with $u_m^k(t,x) \in L^2(I \times \Omega)$. ■

The next lemma focuses on the HF-macroscopic model (3.26)-(3.38) for each $k \in Y^*$ and $n \in \mathbb{N}^*$.

Lemma 36 *For $k \in Y^*$, $n \in \mathbb{N}^*$ and any bounded data as in (3.10), let u^ε be solution of the weak formulation of the wave equation (3.9) satisfying the uniform bound (3.11) and the assumption (3.14). For $\varepsilon \in E_k$ as in Assumption 14, if $u_m^\sigma \in H^1(I \times \Omega)$ for $\sigma \in I^k$, $s \in \{+,-\}$ and $m \in M_{n,s}^\sigma$, then u_m^σ is solution of the HF-macroscopic model (3.26)-(3.38).*

Before continuing with the proof of Lemma 36, we establish an auxiliary result for existence of special test functions. For $k \in Y^* \backslash \left\{0, -\frac{1}{2}\right\}$, $n \in \mathbb{N}^*$ and $\sigma \in I^k$, we consider the two functions $\varphi_n^k(t,x), \varphi_n^{-k}(t,x) \in H^2(I \times \Omega)$ such that

$$\varphi_n^k(t,x)\, \phi_n^k(0)\, e^{2i\pi l^k \frac{x}{\alpha}} + \varphi_n^{-k}(t,x)\, \phi_n^{-k}(0)\, e^{-2i\pi l^k \frac{x}{\alpha}} = 0 \text{ on } I \times \partial\Omega \tag{3.89}$$

where l^k is defined in (1.40).

Lemma 37 *For $k \in Y^* \backslash \left\{0, -\frac{1}{2}\right\}$, let $\varepsilon \in E_k$, there exist $\varphi_n^{k,\varepsilon}$, $\varphi_n^{-k,\varepsilon} \in H^2(I \times \Omega)$ satisfying*

i) the boundary conditions

$$\varphi_n^{k,\varepsilon}(t,x)\, \phi_n^k(0)\, e^{2i\pi k \frac{x}{\varepsilon}} + \varphi_n^{-k,\varepsilon}(t,x)\, \phi_n^{-k}(0)\, e^{-2i\pi k \frac{x}{\varepsilon}} = 0 \textit{ on } I \times \partial\Omega, \tag{3.90}$$

ii) and the strong convergence

$$\varphi_n^{\sigma,\varepsilon} \to \varphi_n^\sigma \textit{ in } H^2(I \times \Omega) \textit{ when } \varepsilon \to 0 \textit{ for } \sigma \in I^k. \tag{3.91}$$

Proof. [Proof of Lemma 37] For any $\varepsilon \in E_k$ and let the two functions $\varphi_n^k(t,x)$, $\varphi_n^{-k}(t,x) \in H^2(I \times \Omega)$ satisfying (3.89), we prove that the following choice satisfies the conditions,

$$\begin{aligned} \varphi_n^{k,\varepsilon}(t,x) &= \varphi_n^k(t,x) \in H^2(I \times \Omega) \\ \text{and } \varphi_n^{-k,\varepsilon}(t,x) &= \varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x) \text{ where } \mu^\varepsilon(t,x) \in H^2(I \times \Omega) \end{aligned} \tag{3.92}$$

with

$$\mu^\varepsilon(t,x) = -\left(1 - e^{4i\pi\left(l_k^\varepsilon - l^k\right)}\right)\varphi_n^{-k}(t,\alpha)\frac{x}{\alpha}$$

where l_ε^k and l^k is defined in (1.39) and (1.40).

i) Replacing (3.92) in (3.90), the boundary conditions are

$$\varphi_n^k(t,x)\,\phi_n^k(0)\,e^{2i\pi k\frac{x}{\varepsilon}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\phi_n^{-k}(0)\,e^{-2i\pi k\frac{x}{\varepsilon}} = 0 \text{ on } I \times \partial\Omega.$$

Using (1.39) and (1.40) with remarking that $e^{2i\pi h_k^\varepsilon \frac{x}{\alpha}} = 1$ at $x \in \partial\Omega$, so

$$\varphi_n^k(t,x)\,\phi_n^k(0)\,e^{2i\pi l_k^\varepsilon \frac{x}{\alpha}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\phi_n^{-k}(0)\,e^{-2i\pi l_k^\varepsilon \frac{x}{\alpha}} = 0 \text{ on } I \times \partial\Omega.$$

Or equivalently,

$$\varphi_n^k(t,x)\,\phi_n^k(0)\,e^{2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\phi_n^{-k}(0)\,e^{-2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} = 0$$

on $I \times \partial\Omega$. Or,

$$\begin{aligned} &\varphi_n^k(t,x)\,\phi_n^k(0)\,e^{2i\pi l^k \frac{x}{\alpha}} e^{2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} \\ &+ \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\phi_n^{-k}(0)\,e^{-2i\pi l^k \frac{x}{\alpha}} e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} = 0 \end{aligned}$$

on $I \times \partial\Omega$. From (3.89),

$$\varphi_n^k(t,x)\,\phi_n^k(0)\,e^{2i\pi l^k \frac{x}{\alpha}} = -\varphi_n^{-k}(t,x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k \frac{x}{\alpha}} \text{ on } I \times \partial\Omega.$$

After replacement, the equation remains,

$$\begin{aligned} &\varphi_n^{-k}(t,x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k \frac{x}{\alpha}}\left(e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} - e^{2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}}\right) \\ &+ \mu^\varepsilon(t,x)\,\phi_n^{-k}(0)\,e^{-2i\pi l^k \frac{x}{\alpha}} e^{-2i\pi\left(l_k^\varepsilon - l^k\right)\frac{x}{\alpha}} = 0 \text{ on } I \times \partial\Omega. \end{aligned}$$

This equation is satisfied with the above μ^ε.

ii) For $\sigma = k$, the strong convergence is true since $\varphi_n^{k,\varepsilon}$ is independent on ε. For $\sigma = -k$, the strong convergence of $\mu^\varepsilon(t,x)$ in $H^2(I \times \Omega)$ is trivial, i.e. $\mu^\varepsilon(t,x) \to 0$ in $H^2(I \times \Omega)$ strongly when $\varepsilon \to 0$. Therefore, $\varphi_n^{-k,\varepsilon} \to \varphi_n^{-k}$ in $H^2(I \times \Omega)$ strongly when $\varepsilon \to 0$. ■

Proof. [Proof of Lemma 36] Let $k \in Y^*$, $s \in \{-,+\}$ and $n \in \mathbb{N}^*$, we consider $\left(\phi_{|q|}^\sigma\right)_{q \in M_{n,s}^\sigma, \sigma \in I^k}$ the Bloch eigenmodes associated to the eigenvalue λ_n^k of the Bloch modes equation. We restart with the weak formulation of the wave equation (3.9) by choosing $w_q^{\sigma,\varepsilon}$ decomposed as

$$w_q^{\sigma,\varepsilon}(t,\tau,x,y) = \varphi_q^{\sigma,\varepsilon}(t,x)\,e^{sign(q)2i\pi\tau}\phi_{|q|}^\sigma(y) \tag{3.93}$$

for any $q \in M_{n,s}^{\sigma}, \sigma \in I^k$ and $\varphi_q^{\sigma,\varepsilon} \in C^{\infty}(I \times \Omega)$ satisfying

$$\partial_t \varphi_q^{\sigma,\varepsilon}(t=0) = 0,\ \varphi_q^{\sigma,\varepsilon}(t=T) = 0 \text{ and } \partial_t \varphi_q^{\sigma,\varepsilon}(t=T) = 0. \tag{3.94}$$

The boundary condition $w^{\varepsilon} = 0$ at $x \in \partial\Omega$ is equivalent to

$$\sum_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \varphi_q^{\sigma,\varepsilon}(t,x)\, e^{sign(q)\frac{2i\pi t}{\varepsilon \alpha_{|q|}^{\sigma}}} \phi_{|q|}^{\sigma}\left(\frac{x}{\varepsilon}\right) = 0 \text{ on } I \times \partial\Omega. \tag{3.95}$$

Since $\alpha_{|q|}^{\sigma} = \alpha_n^k$ for all $q \in M_{n,s}^{\sigma}$ and $\sigma \in I^k$, so $e^{sign(q)\frac{2i\pi t}{\varepsilon \alpha_{|q|}^{\sigma}}} \neq 0$ can be eliminated. In the case of $k \in \left\{0, -\frac{1}{2}\right\}$, thanks to the periodicity or anti-periodicity of $\phi_{|q|}^{\sigma}$, (3.95) is equivalent to

$$\sum_{q \in M_{n,s}^{k}} \varphi_q^{k,\varepsilon}(t,x)\, \phi_{|q|}^{k}(0) = 0 \text{ on } I \times \partial\Omega.$$

Taking $\varphi_q^{k,\varepsilon} = \varphi_q^k \in C^{\infty}(I \times \Omega)$ independent on ε, the boundary conditions of the test function are

$$\sum_{q \in M_{n,s}^{k}} \varphi_q^{k}(t,x)\, \phi_{|q|}^{k}(0) = 0 \text{ on } I \times \partial\Omega. \tag{3.96}$$

In the case of $k \in Y^* \setminus \left\{0, -\frac{1}{2}\right\}$, using the $\sigma-$quasi-periodicity of $\phi_{|q|}^{\sigma}$, (3.95) is equivalent to

$$\sum_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \varphi_q^{\sigma,\varepsilon}(t,x)\, \phi_{|q|}^{\sigma}(0)\, e^{2i\pi\sigma\frac{x}{\varepsilon}} = 0 \text{ on } I \times \partial\Omega. \tag{3.97}$$

For $\varepsilon \in E_k$ and $\sigma \in I^k$, using Lemma 37, there exists a sequence $\varphi_q^{\sigma,\varepsilon} \in C^{\infty}(I \times \Omega)$ such that (3.97) is satisfied and

$$\varphi_q^{\sigma,\varepsilon} \to \varphi_q^{\sigma} \text{ in } C^{\infty}(I \times \Omega) \text{ strongly when } \varepsilon \to 0 \tag{3.98}$$

where $\left(\varphi_q^{\sigma}\right)_{\sigma}$ satisfy (3.89). Thus, the limit w_q^{σ} of the test function $w_q^{\sigma,\varepsilon}$ is

$$w_q^{\sigma}(t,\tau,x,y) = \varphi_q^{\sigma}(t,x)\, e^{sign(q)2i\pi\tau} \phi_{|q|}^{\sigma}(y) \tag{3.99}$$

and the boundary conditions satisfied by the test function are

$$\sum_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \varphi_q^{\sigma}(t,x)\, \phi_{|q|}^{\sigma}(0)\, e^{sign(\sigma)2i\pi l^k \frac{x}{\alpha}} = 0 \text{ on } I \times \partial\Omega. \tag{3.100}$$

Moreover, according to (3.94), (3.98) and (3.99), the test function φ_q^{σ} satisfies

$$\partial_t \varphi_q^{\sigma}(t=0) = \varphi_q^{\sigma}(t=T) = \partial_t \varphi_q^{\sigma}(t=T) = 0. \tag{3.101}$$

We choose $w^{\varepsilon} = \sum\limits_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \mathfrak{B}_q^{\sigma} w_q^{\sigma,\varepsilon}$ as a test functions,

$$w^{\varepsilon} = \sum_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \mathfrak{B}_q^{\sigma} w_q^{\sigma,\varepsilon} \in H^2(I \times \Omega) \cap L^2\left(I; H_0^1(\Omega)\right) \tag{3.102}$$

Hence, w^{ε} reads as

$$w^{\varepsilon}(t,x) = \sum_{\sigma \in I^k, q \in M_{n,s}^{\sigma}} \varphi_q^{\sigma,\varepsilon}(t,x)\, e^{sign(q)\frac{2i\pi t}{\varepsilon \alpha_{|q|}^{\sigma}}} \phi_{|q|}^{\sigma}\left(\frac{x}{\varepsilon}\right) \tag{3.103}$$

and satisfies

$$w^{\varepsilon}(t=T)=0 \text{ and } \partial_t w^{\varepsilon}(t=T)=0. \tag{3.104}$$

From the boundary and the initial conditions of the test functions, and since $\left(\alpha^0_{|q|}\right)^{-2} Q^2 w^{\sigma,\varepsilon}_q + P^2 w^{\sigma,\varepsilon}_q = 0$, the very weak formulation (3.63) yields,

$$\begin{aligned}
&\sum_{\sigma\in I^k, q\in M^\sigma_{n,s}} \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}^\sigma_q \left(\sum_{l=0}^{1}\left(\left(\varepsilon\alpha^\sigma_{|q|}\right)^{-l} Q^l w^{\sigma,\varepsilon}_q - \varepsilon^{-l} P^l w^{\sigma,\varepsilon}_q\right)\right) - f^\varepsilon \cdot \mathfrak{B}^\sigma_q w^{\sigma,\varepsilon}_q \, dtdx \\
&\qquad + \int_\Omega u^\varepsilon_0 \cdot \partial_t \mathfrak{B}^\sigma_q \left(\rho w^{\sigma,\varepsilon}_q\right)(t=0) - v^\varepsilon_0 \cdot \mathfrak{R}^\sigma \left(\rho w^{\sigma,\varepsilon}_q\right)(t=0)\, dx = 0.
\end{aligned} \tag{3.105}$$

For each $\sigma \in I^k$ and $q \in M^\sigma_{n,s}$, thanks to (1.21) and (1.13) the relations between $\mathfrak{B}^\sigma_n$ and $T^{\varepsilon\alpha^\sigma_n *} S^{\varepsilon *}_\sigma$, $\mathfrak{R}^\sigma$ and $S^{\varepsilon *}_\sigma$, the second and the third term in (3.105) are approximated by,

$$\begin{aligned}
\int_{I\times\Omega} f^\varepsilon \cdot \mathfrak{B}^k_n w^{\sigma,\varepsilon}_q dtdx &= \int_{I\times\Omega} f^\varepsilon \cdot T^{\varepsilon\alpha^\sigma_n *} S^{\varepsilon *}_\sigma w^{\sigma,\varepsilon}_q dtdx + O(\varepsilon) \\
&= \int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha^\sigma_n} S^\varepsilon_\sigma f^\varepsilon \cdot w^{\sigma,\varepsilon}_q dtd\tau dxdy + O(\varepsilon),
\end{aligned} \tag{3.106}$$

and

$$\begin{aligned}
\int_\Omega v^\varepsilon_0 \cdot \mathfrak{R}^\sigma \left(\rho w^{\sigma,\varepsilon}_q\right)(t=0)\, dx &= \int_\Omega v^\varepsilon_0 \cdot S^{\varepsilon *}_\sigma \left(\rho w^{\sigma,\varepsilon}_q\right)(t=0)\, dx + O(\varepsilon) \\
&= \int_{\Omega\times Y} S^\varepsilon_\sigma v^\varepsilon_0 \cdot \rho w^{\sigma,\varepsilon}_q (t=0, \tau=0)\, dxdy + O(\varepsilon) \\
&= \int_{\Omega\times Y} S^\varepsilon_\sigma v^\varepsilon_0 \cdot \rho \phi^\sigma_{|q|} \varphi^{\sigma,\varepsilon}_q (t=0)\, dxdy + O(\varepsilon).
\end{aligned} \tag{3.107}$$

Moreover, the third term in (3.105) yields,

$$\begin{aligned}
&\int_\Omega u^\varepsilon_0 \cdot \partial_t \mathfrak{B}^\sigma_q \left(\rho w^{\sigma,\varepsilon}_q\right)(t=0)\, dx \\
&= \int_\Omega u^\varepsilon_0 \cdot \mathfrak{B}^\sigma_q \left(\partial_t\left(\rho \varphi^{\sigma,\varepsilon}_q e^{sign(q)2i\pi\tau}\phi^\sigma_{|q|}\right) + \frac{1}{\varepsilon\alpha^\sigma_{|q|}}\partial_\tau\left(\rho \varphi^{\sigma,\varepsilon}_q e^{sign(q)2i\pi\tau}\phi^\sigma_{|q|}\right)\right)(t=0)\, dx \\
&= \int_\Omega u^\varepsilon_0 \cdot \mathfrak{B}^\sigma_q \left(\rho\partial_t \varphi^{\sigma,\varepsilon}_q e^{sign(q)2i\pi\tau}\phi^\sigma_{|q|} + \frac{sign(q)\, 2i\pi}{\varepsilon\alpha^\sigma_{|q|}}\rho \varphi^{\sigma,\varepsilon}_q e^{sign(q)2i\pi\tau}\phi^\sigma_{|q|}\right)(t=0)\, dx.
\end{aligned}$$

Since $\partial_t \varphi^{\sigma,\varepsilon}_q (0,x) = 0$, $\tau = 0$, $\alpha^\sigma_{|q|} = \frac{2\pi}{\sqrt{\lambda^k_n}}$, $\phi^\sigma_{|q|}$ is solution of Bloch wave equation (1.5), and $\mathfrak{B}^\sigma_q := \mathfrak{R}^\sigma$ at $t=0$,

$$= \frac{sign(q)\, i}{\varepsilon\sqrt{\lambda^k_n}} \int_\Omega u^\varepsilon_0 \cdot \mathfrak{R}^\sigma \left(P^2\left(\varphi^{\sigma,\varepsilon}_q (t=0)\, \phi^\sigma_{|q|}\right)\right) dx.$$

Thanks to the decomposition (1.23), we have

$$\mathfrak{R}^\sigma P^2(.) = \varepsilon^2 P^\varepsilon \mathfrak{R}^\sigma(.) - \varepsilon^2 \mathfrak{R}^\sigma P^0(.) - \varepsilon \mathfrak{R}^\sigma P^1(.),$$

so,

$$= -\frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}} \int_\Omega u_0^\varepsilon \cdot \mathfrak{R}^\sigma \left[P^1 \left(\varphi_q^{\sigma,\varepsilon} (t=0)\, \phi_{|q|}^\sigma \right) \right] dx + O(\varepsilon) .$$

Or equivalently,

$$= -\frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}} \int_{\Omega\times Y} S_\sigma^\varepsilon u_0^\varepsilon \cdot \partial_x \varphi_q^{\sigma,\varepsilon} (t=0) \left(a\partial_y \phi_{|q|}^\sigma + \partial_y \left(a\phi_{|q|}^\sigma \right) \right) dxdy + O(\varepsilon) .$$

Taking the integration by part and remarking that the boundary condition $S_\sigma^\varepsilon u_0^\varepsilon = 0$ on $\partial\Omega \times Y$ based on $u_0^\varepsilon = 0$ on $\partial\Omega$,

$$= -\frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}} \int_{\Omega\times Y} \partial_x S_\sigma^\varepsilon u_0^\varepsilon \cdot \left(a\partial_y \phi_{|q|}^\sigma + \partial_y \left(a\phi_{|q|}^\sigma \right) \right) \varphi_q^{\sigma,\varepsilon} (t=0)\, dxdy + O(\varepsilon) . \quad (3.108)$$

Multiplying by ε, using (3.14), (3.106) and (3.107), and remarking that $u_n^{0,\sigma} = \chi_0(\sigma)\, u^0$, Equation (3.105) becomes,

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma} [\int_{I\times\Lambda\times\Omega\times Y} [\left(\chi_0(\sigma)\, u^0 + \varepsilon u_n^{1,\sigma} \right) \cdot \left(\left(\alpha_{|q|}^k \right)^{-1} Q^1 w_q^{\sigma,\varepsilon} - P^1 w_q^{\sigma,\varepsilon} + \varepsilon Q^0 w_q^{\sigma,\varepsilon} - \varepsilon P^0 w_q^{\sigma,\varepsilon} \right)$$

$$-\varepsilon T^{\varepsilon\alpha_n^\sigma} S_\sigma^\varepsilon f^\varepsilon \cdot w_q^{\sigma,\varepsilon}] dtd\tau dxdy - \varepsilon \int_{\Omega\times Y} [S_\sigma^\varepsilon v_0^\varepsilon \cdot \rho \phi_{|q|}^\sigma \varphi_q^{\sigma,\varepsilon} (t=0)$$

$$+ \frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}} \partial_x S_\sigma^\varepsilon u_0^\varepsilon \cdot \left(a\partial_y \phi_{|q|}^\sigma + \partial_y \left(a\phi_{|q|}^\sigma \right) \right) \varphi_q^{\sigma,\varepsilon} (t=0)\, dxdy] = \varepsilon O(\varepsilon) .$$

Since $Q^1 u^0 = 0$ and due to the special form of $w_q^{\sigma,\varepsilon}$ in τ so,

$$\int_\Lambda \chi_0(\sigma)\, u^0 \cdot P^0 w_q^{\sigma,\varepsilon} d\tau = 0 \text{ and } \int_\Lambda \chi_0(\sigma)\, u^0 \cdot Q^0 w_q^{\sigma,\varepsilon} d\tau = 0, \quad (3.109)$$

$$\int_\Lambda \chi_0(\sigma)\, u^0 \cdot Q^1 w_q^{\sigma,\varepsilon} d\tau = 0 \text{ and } \int_\Lambda \chi_0(\sigma)\, u^0 \cdot P^1 w_q^{\sigma,\varepsilon} d\tau = 0,$$

hence, dividing by ε, the equation reads

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma} [\int_{I\times\Lambda\times\Omega\times Y} [u_n^{1,\sigma} \cdot \left(\left(\alpha_{|q|}^k \right)^{-1} Q^1 w_q^{\sigma,\varepsilon} - P^1 w_q^{\sigma,\varepsilon} \right) - T^{\varepsilon\alpha_n^\sigma} S_\sigma^\varepsilon f^\varepsilon \cdot w_q^{\sigma,\varepsilon}] dtd\tau dxdy$$

$$- \int_{\Omega\times Y} [\frac{sign\,(q)\,i}{\sqrt{\lambda_n^\sigma}} \partial_x S_\sigma^\varepsilon u_0^\varepsilon \cdot \left(a\partial_y \phi_{|q|}^\sigma + \partial_y \left(a\phi_{|q|}^\sigma \right) \right) \varphi_q^{\sigma,\varepsilon} + S_\sigma^\varepsilon v_0^\varepsilon \cdot \rho \phi_{|q|}^\sigma \varphi_q^{\sigma,\varepsilon}] (t=0)\; dxdy] = O(\varepsilon) .$$

Moreover, according to (3.21), (3.98) and (3.99), and passing to the limit, the equation remains,

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma} \left[\int_{I\times\Lambda\times\Omega\times Y} u_n^{1,\sigma} \cdot \left(\alpha_{|q|}^\sigma \right)^{-1} Q^1 w_q^\sigma + u_n^{1,\sigma} \cdot P^1 w_q^\sigma dtd\tau dxdy \right.$$

$$\left. - \int_{I\times\Omega} F_q^\sigma \cdot \varphi_q^\sigma dtdx - \int_\Omega g_q^\sigma \cdot \varphi_q^\sigma (t=0)\, dx \right] = 0.$$

Using the decomposition (3.78) of $u_n^{1,k}$ and the definition (1.4),

$$\sum_{\sigma\in I^k, q\in M_{n,s}^{\sigma}} [\int_{I\times\Lambda\times\Omega\times Y} \left(\chi_0(\sigma) u_0^{1,0} + \widehat{u}_n^{1,\sigma}\right) \cdot [\left(\alpha_{|q|}^{\sigma}\right)^{-1} 2\rho\partial_{t\tau} w_q^{\sigma} - \partial_x \left(a\partial_y w_q^{\sigma}\right)$$

$$-\partial_y \left(a\partial_x w_q^{\sigma}\right)] dt d\tau dx dy - \int_{I\times\Omega} F_q^{\sigma}\cdot\varphi_q^{\sigma} dt dx - \int_{\Omega} g_q^{\sigma}\cdot\varphi_q^{\sigma}(t=0)\, dx] = 0.$$

Since w_q^{σ} is periodic in τ, $u_0^{1,0}$ is independent in τ and due to the special form of w_q^{σ} in τ, $\int_{\Lambda} w_q^{\sigma} d\tau = 0$, so

$$\int_{\Lambda} \chi_0(\sigma) u_0^{1,0} \cdot \left(\left(\alpha_{|q|}^{\sigma}\right)^{-1} 2\rho\partial_{t\tau} w_q^{\sigma} - \partial_x \left(a\partial_y w_q^{\sigma}\right) - \partial_y \left(a\partial_x w_q^{\sigma}\right)\right) d\tau = 0. \tag{3.110}$$

Moreover, using the decompositions (3.80) and (3.99) of $\widehat{u}_q^{1,\sigma}$ and w_q^{σ} with remarking that the index m in (3.80) is changed by p and

$$\int_{\Lambda} e^{sign(p)2i\pi\sqrt{\frac{\lambda_{|p|}^{\sigma}}{\lambda_n^{\sigma}}}\tau} \cdot e^{sign(q)2i\pi\tau} d\tau = 0 \text{ if } sign(p)\sqrt{\frac{\lambda_{|p|}^{\sigma}}{\lambda_n^{\sigma}}} \neq sign(q) \text{ and } = 1 \text{ otherwise,} \tag{3.111}$$

so the equation is equivalent to,

$$\sum_{\sigma\in I^k, p\in M_{n,s}^{\sigma}, q\in M_{n,s}^{\sigma}} \int_{I\times\Omega} \left[-sign(q)\, 4i\pi \left(\alpha_{|q|}^{\sigma}\right)^{-1} \left(\int_Y \rho\phi_{|p|}^{\sigma}\overline{\phi_{|q|}^{\sigma}} dy\right) u_p^{\sigma}\cdot\partial_t\varphi_q^{\sigma}\right.$$

$$\left. - \left(\int_Y \phi_{|p|}^{\sigma}\left(a\partial_y\overline{\phi_{|q|}^{\sigma}} + \partial_y\left(a\overline{\phi_{|q|}^{\sigma}}\right)\right) dy\right) u_p^{\sigma}\cdot\partial_x\varphi_q^{\sigma}\right] dt dx$$

$$- \sum_{\sigma\in I^k, q\in M_{n,s}^{\sigma}} \left[\int_{\Omega} g_q^{\sigma}\cdot\varphi_q^{\sigma}(t=0)\, dx + \int_{I\times\Omega} F_q^{\sigma}\cdot\varphi_q^{\sigma} dt dx\right] = 0.$$

We observe that,

$$\left[a\phi_{|p|}^{\sigma}\overline{\phi_{|q|}^{\sigma}}\right]_{y=0}^{y=1} = 0 \text{ and } \int_Y \phi_{|p|}^{\sigma}\left(a\partial_y\overline{\phi_{|q|}^{\sigma}} + \partial_y\left(a\overline{\phi_{|q|}^{\sigma}}\right)\right) dy = \int_Y \phi_{|p|}^{\sigma} a\partial_y\overline{\phi_{|q|}^{\sigma}} - a\partial_y\phi_{|p|}^{\sigma}\overline{\phi_{|q|}^{\sigma}} dy. \tag{3.112}$$

Therefore,

$$\sum_{\sigma\in I^k, p\in M_{n,s}^{\sigma}, q\in M_{n,s}^{\sigma}} \int_{I\times\Omega} -b(\sigma,p,q)\, u_p^{\sigma}\cdot\partial_t\varphi_q^{\sigma} - c(\sigma,p,q)\, u_p^{\sigma}\cdot\partial_x\varphi_q^{\sigma} dt dx$$

$$- \sum_{\sigma\in I^k, q\in M_{n,s}^{\sigma}} \left[\int_{\Omega} g_q^{\sigma}\cdot\varphi_q^{\sigma}(t=0)\, dx + \int_{I\times\Omega} F_q^{\sigma}\cdot\varphi_q^{\sigma} dt dx\right] = 0$$

where $b(\sigma,p,q)$ and $c(\sigma,p,q)$ are defined in (3.25). Assuming that $u_p^{\sigma}\in H^1(I\times\Omega)$, using (3.101) and applying integrations by parts,

$$\sum_{\sigma\in I^k, p\in M_{n,s}^{\sigma}, q\in M_{n,s}^{\sigma}} \left[\int_{I\times\Omega} b(\sigma,p,q)\,\partial_t u_p^{\sigma}\cdot\varphi_q^{\sigma} + c(\sigma,p,q)\,\partial_x u_p^{\sigma}\cdot\varphi_q^{\sigma} dt dx\right.$$

$$\left. + \int_{\Omega} b(\sigma,p,q)\, u_p^{\sigma}\cdot\varphi_q^{\sigma}(t=0)\, dx - \int_I \left[c(\sigma,p,q)\, u_p^{\sigma}\cdot\varphi_q^{\sigma}\right]_{x=0}^{x=\alpha} dt\right]$$

$$- \sum_{\sigma\in I^k, q\in M_{n,s}^{\sigma}} \left[\int_{\Omega} g_q^{\sigma}\cdot\varphi_q^{\sigma}(t=0)\, dx + \int_{I\times\Omega} F_q^{\sigma}\cdot\varphi_q^{\sigma} dt dx\right] = 0.$$

For each $\sigma \in I^k$ and $q \in M^{\sigma}_{n,s}$, by choosing test functions $\varphi^{\sigma}_q \in H^1_0(I \times \Omega)$ the strong form comes

$$\sum_{p\in M^{\sigma}_{n,s}} b(\sigma,p,q)\,\partial_t u^{\sigma}_p + \sum_{p\in M^{\sigma}_{n,s}} c(\sigma,p,q)\,\partial_x u^{\sigma}_p = F^{\sigma}_q \text{ in } I\times\Omega.$$

It remains,

$$\sum_{\sigma\in I^k, q\in M^{\sigma}_{n,s}} \left[\int_{\Omega}\left(\sum_{p\in M^{\sigma}_{n,s}} b(\sigma,p,q)\,u^{\sigma}_p - g^{\sigma}_q\right)\cdot\varphi^{\sigma}_q(t=0)\,dx \right. \tag{3.113}$$
$$\left. -\int_I \sum_{p\in M^{\sigma}_{n,s}} \left[c(\sigma,p,q)\,u^{\sigma}_p\cdot\varphi^{\sigma}_q\right]^{x=\alpha}_{x=0} dt\right] = 0.$$

The initial condition is deduced,

$$\sum_{p\in M^{\sigma}_{n,s}} b(\sigma,p,q)\,u^{\sigma}_p(t=0) = g^{\sigma}_q \text{ on } \Omega \text{ for each } q\in M^{\sigma}_n \tag{3.114}$$

and the boundary term is,

$$\sum_{\sigma\in I^k,p,q\in M^{\sigma}_{n,s}} \int_I \left[c(\sigma,p,q)\,u^{\sigma}_p\cdot\varphi^{\sigma}_q\right]^{x=\alpha}_{x=0} dt = 0 \text{ for } \varphi^{\sigma}_q \text{ satisfying (3.100).} \tag{3.115}$$

The remaining of this proof focuses on finding the boundary conditions of $\left(u^{\sigma}_p\right)_p$ the macroscopic solutions. We distinguish between the three cases $k=0$, $k=-\frac{1}{2}$ and $k\neq\left\{0,-\frac{1}{2}\right\}$. For notational convenience, we here understand $n := sn$ and $n' := sn'$ for both two cases $s=-$ and $s=+$.

(i) **Case** $k=0$ with λ^0_n be a double eigenvalue and the condition (3.31). Introducing the matrices $C=(c(0,p,q))_{p,q}$, $B=(b(0,p,q))_{p,q}$ and the vectors $U=\left(u^0_p\right)_p$, $F=\left(F^0_p\right)_p$, $G=\left(g^0_p\right)_p$, $\varphi=\left(\varphi^0_p\right)_p$, $\phi=\left(\phi^0_{|p|}\right)_p$, we get the matrix form,

$$B\partial_t U + C\partial_x U = F \text{ in } I\times\Omega, \tag{3.116}$$
$$BU(0,x) = G \text{ in } \Omega,$$
$$\text{and } CU(t,x).\overline{\varphi}(t,x) = 0 \text{ on } I\times\partial\Omega \text{ for all } \varphi \text{ such that } \overline{\phi}(0).\overline{\varphi}(t,x)=0 \text{ on } I\times\partial\Omega.$$

The boundary condition is equivalent to the fact that $CU(t,x)$ is collinear to $\overline{\phi}(0)$ on $\partial\Omega$ for $t\in I$. It means that,

$$\det\left(CU(t,x),\overline{\phi}(0)\right) = 0 \text{ on } I\times\partial\Omega. \tag{3.117}$$

But $c(0,p,p)=0$ for $p\in M^0_n$, the equation (3.117) yields,

$$c(0,n,n')\,u^0_{n'}(t,x)\,\phi^0_{|n'|}(0) - c(0,n',n)\,u^0_n(t,x)\,\phi^0_{|n|}(0) = 0 \text{ on } I\times\partial\Omega.$$

Therefore, since $c(0,n,n') = -c(0,n',n)$ and assume that $c(0,n,n')\neq 0$, the boundary conditions of the HF-macroscopic equation are

$$u^0_n(t,x)\,\phi^0_{|n|}(0) + u^0_{n'}(t,x)\,\phi^0_{|n'|}(0) = 0 \text{ on } I\times\partial\Omega. \tag{3.118}$$

Moreover, if $c(0,n,n')=0$ then the matrix form (3.116) yields,

$$B\partial_t U = F \text{ in } I\times\Omega \text{ and } BU(0,x)=G \text{ in } \Omega$$

where the boundary condition has disappeared,

Still for $k=0$ with $\phi^0_{|n|}(0)=\phi^0_{|n'|}(0)=0$ and whatever the values of $c(0,n,n')$, the matrix form is similar to (3.116) but without boundary condition.

Finally, if the eigenvalue λ^0_n is simple, then the matrix $C=0$ and the HF-macroscopic equation is stated as

$$b(0,n,n)\,\partial_t u^0_n = F^0_n \text{ in } I\times\Omega \text{ with } b(0,n,n)\,u^0_n(t=0)=g^0_n \text{ in } \Omega$$

without boundary conditions.

(ii) **Case** $k=-\frac{1}{2}$ with λ^k_n be a double eigenvalue. Introducing the matrices $C=(c(k,p,q))_{p,q}$, $B=(b(k,p,q))_{p,q}$ and the vectors $U=\left(u^k_p\right)_p$, $F=\left(F^k_p\right)_p$, $G=\left(g^k_p\right)_p$, $\varphi=\left(\varphi^k_p\right)_p$, $\phi=\left(\phi^k_{|p|}e^{2i\pi x\frac{l^k}{\alpha}}\right)_p$, we get the same matrix form as (3.116). The boundary conditions are equivalent to the fact that $CU(t,x)$ is collinear to $\overline{\phi}(0)$ on $\partial\Omega$ for $t\in I$, or equivalently,

$$\det\left(CU(t,x),\overline{\phi}(0)\right)=0 \text{ on } I\times\partial\Omega.$$

Therefore,

$$\begin{gathered}\left[c(k,n,n)\,\phi^k_{|n'|}(0)\,u^k_n(t,x)+c(k,n,n')\,\phi^k_{|n'|}(0)\,u^k_{n'}(t,x)\right.\\ \left.-c(k,n',n)\,\phi^k_{|n|}(0)\,u^k_n(t,x)-c(k,n',n')\,\phi^k_{|n|}(0)\,u^k_{n'}(t,x)\right]e^{2i\pi x\frac{l^k}{\alpha}}=0 \text{ on } I\times\partial\Omega.\end{gathered}$$

Since $e^{2i\pi x\frac{l^k}{\alpha}}\neq 0$ for all $x\in\partial\Omega$, the boundary condition of the HF-macroscopic equation is,

$$\begin{gathered}\left(c(k,n,n)\,\phi^k_{|n'|}(0)-c(k,n',n)\,\phi^k_{|n|}(0)\right)u^k_n(t,x)\\ +\left(c(k,n,n')\,\phi^k_{|n'|}(0)-c(k,n',n')\,\phi^k_{|n|}(0)\right)u^k_{n'}(t,x)=0 \text{ on } I\times\partial\Omega.\end{gathered}$$

For $k=-\frac{1}{2}$ with a simple eigenvalue λ^0_n, the HF-macroscopic equation is

$$\begin{gathered}b(k,n,n)\,\partial_t u^k_n + c(k,n,n)\,\partial_x u^k_n = F^k_n \text{ in } I\times\Omega,\\ \text{with } b(k,n,n)\,u^k_n(t=0)=g^k_n \text{ in } \Omega,\\ \text{and } c(k,n,n)\,u^k_n.\overline{\varphi^k_n}=0 \text{ on } I\times\partial\Omega \text{ for all } \varphi^k_n \text{ such that } \overline{\phi^k_n}(0).\overline{\varphi^k_n}=0 \text{ on } I\times\partial\Omega.\end{gathered}$$

If $c(k,n,n)=0$ or $\varphi^k_n=0$ on $I\times\partial\Omega$ then the boundary condition is vanished. Otherwise, $\phi^k_n(0)=0$ and $u^k_n=0$ on $I\times\partial\Omega$.

(iii) **Case** $k\notin\left\{0,-\frac{1}{2}\right\}$. According to Remark 2, the Bloch eigenvalue λ^k_n is simple. Similarly to the case $k=0$, by introducing the matrices $C=diag\,(c(\sigma,n,n))_\sigma$, $B=diag\,(b(\sigma,n,n))_\sigma$ and the vectors $U=(u^\sigma_n)_\sigma$, $F=(F^\sigma_n)_\sigma$, $G=(g^\sigma_n)_\sigma$, $\varphi=(\varphi^\sigma_n)_\sigma$, $\phi=\left(\phi^\sigma_{|n|}e^{sign(\sigma)2i\pi x\frac{l^k}{\alpha}}\right)_\sigma$, we get the same matrix form as (3.116). The boundary conditions are equivalent to the fact that $CU(t,x)$ is collinear to $\overline{\phi}(0)$ on $\partial\Omega$ for all $t\in I$, or equivalently,

$$\det\left(CU(t,x),\overline{\phi}(0)\right)=0 \text{ on } I\times\partial\Omega.$$

Thus,

$$c(k,n,n)\,u_n^k(t,x)\,\phi_{|n|}^k(0)\,e^{2i\pi x\frac{l^k}{\alpha}} - c(-k,n,n)\,u_n^{-k}(t,x)\,\phi_{|n|}^{-k}(0)\,e^{-2i\pi x\frac{l^k}{\alpha}} = 0 \text{ on } I\times\partial\Omega.$$

Therefore, from the assumption $c(k,n,n)\neq 0$ and from $c(k,n,n) = -c(-k,n,n)$, the boundary conditions turn to be

$$u_n^k(t,x)\,\phi_{|n|}^k(0)\,e^{2i\pi x\frac{l^k}{\alpha}} + u_n^{-k}(t,x)\,\phi_{|n|}^{-k}(0)\,e^{-2i\pi x\frac{l^k}{\alpha}} = 0 \text{ on } I\times\partial\Omega.$$

Finally, if $c(k,n,n) = 0$ then whatever the value of $\phi_{|n|}^k(0)$, the matrix form of HF-macroscopic equation is stated by

$$B\partial_t U = F \text{ in } I\times\Omega \text{ and } BU(0,x) = G \text{ in } \Omega$$

which does not require any boundary condition. ■

3.3.2 Proof of main Theorem

For any $K\in\mathbb{N}^*$ and any test function $\varphi\in C^2(\Lambda\times Y; C_c(I\times\Omega)\cap C_c^0(I\times\Omega))$ being periodic in τ, let a bounded sequence u^ε be solution of the weak formulation of the wave equation (3.9). For $k\in L_K^*$ and $n\in M^k$, we already have shown the (n,k)-mode wave-two-scale approximations

$$u^\varepsilon =^{TSW(k,n)} \chi_{0(k)}u^0 + \varepsilon(\chi_{0(k)}\theta\partial_x u^0 + \sum_{m\in M_{n,int}^k} u_m^k e^{sign(m)2i\pi\tau}\phi_{|m|}^k) + \varepsilon O(\varepsilon) \qquad (3.119)$$

and it remains to check that this approximation holds in the wave-two-scale sense,

$$\int_{I\times\Omega} u^\varepsilon(t,x)\cdot\sum_{k\in L_K^*, n\in M^k}(\mathfrak{B}_n^k\Pi_n^k\varphi)(t,x)\ dtdx$$
$$= \int_{I\times\Lambda\times\Omega\times Y}(u^0+\varepsilon[\theta\partial_x u^0 + \sum_{k\in L_K^*, n\in\mathbb{Z}^*}\int_{I\times\Lambda\times\Omega\times Y} u_n^k e^{sign(n)2i\pi\tau}\phi_{|n|}^k])\cdot\varphi\ dtd\tau dxdy + \varepsilon O(\varepsilon).$$

But

$$\int_{I\times\Omega} u^\varepsilon(t,x)\cdot\sum_{k\in L_K^*, n\in M^k}\mathfrak{B}_n^k\Pi_n^k\varphi(t,x)\ dtdx = \sum_{k\in L_K^*, n\in M^k}\int_{I\times\Omega} u^\varepsilon(t,x)\cdot\mathfrak{B}_n^k\Pi_n^k\varphi(t,x)\ dtdx$$

with $\Pi_n^k\varphi$ a periodic function in τ and k-quasi-periodic function in y. Using the decomposition (3.119), the fact that the projections are self-adjoint operators, and

$$\sum_{n\in\mathbb{N}}\Pi_n^0 u^0 = u^0,\ \sum_{n\in\mathbb{N}}\Pi_n^0\theta = \theta,$$
$$\Pi_n^k\sum_{m\in M_{n,int}^k} u_m^k(t,x)\,e^{sign(m)2i\pi\tau}\phi_{|m|}^k(y) = \sum_{m\in\{n,-n\}} u_m^k e^{sign(m)2i\pi\tau}\phi_{|m|}^k$$

yield the expected result.

Furthermore, if (3.19) is satisfied then u^0 is the solution of the weak formulation of the LF-homogenized equation (3.24) as proved in Lemma 34.

For any $k \in L_K^*$, we pose $k = \frac{p}{K}$ for $p \in KL_K^*$. For $\varepsilon \in E_{1/K}$ as in (1.39) and (1.40), it implies that

$$\frac{\alpha p}{\varepsilon K} = p\left[\frac{\alpha p}{\varepsilon K}\right] + pl_\varepsilon^{1/K} \text{ and } pl_\varepsilon^{1/K} \to l^k := pl^{1/K} \text{ when } \varepsilon \to 0$$

and the data (3.21) is satisfied for all $k \in L_K^*$ with the same sequence $\varepsilon \in E_{1/K}$. If $u_n^k \in H^1(I \times \Omega)$ then u_n^k is solution of the HF-macroscopic equation for all $k \in L_K^*$ based on the proof of Lemma 36.

3.4 Other cases

In this Section, we study the homogenization of the wave equation in two other cases.

1. The wave equation with Neumann boundary conditions in Section 3.4.1.
2. The wave equation with additional zero and first order time and space derivatives in Section 3.4.2.

The process of homogenization is similar to Section 3.2 and 3.3. The final results state similarly as in Theorem 29 but with different homogenized models. The differences are in the detail of some homogenized terms. Here we focus on discussing the differences in the homogenization of each pair of fibers.

3.4.1 Neumann boundary conditions

We consider the wave equation with Neumann boundary conditions,

$$\begin{aligned}
&\rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x \left(a^\varepsilon \partial_x u^\varepsilon\right) = f^\varepsilon \text{ in } I \times \Omega,\\
&u^\varepsilon(t=0,x) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon \text{ in } \Omega,\\
&\partial_x u^\varepsilon = 0 \text{ on } I \times \partial\Omega.
\end{aligned}$$

Here the test function w^ε satisfies $\partial_x w^\varepsilon = 0$ at $x \in \partial\Omega$. The LF-homogenized equation with initial conditions, the HF-microscopic equation and the internal HF-macroscopic equation are unchanged, while the boundary conditions of the LF-homogenized equation are replaced by

$$\partial_x u^0 = 0 \text{ on } I \times \partial\Omega. \tag{3.120}$$

Moreover, the boundary conditions of the HF-macroscopic equation are,

$$\sum_{\sigma \in I^k, p \in M_{n,s}^\sigma} u_p^\sigma(t,x)\, \partial_y \phi_{|p|}^\sigma(0)\, e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } I \times \partial\Omega \text{ if } k \neq -\frac{1}{2}$$

and

$$\begin{aligned}
&\left(c(k,n,n)\,\partial_y \phi_{|n'|}^k(0) - c(k,n',n)\,\partial_y \phi_{|n|}^k(0)\right) u_n^k(t,x) + \left(c(k,n,n')\,\partial_y \phi_{|n'|}^k(0)\right.\\
&\left.-c(k,n',n')\,\partial_y \phi_{|n|}^k(0)\, u_{n'}^k(t,x)\right) = 0 \text{ on } I \times \partial\Omega \text{ otherwise}
\end{aligned}$$

for any $n \in \mathbb{N}^*$ and $s \in \{+,-\}$. Their derivation follows the same steps, so we only mention the boundary condition satisfied by the test functions. They are chosen to satisfy $\partial_x w^\varepsilon(t,x) = 0$ on $I \times \partial\Omega$ or equivalently,

$$\sum_{\sigma \in I^k, q \in M_{n,s}^\sigma} \left(\partial_x \varphi_q^{\sigma,\varepsilon}(t,x)\, \phi_{|q|}^\sigma\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\varphi_q^{\sigma,\varepsilon}(t,x)\, \partial_y \phi_{|q|}^\sigma\left(\frac{x}{\varepsilon}\right)\right) e^{sign(q)\frac{2i\pi t}{\varepsilon \alpha_q^\sigma}} = 0 \text{ on } I \times \partial\Omega.$$

Since $\alpha_q^\sigma = \alpha_n^k$ for all $q \in M_{n,s}^\sigma$ and $\sigma \in I^k$, so $e^{sign(q)\frac{2i\pi t}{\varepsilon\alpha_q^\sigma}} \neq 0$ can be eliminated. Moreover, $\phi_{|q|}^\sigma$ is $\sigma-$quasi-periodic in y, so $\partial_y \phi_{|q|}^\sigma$ is also $\sigma-$quasi-periodic in y. Multiplying by ε and using the $\sigma-$quasi-periodicity of $\partial_y \phi_{|q|}^\sigma$

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma} \varphi_q^{\sigma,\varepsilon}(t,x)\,\partial_y \phi_{|q|}^\sigma(0)\, e^{2i\pi\frac{\sigma x}{\varepsilon}} + O(\varepsilon) = 0 \text{ on } I \times \partial\Omega. \tag{3.121}$$

Choosing a sequence $\varepsilon \in E_k$, using (1.39, 1.40), taking $\varphi_q^{\sigma,\varepsilon} = \varphi_q^\sigma$ for $k \in \left\{-\frac{1}{2}, 0\right\}$ and considering the strong convergence of $\varphi_q^{\sigma,\varepsilon}$ in $H^2(I \times \Omega)$ similar to Lemma 37 for $k \notin \left\{-\frac{1}{2}, 0\right\}$ but $\phi_{|q|}^\sigma(0)$ being replaced by $\partial_y \phi_{|q|}^\sigma(0)$, see also in Lemma 58 in Appendix, the boundary conditions of the test function are,

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma} \varphi_q^{\sigma}(t,x)\,\partial_y \phi_{|q|}^\sigma(0)\, e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } I \times \partial\Omega.$$

3.4.2 Generalization of the wave equation

We consider the wave equation with a damping term, a convection term and a potential term with homogeneous Dirichlet boundary conditions,

$$\begin{aligned}
&\rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x\left(a^\varepsilon \partial_x u^\varepsilon\right) + \gamma^\varepsilon \partial_t u^\varepsilon + \zeta^\varepsilon \partial_x u^\varepsilon + \xi^\varepsilon u^\varepsilon = f^\varepsilon \text{ in } I \times \Omega,\\
&u^\varepsilon(t=0,x) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon \text{ in } \Omega,\\
&u^\varepsilon = 0 \text{ on } I \times \partial\Omega.
\end{aligned} \tag{3.122}$$

Here three functions $(\gamma^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ are assumed to obey a prescribed profile

$$\gamma^\varepsilon := \gamma\left(\frac{x}{\varepsilon}\right), \zeta^\varepsilon(x) := \zeta\left(\frac{x}{\varepsilon}\right) \text{ and } \xi^\varepsilon := \xi\left(\frac{x}{\varepsilon}\right)$$

where γ, ζ and ξ are $Y-$periodic and bounded in $L^\infty(\mathbb{R})$. The equation (3.122) is also taken in the weak sense,

$$\begin{aligned}
&\int_{I\times\Omega} \rho^\varepsilon \partial_{tt} u^\varepsilon \cdot w^\varepsilon + a^\varepsilon \partial_x u^\varepsilon \cdot \partial_x w^\varepsilon + \partial_t u^\varepsilon \cdot w^\varepsilon + \partial_x u^\varepsilon \cdot w^\varepsilon + u^\varepsilon \cdot w^\varepsilon dt dx \\
&= \int_{I\times\Omega} f^\varepsilon \cdot w^\varepsilon dt dx\\
&\text{with } u^\varepsilon(t=0,x) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t=0,x) = v_0^\varepsilon \text{ in } \Omega,
\end{aligned} \tag{3.123}$$

for all the admissible test functions

$$w^\varepsilon \in L^2\left(I; H_0^1(\Omega)\right). \tag{3.124}$$

From the assumption (3.10), the uniform bounds (3.11) can be derived.

Statement of the models

The process of homogenization is also similar to the case in Section 3.2. The HF-microscopic equation is unchanged while the LF-homogenized equation and the HF-macroscopic equations include other terms related to the first and zero order terms. After extraction of a subsequence, similarly to (3.21), we introduce

$$h_n^k = \lim_{\varepsilon\to 0} \int_Y S_k^\varepsilon u_0^\varepsilon \cdot \gamma \phi_{|n|}^k dy \in L^2(\Omega). \tag{3.125}$$

Moreover, we define the coefficients

$$\widehat{\gamma} = \int_Y \gamma dy,\ \widehat{\zeta} = \int_Y \zeta \left(1 + \partial_y \theta\right) dy \text{ and } \widehat{\xi} = \int_Y \xi dy, \tag{3.126}$$

and

$$d\left(k,n,m\right) = -\left(-sign\left(n\right) 2i\pi \left(\alpha_n^k\right)^{-1} \int_Y \gamma \phi_{|n|}^k \cdot \phi_{|m|}^k dy + \int_Y \phi_{|n|}^k \cdot \partial_y \left(\zeta \overline{\phi_{|m|}^k}\right) dy \right). \tag{3.127}$$

Thus, the LF-homogenized equation states

$$\begin{gathered} \widehat{\rho}\partial_{tt}u^0 - \partial_x\left(\widehat{a}\partial_x u^0\right) + \widehat{\gamma}\partial_t u^0 + \widehat{\zeta}\partial_x u^0 + \widehat{\xi}u^0 = \widehat{f^0} \text{ in } I \times \Omega, \\ u^0\left(t=0\right) = \widehat{h^0} \text{ and } \partial_t u^0 = \widehat{g^0} \text{ in } \Omega, \\ u^0 = 0 \text{ on } I \times \partial\Omega, \end{gathered} \tag{3.128}$$

where $\widehat{a}$, $\widehat{\rho}$, $\widehat{f^0}$, $\widehat{h^0}$ and $\widehat{g^0}$ are defined in (3.22) and (3.20) in Section 3.2.2.

Before to state the HF-macroscopic model, we remarking that it is stated under the assumptions (3.26) and (3.31). For each $k \in Y^*$, $\sigma \in I^k$, $s \in \{+,-\}$ and $q \in M_{n,s}^\sigma$, the macroscopic system is stated by,

$$\sum_{p \in M_{n,s}^\sigma} \left(b\left(\sigma,p,q\right)\partial_t u_p^\sigma + c\left(\sigma,p,q\right)\partial_x u_p^\sigma + d\left(\sigma,p,q\right)u_p^\sigma\right) = F_q^\sigma \text{ in } I \times \Omega, \tag{3.129}$$

$$\sum_{p \in M_{n,s}^\sigma} b\left(\sigma,p,q\right) u_p^\sigma\left(t=0\right) = g_q^\sigma + h_q^\sigma \text{ in } \Omega,$$

with the boundary conditions as in (3.29), (3.34), and (3.35) in Section 3.3.

Homogenization results and their proofs

We only state the LF-homogenized equation and HF-macroscopic equation since the others parts of the model remain unchanged. We observe that the HF-microscopic equation relies to the second order part only, thus it is unchanged in this case, see also Remark 39 after the proof of Lemma 38 regarding its derivation.

Lemma 38 *For $k \in Y^*, n \in M^k$ and any bounded data as in (3.10), let u^ε be solution of the wave equation (3.123) satisfying the uniform bound (3.11) and the assumption (3.12). Then,*

$$u_n^{0,k} = \chi_0\left(k\right) u^0 \text{ in } L^2\left(I \times \Omega\right) \text{ weakly,}$$

where u^0 is the unique solution of the LF - homogenized equation (3.128).

Proof. [Proof of Lemma 38] For any $k \in Y^*$ and for each $n \in M^k$, the test functions w^ε of the weak formulation (3.123) are chosen as in (3.61, 3.59, 3.60) in Subsection 3.3.2. Applying two integrations by parts and the boundary conditions satisfied by u^ε and by $\mathfrak{B}_n^k w$, it remains,

$$\begin{aligned} \int_{I\times\Omega} u^\varepsilon \cdot \left(\rho^\varepsilon \partial_{tt}\mathfrak{B}_n^k w - \partial_x\left(a^\varepsilon \partial_x \mathfrak{B}_n^k w\right) - \gamma^\varepsilon \partial_t \mathfrak{B}_n^k w - u^\varepsilon \cdot \partial_x\left(\zeta^\varepsilon \mathfrak{B}_n^k w\right) + \xi^\varepsilon \mathfrak{B}_n^k w\right) dtdx \\ + \int_\Omega -v_0^\varepsilon \cdot \rho^\varepsilon \left(\mathfrak{B}_n^k w\right)\left(t=0\right) + u_0^\varepsilon \cdot \rho^\varepsilon \partial_t \left(\mathfrak{B}_n^k w\right)\left(t=0\right) \\ -u_0^\varepsilon \cdot \gamma^\varepsilon \mathfrak{B}_n^k w\left(t=0\right) dx = \int_{I\times\Omega} f^\varepsilon \cdot \mathfrak{B}_n^k w dtdx. \end{aligned}$$

Using the decomposition (1.23) of the two-scale operators with remarking that $\left(\mathfrak{B}_n^k w\right)(t=0) = \left(\mathfrak{R}^k w\right)(t=0)$,

$$\int_{I\times\Omega} u^\varepsilon\cdot\, \mathfrak{B}_n^k \left(\sum_{l=0}^{2}\left(\left(\varepsilon\alpha_n^k\right)^{-l} Q^l w + \varepsilon^{-l} P^l w\right)\right) - f^\varepsilon \cdot \mathfrak{B}_n^k w \, dtdx \tag{3.130}$$
$$- \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k \left(\gamma\partial_t w + \frac{1}{\varepsilon\alpha_n^k}\gamma\partial_\tau w + \zeta\partial_x w + \frac{1}{\varepsilon}\partial_y(\zeta w) - \xi w\right) dtdx$$
$$+ \int_\Omega \left(u_0^\varepsilon\cdot\, \mathfrak{R}^k\rho\left(\partial_t w + \frac{1}{\varepsilon\alpha_n^k}\partial_\tau w\right) - v_0^\varepsilon \cdot \mathfrak{R}^k\rho w - u_0^\varepsilon \cdot \mathfrak{R}^k\gamma w\right)(t=0,\tau=0)\, dtdx = 0.$$

Moreover, from (1.4), (3.59) and $\partial_\tau w_0 = \partial_y w_0 = 0$ so

$$\int_{I\times\Omega} [u^\varepsilon \cdot \mathfrak{B}_n^k[\rho\left(\partial_{tt}w_0 + \frac{2}{\alpha_n^k}\partial_{t\tau}w_1 + \frac{1}{\varepsilon\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1\right) - \partial_x(a\partial_x w_0) \tag{3.131}$$
$$-\partial_x(a\partial_y w_1) - \frac{1}{\varepsilon}\partial_y(a\partial_x w_0) - \partial_y(a\partial_x w_1) - \frac{1}{\varepsilon}\partial_y(a\partial_y w_1)] - f^\varepsilon \cdot \mathfrak{B}_n^k w_0]\, dtdx$$
$$- \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k \left(\gamma\partial_t w_0 + \frac{1}{\alpha_n^k}\gamma\partial_\tau w_1 + \zeta\partial_x w_0 + \frac{1}{\varepsilon}\partial_y(\zeta w_0) + \partial_y(\zeta w_1) - \xi w_0\right) dtdx$$
$$+ \int_\Omega \left(u_0^\varepsilon\cdot \mathfrak{R}^k\rho\left(\partial_t w_0 + \frac{1}{\alpha_n^k}\partial_\tau w_1\right) - v_0^\varepsilon \cdot \mathfrak{R}^k\rho w_0 - u_0^\varepsilon \cdot \mathfrak{R}^k\gamma w_0\right)(t=0,\tau=0)\, dtdx = 0.$$

Multiplying by ε, then using the assumption (3.12), Equation (3.131) yields,

$$\int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k}\cdot [\frac{\rho}{\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1 - \partial_y(a\partial_x w_0) - \partial_y(a\partial_y w_1) - \partial_y(\zeta w_0)]\, dtd\tau dxdy = O(\varepsilon). \tag{3.132}$$

Passing to the limit and observing that

$$\int_{\Lambda\times Y} u_n^{0,k}\cdot \left(\rho\left(\frac{1}{\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1\right) - \partial_y(a\partial_x w_0) - \partial_y(\zeta w_0)\right) d\tau dy = 0,$$

so, Equation (3.132) is equivalent to (3.67) in the proof i) of Lemma 34,

$$\int_{I\times\Lambda\times\Omega\times Y} u_n^{0,k}\cdot \partial_y(a\partial_y w_1)\, dtd\tau dxdy = 0.$$

Thus, we also obtain $u_n^{0,k} = 0$ in $I\times\Omega$ if $k \neq 0$ and $u_n^{0,k} = \chi_0(k)\, u^0$.

In order to find the LF-homogenized equation, we also consider $k=0$ and $n=0$ as in Section 3.3. We restart with the weak formulation (3.123) by choosing the test functions as in (3.70, 3.68, 3.69) in the proof ii) of Lemma 34. So the very weak formulation (3.131) is equivalent to,

$$\int_{I\times\Omega} [u^\varepsilon \cdot \mathfrak{R}^0[\rho\partial_{tt}w_0 - \partial_x(a\partial_x w_0) - \partial_x(a\partial_y w_{1,0}) \tag{3.133}$$
$$-\frac{1}{\varepsilon}\partial_y(a\partial_x w_0) - \partial_y(a\partial_x w_{1,0}) - \frac{1}{\varepsilon}\partial_y(a\partial_y w_{1,0})] - f^\varepsilon \cdot \mathfrak{R}^0 w_0]\, dtdx$$
$$- \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{R}^0 \left(\gamma\partial_t w_0 + \zeta\partial_x w_0 + \frac{1}{\varepsilon}\partial_y(\zeta w_0) + \partial_y(\zeta w_{1,0}) - \xi w_0\right) dtdx$$
$$+ \int_\Omega \left(u_0^\varepsilon\cdot \mathfrak{R}^0\rho\partial_t w_0 - v_0^\varepsilon \cdot \mathfrak{R}^0\rho w_0 - u_0^\varepsilon \cdot \mathfrak{R}^0\gamma w_0\right)(t=0)\, dx = 0.$$

Choosing $w_0 = 0$ and multiplying by ε,

$$\int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{R}^0 \left(-\varepsilon\partial_x \left(a\partial_y w_{1,0}\right) - \varepsilon\partial_y \left(a\partial_x w_{1,0}\right) - \partial_y \left(a\partial_y w_{1,0}\right) - \varepsilon\partial_y \left(\zeta w_{1,0}\right)\right) dtdx = 0.$$

Using (3.54) the $(0,0)$-mode two-scale approximation of u^ε and observing that

$$\int_Y u^0 \cdot \left(\varepsilon\partial_y \left(a\partial_x w_{1,0}\right) - \partial_y \left(a\partial_y w_{1,0}\right) - \varepsilon\partial_y w_{1,0}\right) dy = 0,$$

the equation remains,

$$\int_{I\times\Omega\times Y} -\varepsilon u^0 \cdot \partial_x \left(a\partial_y w_{1,0}\right) - \varepsilon u_0^{1,0} \cdot \partial_y \left(a\partial_y w_{1,0}\right) dtdxdy = \varepsilon O\left(\varepsilon\right).$$

Dividing by ε and passing to the limit, the equation becomes (3.72) in the proof ii) of Lemma 34,

$$\int_{I\times\Omega\times Y} u^0 \cdot \partial_x \left(a\partial_y w_{1,0}\right) + u_0^{1,0} \cdot \partial_y \left(a\partial_y w_{1,0}\right) \; dtdxdy = 0.$$

Therefore, the equation (3.73) is obtained in this case also.

Furthermore, by choosing $w_{1,0} = 0$ and multiplying by ε, Equation (3.133) becomes

$$\begin{aligned}
&\int_{I\times\Omega} \left[u^\varepsilon \cdot \mathfrak{R}^0[\varepsilon\rho\partial_{tt} w_0 - \varepsilon\partial_x \left(a\partial_x w_0\right) - \partial_y \left(a\partial_x w_0\right)] - \varepsilon f^\varepsilon \cdot \mathfrak{R}^0 w_0\right] dtdx \\
&\quad -\varepsilon \int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{R}^0 \left(\gamma\partial_t w_0 + \zeta\partial_x w_0 + \partial_y \left(\zeta w_0\right) - \xi w_0\right) dtdx \\
&\quad +\varepsilon \int_\Omega \left(u_0^\varepsilon \cdot \mathfrak{R}^0 \rho\partial_t w_0 - v_0^\varepsilon \cdot \mathfrak{R}^0 \rho w_0 - u_0^\varepsilon \cdot \mathfrak{R}^0 \gamma w_0\right) (t=0)\, dx = 0.
\end{aligned}$$

According to (3.54), the data (3.19) and remarking that

$$\int_Y u^0 \cdot \left(\partial_y \left(a\partial_x w_0\right) + \partial_y \left(\zeta w_0\right)\right) dy = 0,$$

so,

$$\begin{aligned}
&\int_{I\times\Omega\times Y} [\varepsilon u^0 \cdot \left(\rho\partial_{tt} w_0 - \partial_x \left(a\partial_x w_0\right)\right) - \varepsilon u_0^{1,0} \cdot \left(\partial_y \left(a\partial_x w_0\right) + \partial_y \left(\zeta w_0\right)\right) \\
&\quad -\varepsilon f^0 \cdot w_0 dtdxdy] - \int_{I\times\Omega\times Y} \varepsilon u^0 \cdot \left(\gamma\partial_t w_0 + \zeta\partial_x w_0 - \xi w_0\right) dtdxdy \\
&\quad +\varepsilon \int_{\Omega\times Y} \left(\widehat{h^0} \cdot \rho\partial_t w_0 - g^0 \cdot \rho w_0 - \widehat{h^0} \cdot \gamma w_0\right) (t=0)\, dt\, dx = \varepsilon O\left(\varepsilon\right).
\end{aligned}$$

Dividing by ε and passing to the limit,

$$\begin{aligned}
&\int_{I\times\Omega\times Y} [u^0 \cdot \left(\rho\partial_{tt} w_0 - \partial_x \left(a\partial_x w_0\right)\right) - u_0^{1,0} \cdot \left(\partial_y \left(a\partial_x w_0\right) + \partial_y \left(\zeta w_0\right)\right) \\
&\quad -f^0 \cdot w_0 dtdxdy] - \int_{I\times\Omega\times Y} u^0 \cdot \left(\gamma\partial_t w_0 + \zeta\partial_x w_0 - \xi w_0\right) dtdxdy \\
&\quad + \int_{\Omega\times Y} \left(\widehat{h^0} \cdot \rho\partial_t w_0 - g^0 \cdot \rho w_0 - \widehat{h^0} \cdot \gamma w_0\right) (t=0)\, dt\, dx = 0.
\end{aligned}$$

Assuming that $u^0 \in H^2(I\times\Omega)$ and $u_0^{1,0} \in L^2(I; H^1(\Omega\times Y))$, taking integrations by parts, using (3.20), $w_0 \in L^2(I; H_0^1(\Omega))$ and periodicity of $u_0^{1,0}$, it remains,

$$\int_{I\times\Omega}\left(\widehat{\rho}\partial_{tt}u^0 - \partial_x\left(\left(\int_Y a dy\right)\partial_x u^0\right) - \partial_x\left(\int_Y a\partial_y u_0^{1,0}dy\right) - \widehat{f^0}\right)\cdot w_0 dtdx \tag{3.134}$$
$$+\int_{I\times\Omega}\left(\widehat{\gamma}\partial_t u^0 + \left(\int_Y \zeta dy\right)\partial_x u^0 + \int_Y \zeta\partial_y u_0^{1,0}dy + \widehat{\xi}u^0\right)\cdot w_0 dtdx$$
$$-\int_I\left[\left(\int_Y a dy\right)u^0\cdot\partial_x w_0\right]_{x=0}^{x=\alpha}dt + \int_\Omega\left[\widehat{\rho}\left(-u^0\cdot\partial_t w_0(t=0) + \partial_t u^0\cdot w_0(t=0)\right)\right.$$
$$\left.+\widehat{\gamma}u^0\cdot w_0\right]dx + \int_\Omega\left(\widehat{\rho}\widehat{h^0}\cdot\partial_t w_0 - \widehat{\rho}\widehat{g^0}\cdot w_0 - \widehat{\gamma}\widehat{h^0}\cdot w_0\right)(t=0)\,dx = 0.$$

According to the proof iii) of Lemma 34,

$$\partial_x\left(\left(\int_Y a dy\right)\partial_x u^0\right) + \partial_x\left(\int_Y a\partial_y u_0^{1,0}dy\right) = \partial_x\left(\left(\int_Y a(1+\partial_y\theta)\cdot(1+\partial_y\theta)\,dy\right)\partial_x u^0\right),$$

and

$$\left(\int_Y \zeta dy\right)\partial_x u^0 + \int_Y \zeta\partial_y u_0^{1,0}dy = \int_Y(\zeta+\zeta\partial_y\theta)\,dy\partial_x u^0 = \widehat{\zeta}\partial_x u^0.$$

Therefore, (3.134) is equivalent to,

$$\int_{I\times\Omega}\left(\widehat{\rho}\partial_{tt}u^0 - \partial_x\left(\widehat{a}\partial_x u^0\right) - \widehat{f^0}\right)\cdot w_0 + \left(\widehat{\gamma}\partial_t u^0 + \widehat{\zeta}\partial_x u^0 + \widehat{\xi}u^0\right)\cdot w_0 dtdx \tag{3.135}$$
$$-\int_I\left[\widehat{a}u^0\cdot\partial_x w_0\right]_{x=0}^{x=\alpha}dt + \int_\Omega -u^0\cdot\left(\widehat{\rho}\partial_t w_0 - \widehat{\gamma}w_0\right)(t=0) + \widehat{\rho}\partial_t u^0\cdot w_0(t=0)\,dx$$
$$+\int_\Omega \widehat{h^0}\cdot\left(\widehat{\rho}\partial_t w_0 - \widehat{\gamma}w_0\right)(t=0) - \widehat{\rho}\widehat{g^0}\cdot w_0(t=0)\,dx = 0.$$

Choosing test functions $w_0 \in H_0^1(I\times\Omega)$, then the strong form of the homogenized equation is

$$\widehat{\rho}\partial_{tt}u^0 - \partial_x\left(\widehat{a}\partial_x u^0\right) + \widehat{\gamma}\partial_t u^0 + \widehat{\zeta}\partial_x u^0 + \widehat{\xi}u^0 = \widehat{f^0} \text{ in } I\times\Omega.$$

So, in (3.135) it remains,

$$-\int_I\left[\widehat{a}u^0\cdot\partial_x w_0\right]_{x=0}^{x=\alpha}dt + \int_\Omega -u^0\cdot\left(\widehat{\rho}\partial_t w_0 - \widehat{\gamma}w_0\right)(t=0) + \widehat{\rho}\partial_t u^0\cdot w_0(t=0)\,dx$$
$$+\int_\Omega \widehat{h^0}\cdot\left(\widehat{\rho}\partial_t w_0 - \widehat{\gamma}w_0\right)(t=0) - \widehat{\rho}\widehat{g^0}\cdot w_0(t=0)\,dx = 0.$$

Then, the initial conditions and the boundary conditions of the LF-homogenized equation follow. ■

Remark 39 *The derivation of the HF-microscopic equation is done for a given $k \in Y^*$ by restarting from the weak formulation (3.123) and by following the proof of Lemma 38. Choosing test functions as (3.61, 3.59, 3.60) with $w_0 = 0$ in $I\times\Omega$ and $w_1 \in$*

$L^2\left(C_c^\infty\left(I\times\Omega\right);\Lambda\times Y\right)$, the very weak formulation (3.131) becomes,

$$\begin{aligned}\int_{I\times\Omega} u^\varepsilon\cdot\mathfrak{B}_n^k[\rho\left(\frac{2}{\alpha_n^k}\partial_{t\tau}w_1+\frac{1}{\varepsilon\left(\alpha_n^k\right)^2}\partial_{\tau\tau}w_1\right)-\partial_x\left(a\partial_y w_1\right)\\ -\partial_y\left(a\partial_x w_1\right)-\frac{1}{\varepsilon}\partial_y\left(a\partial_y w_1\right)]dtdx\\ -\int_{I\times\Omega}u^\varepsilon\cdot\mathfrak{B}_n^k\left(\frac{1}{\alpha_n^k}\gamma\partial_\tau w_1+\partial_y\left(\zeta w_1\right)\right)dtdx=O\left(\varepsilon\right).\end{aligned}$$

Multiplying by ε, observing that

$$\begin{aligned}&\varepsilon\int_{I\times\Omega}u^\varepsilon\cdot\mathfrak{B}_n^k\left(\frac{1}{\alpha_n^k}\gamma\partial_\tau w_1+\partial_y\left(\zeta w_1\right)\right)dtd\tau dxdy\\ =\;&\varepsilon\int_{I\times\Lambda\times\Omega\times Y}\left(\chi_0\left(k\right)u^0+\varepsilon u_n^{1,k}\right)\cdot\left(\frac{1}{\alpha_n^k}\gamma\partial_\tau w_1+\partial_y\left(\zeta w_1\right)\right)=\varepsilon O\left(\varepsilon\right),\end{aligned}$$

the equation is recovered to (3.81) in the proof of Lemma 35. Therefore, the HF - microscopic equation is then the same as in Section 3.3.1.

In order to be easier to follow the next lemma, we recall the modal decomposition (3.80) of the HF-macroscopic solution $\widehat{u}_n^{1,k}$

$$\widehat{u}_n^{1,k}\left(t,\tau,x,y\right)=\sum_{p\in M_{n,int}^k}u_p^k\left(t,x\right)e^{sign(p)2i\pi\sqrt{\frac{\lambda_{|p|}^k}{\lambda_n^k}}\tau}\phi_{|p|}^k\left(y\right)\text{ with }u_p^k\left(t,x\right)\in L^2\left(I\times\Omega\right). \tag{3.136}$$

Lemma 40 *For each $k\in Y^*$, $n\in\mathbb{N}^*$ and any bounded data as in (3.10), let u^ε be solution of the wave equation (3.123) satisfying the uniform bound (3.11) and the assumption (3.12). For $\varepsilon\in E_k$ as in Assumption 14, if $u_p^\sigma\in H^1\left(I\times\Omega\right)$ for each $\sigma\in I^k$, $s\in\{+,-\}$ and $q\in M_{n,s}^\sigma$ with (3.26) and (3.31) satisfying, then u_p^σ is solution of the HF-macroscopic model (3.129) including the boundary conditions.*

Proof. [Proof of Lemma 40] Let $k\in Y^*$, $n\in\mathbb{N}^*$ and $s\in\{-,+\}$, we consider $\left(\phi_{|q|}^\sigma\right)_{q\in M_{n,s}^\sigma,\sigma\in I^k}$ the Bloch eigenmodes corresponding to the eigenvalue λ_n^k of the Bloch mode equation. We restart with the very weak formulation (3.130) by choosing test functions as in (3.102, 3.93, 3.94, 3.103, 3.104, 3.95, 3.96, 3.98, 3.99, 3.100) in the proof of Lemma 36. Using $\left(\alpha_n^\sigma\right)^{-2}Q^2w_q^{\sigma,\varepsilon}+P^2w_q^{\sigma,\varepsilon}=0$, multiplying by ε and applying the decomposition (1.4), so the very weak formulation (3.130) yields,

$$\begin{aligned}\sum_{\sigma\in I^k,q\in M_{n,s}^\sigma}\int_{I\times\Omega}u^\varepsilon\cdot\mathfrak{B}_n^\sigma[\frac{2}{\alpha_n^k}\rho\partial_{t\tau}w_q^{\sigma,\varepsilon}+\varepsilon\rho\partial_{tt}w_q^{\sigma,\varepsilon}-\partial_x\left(a\partial_y w_q^{\sigma,\varepsilon}\right)\\ -\partial_y\left(a\partial_x w_q^{\sigma,\varepsilon}\right)-\varepsilon\partial_x\left(a\partial_x w_q^{\sigma,\varepsilon}\right)]-\varepsilon f^\varepsilon\cdot\mathfrak{B}_q^\sigma w_q^{\sigma,\varepsilon}-\\ u^\varepsilon\cdot\mathfrak{B}_n^\sigma\left(\varepsilon\gamma\partial_t w_q^{\sigma,\varepsilon}+\frac{1}{\alpha_n^k}\gamma\partial_\tau w_q^{\sigma,\varepsilon}+\varepsilon\zeta\partial_x w_q^{\sigma,\varepsilon}+\partial_y\left(\zeta w_q^{\sigma,\varepsilon}\right)-\varepsilon\xi w_q^{\sigma,\varepsilon}\right)dtdx\\ +\varepsilon\int_\Omega\left(u_0^\varepsilon\cdot\rho^\varepsilon\partial_t\left(\mathfrak{B}_n^k w\right)-v_0^\varepsilon\cdot\mathfrak{R}^\sigma\left(\rho w_q^{\sigma,\varepsilon}\right)-u_0^\varepsilon\cdot\mathfrak{R}^\sigma\left(\gamma w_q^{\sigma,\varepsilon}\right)\right)\left(t=0\right)dx=0.\end{aligned} \tag{3.137}$$

Thanks to (1.21) and (1.13) the relation between $\mathfrak{B}_n^\sigma$ and $T^{\varepsilon\alpha_n^\sigma *}S_\sigma^{\varepsilon *}$, $\mathfrak{R}^\sigma$ and $S_\sigma^{\varepsilon *}$, we observe that

$$\begin{aligned}\int_{I\times\Omega} f^\varepsilon\cdot\mathfrak{B}_n^k w_q^{\sigma,\varepsilon}dtdx &= \int_{I\times\Omega} f^\varepsilon\cdot T^{\varepsilon\alpha_n^\sigma *}S_\sigma^{\varepsilon *}w_q^{\sigma,\varepsilon}dtdx+O(\varepsilon) \\ &= \int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^\sigma}S_\sigma^\varepsilon f^\varepsilon\cdot w_q^{\sigma,\varepsilon}dtd\tau dxdy+O(\varepsilon),\end{aligned}\tag{3.138}$$

$$\begin{aligned}\int_\Omega v_0^\varepsilon\cdot\mathfrak{R}^\sigma\left(\rho w_q^{\sigma,\varepsilon}\right)dx &= \int_\Omega v_0^\varepsilon\cdot S_\sigma^{\varepsilon *}\left(\rho w_q^{\sigma,\varepsilon}\right)dx+O(\varepsilon) \\ &= \int_{\Omega\times Y} S_\sigma^\varepsilon v_0^\varepsilon\cdot\rho w_q^{\sigma,\varepsilon}\,dxdy+O(\varepsilon) \\ &= \int_{\Omega\times Y} S_\sigma^\varepsilon v_0^\varepsilon\cdot\rho\phi_{|q|}^\sigma\varphi_q^{\sigma,\varepsilon}(t=0)\,dxdy+O(\varepsilon),\end{aligned}\tag{3.139}$$

$$\begin{aligned}&\int_\Omega u_0^\varepsilon\cdot\partial_t\mathfrak{B}_q^\sigma\left(\rho w_q^{\sigma,\varepsilon}\right)(t=0)\,dx \\ &= -\frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}}\int_{\Omega\times Y}\partial_x S_\sigma^\varepsilon u_0^\varepsilon\cdot\left(a\partial_y\phi_{|q|}^\sigma+\partial_y\left(a\phi_{|q|}^\sigma\right)\right)\varphi_q^{\sigma,\varepsilon}(t=0)\,dxdy+O(\varepsilon),\end{aligned}$$

and

$$\begin{aligned}\int_\Omega u_0^\varepsilon\cdot\mathfrak{R}^\sigma\left(\gamma w_q^{\sigma,\varepsilon}\right)dx &= \int_\Omega u_0^\varepsilon\cdot S_\sigma^{\varepsilon *}\left(\gamma w_q^{\sigma,\varepsilon}\right)dx+O(\varepsilon) \\ &= \int_{\Omega\times Y} S_\sigma^\varepsilon u_0^\varepsilon\cdot\gamma w_q^{\sigma,\varepsilon}\,dxdy+O(\varepsilon) \\ &= \int_{\Omega\times Y} S_\sigma^\varepsilon u_0^\varepsilon\cdot\gamma\phi_{|q|}^\sigma\varphi_q^{\sigma,\varepsilon}(t=0)\,dxdy+O(\varepsilon).\end{aligned}\tag{3.140}$$

Using the assumption (3.12), the decompositions (3.138)-(3.140), (3.77) and (3.78), and remarking that $u_n^{0,\sigma}=\chi_0(\sigma)\,u^0$, Equation (3.137) is rewritten by

$$\begin{aligned}&\sum_{\sigma\in I^k,q\in M_{n,s}^\sigma}\int_{I\times\Lambda\times\Omega\times Y}\left(\chi_0(\sigma)\,u^0+\varepsilon\chi_0(\sigma)\,u_0^{1,0}+\varepsilon\widehat{u}_n^{1,\sigma}\right)\cdot[\frac{2}{\alpha_n^k}\rho\partial_{t\tau}w_q^{\sigma,\varepsilon} \\ &+\varepsilon\rho\partial_{tt}w_q^{\sigma,\varepsilon}-\partial_x\left(a\partial_y w_q^{\sigma,\varepsilon}\right)-\partial_y\left(a\partial_x w_q^{\sigma,\varepsilon}\right)-\varepsilon\partial_x\left(a\partial_x w_q^{\sigma,\varepsilon}\right)] \\ &-\varepsilon T^{\varepsilon\alpha_n^\sigma}S_\sigma^\varepsilon f^\varepsilon\cdot w_q^{\sigma,\varepsilon}-\left(\chi_0(\sigma)\,u^0+\varepsilon\chi_0(\sigma)\,u_0^{1,0}+\varepsilon\widehat{u}_n^{1,\sigma}\right)\cdot[\varepsilon\gamma\partial_t w_q^{\sigma,\varepsilon} \\ &+\frac{1}{\alpha_n^k}\gamma\partial_\tau w_q^{\sigma,\varepsilon}+\varepsilon\zeta\partial_x w_q^{\sigma,\varepsilon}+\partial_y\left(\zeta w_q^{\sigma,\varepsilon}\right)-\varepsilon\xi w_q^{\sigma,\varepsilon}]dtd\tau dxdy \\ &-\varepsilon\int_\Omega[\frac{sign\,(q)\,i}{\sqrt{\lambda_n^k}}\partial_x S_\sigma^\varepsilon u_0^\varepsilon\cdot\left(a\partial_y\phi_{|q|}^\sigma+\partial_y\left(a\phi_{|q|}^\sigma\right)\right)\varphi_q^{\sigma,\varepsilon}(t=0) \\ &+\left(S_\sigma^\varepsilon v_0^\varepsilon\cdot\rho\phi_{|q|}^\sigma\varphi_q^{\sigma,\varepsilon}+S_\sigma^\varepsilon u_0^\varepsilon\cdot\gamma\phi_{|q|}^\sigma\varphi_q^{\sigma,\varepsilon}\right)(t=0)\,dxdy=0.\end{aligned}$$

Moreover, since $\int_\Lambda w_q^{\sigma,\varepsilon}d\tau=0$, $\int_\Lambda\partial_\tau w_q^{\sigma,\varepsilon}d\tau=0$, $w_q^{\sigma,\varepsilon}$ is periodic in τ, u^0 and $u_0^{1,0}$ are independent on τ, so

$$\begin{aligned}&\int_\Lambda\chi_0(\sigma)\left(u^0+\varepsilon u_0^{1,0}\right)\cdot(\frac{2}{\alpha_n^k}\rho\partial_{t\tau}w_q^{\sigma,\varepsilon}+\varepsilon\rho\partial_{tt}w_q^{\sigma,\varepsilon}-\partial_x\left(a\partial_y w_q^{\sigma,\varepsilon}\right)-\partial_y\left(a\partial_x w_q^{\sigma,\varepsilon}\right) \\ &-\varepsilon\partial_x\left(a\partial_x w_q^{\sigma,\varepsilon}\right)+\varepsilon\partial_t w_q^{\sigma,\varepsilon}+\frac{1}{\alpha_n^k}\partial_\tau w_q^{\sigma,\varepsilon}+\varepsilon\partial_x w_q^{\sigma,\varepsilon}+\partial_y w_q^{\sigma,\varepsilon}-\varepsilon w_q^{\sigma,\varepsilon})d\tau=0.\end{aligned}$$

Moreover, according to (3.98), (3.21) and (3.125)

$$\lim_{\varepsilon\to 0}\int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^\sigma} S_\sigma^\varepsilon f^\varepsilon \cdot w_q^{\sigma,\varepsilon}\, dtd\tau dxdy = \int_{I\times\Omega} F_q^\sigma \cdot \varphi_q^\sigma dtdx,$$
$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y} \frac{sign\,(q)\, i}{\sqrt{\lambda_n^k}} \partial_x S_\sigma^\varepsilon u_0^\varepsilon \cdot \left(a\partial_y \phi_{|q|}^\sigma + \partial_y\left(a\phi_{|q|}^\sigma\right)\right)\varphi_q^{\sigma,\varepsilon}\,(t=0)$$
$$+S_\sigma^\varepsilon v_0^\varepsilon \cdot \rho\phi_{|q|}^\sigma \varphi_q^{\sigma,\varepsilon} + S_\sigma^\varepsilon u_0^\varepsilon \cdot \gamma\phi_{|q|}^\sigma \varphi_q^{\sigma,\varepsilon}\,(t=0,\tau=0)\, dxdy = \int_\Omega \left(g_q^\sigma + h_q^\sigma\right)\cdot \varphi_q^\sigma\,(t=0)\, dx.$$

Dividing by ε and passing to the limit, the equation (3.137) reads,

$$\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma}\int_{I\times\Lambda\times\Omega\times Y} \widehat{u}_n^{1,\sigma} \cdot [\frac{2}{\alpha_n^k}\rho\partial_{t\tau} w_q^\sigma - \partial_x\left(a\partial_y w_q^\sigma\right) - \partial_y\left(a\partial_x w_q^\sigma\right)]\, dtd\tau dxdy$$
$$-\int_{I\times\Omega} F_q^\sigma \cdot \varphi_q^\sigma dtdx - \int_{I\times\Lambda\times\Omega\times Y} \widehat{u}_n^{1,\sigma}\cdot\left(\frac{1}{\alpha_n^k}\gamma\partial_\tau w_q^\sigma + \partial_y\left(\zeta w_q^\sigma\right)\right) dtd\tau dxdy$$
$$-\int_\Omega \left(g_q^\sigma + h_q^\sigma\right)\cdot\varphi_q^\sigma\,(t=0)\, dx = 0.$$

From the decomposition (3.136) of $\widehat{u}_n^{1,\sigma}$ and (3.111),

$$\sum_{\sigma\in I^k, p\in M_{n,s}^\sigma, q\in M_{n,s}^\sigma}\int_{I\times\Omega}\Bigg[-sign\,(q)\, 4i\pi\left(\alpha_q^\sigma\right)^{-1}\left(\int_Y \rho\phi_{|p|}^\sigma\overline{\phi_{|q|}^\sigma}dy\right) u_p^\sigma\cdot\partial_t\varphi_q^\sigma$$
$$-\left(\int_Y \phi_p^\sigma\left(a\partial_y\overline{\phi_{|q|}^\sigma} + \partial_y\left(a\overline{\phi_{|q|}^\sigma}\right)\right)dy\right) u_p^\sigma\cdot\partial_x\varphi_q^\sigma$$
$$-\left(-sign\,(q)\, 2i\pi\left(\alpha_q^\sigma\right)^{-1}\int_Y \gamma\phi_{|p|}^\sigma\overline{\phi_{|q|}^\sigma}dy + \int_Y \phi_{|p|}^\sigma\partial_y\left(\zeta\overline{\phi_{|q|}^\sigma}\right)dy\right) u_p^\sigma\cdot\varphi_q^\sigma\Bigg] dtdx$$
$$-\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma}\left[\int_\Omega\left(g_q^\sigma + h_q^\sigma\right)\cdot\varphi_q^\sigma\,(t=0)\, dx + \int_{I\times\Omega} F_q^\sigma\cdot\varphi_q^\sigma dtdx\right] = 0.$$

From (3.112) and (3.127), so

$$\sum_{\sigma\in I^k, p\in M_{n,s}^\sigma, q\in M_{n,s}^\sigma}\int_{I\times\Omega} -b\,(\sigma,p,q)\, u_p^\sigma\cdot\partial_t\varphi_q^\sigma - c\,(\sigma,p,q)\, u_p^\sigma\cdot\partial_x\varphi_q^\sigma + d\,(\sigma,p,q)\, u_p^\sigma\cdot\varphi_q^\sigma dtdx$$
$$-\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma}\left[\int_\Omega\left(g_q^\sigma + h_q^\sigma\right)\cdot\varphi_q^\sigma\,(t=0)\, dx + \int_{I\times\Omega} F_q^\sigma\cdot\varphi_q^\sigma dtdx\right] = 0.$$

Assuming that $u_p^\sigma \in H^1\left(I\times\Omega\right)$ and taking integrations by parts,

$$\sum_{\sigma\in I^k, p\in M_{n,s}^\sigma, q\in M_{n,s}^\sigma}\Bigg[\int_{I\times\Omega} b\,(\sigma,p,q)\,\partial_t u_p^\sigma\cdot\varphi_q^\sigma + c\,(\sigma,p,q)\,\partial_x u_p^\sigma\cdot\varphi_q^\sigma$$
$$+d\,(\sigma,p,q)\, u_p^\sigma\cdot\varphi_q^\sigma dtdx + \int_\Omega b\,(\sigma,p,q)\, u_p^\sigma\cdot\varphi_q^\sigma\,(t=0)\, dx - \int_I\left[c\,(\sigma,p,q)\, u_p^\sigma\cdot\varphi_q^\sigma\right]_{x=0}^{x=\alpha} dtdx\Bigg]$$
$$-\sum_{\sigma\in I^k, q\in M_{n,s}^\sigma}\left[\int_\Omega\left(g_q^\sigma + h_q^\sigma\right)\cdot\varphi_q^\sigma\,(t=0)\, dx + \int_{I\times\Omega} F_q^\sigma\cdot\varphi_q^\sigma dtdx\right] = 0.$$

For each $\sigma \in I^k$ and $q \in M^\sigma_{n,s}$, by choosing test functions $\varphi^\sigma_q \in H^1_0(I \times \Omega)$ the strong form comes

$$\sum_{p \in M^\sigma_{n,s}} b(\sigma,p,q)\,\partial_t u^\sigma_p + c(\sigma,p,q)\,\partial_x u^\sigma_p + d(\sigma,p,q)\,u^\sigma_p = F^\sigma_q \text{ in } I \times \Omega.$$

It remains,

$$\sum_{\sigma \in I^k, q \in M^\sigma_{n,s}} \left[\int_\Omega \left(\sum_{p \in M^\sigma_{n,s}} b(\sigma,p,q)\,u^\sigma_p - g^\sigma_q - h^\sigma_q \right) \cdot \varphi^\sigma_q(t=0)\,dx \right. \quad (3.141)$$

$$\left. - \int_I \sum_{p \in M^\sigma_{n,s}} \left[c(\sigma,p,q)\,u^\sigma_p \cdot \varphi^\sigma_q \right]_{x=0}^{x=\alpha} dt \right] = 0.$$

Therefore, the initial condition is deduced,

$$\sum_{p \in M^\sigma_{n,s}} b(\sigma,p,q)\,u^\sigma_p(t=0) = g^\sigma_q + h^\sigma_q \text{ in } \Omega, \quad (3.142)$$

and the boundary term is,

$$\sum_{\sigma \in I^k, p \in M^\sigma_{n,s}, q \in M^\sigma_{n,s}} \int_I \left[c(\sigma,p,q)\,u^\sigma_p \cdot \varphi^\sigma_q \right]_{x=0}^{x=\alpha} dt = 0 \text{ for } \varphi^\sigma_q \text{ satisfies (3.100).} \quad (3.143)$$

This formula is the same as in the proof of Lemma 36. Finally, the boundary conditions are found exactly on the same way. ■

3.5 Homogenization based on a first order formulation

In this section, the homogenization is studied based on the first order formulation of the wave equation (3.9). This result has already been published in [94] and is to appear in proceeding of ENUMATH 2013.

In fact, the method introduced here is inspired from in [35], except that in the present work the two-scale transform $T^{\varepsilon\alpha^k_n} S^\varepsilon_k$ and $T^{\varepsilon\alpha^k_n} S^\varepsilon_{-k}$ are analyzed separately. In [35], the homogenization is studied based on the first order, but the boundary conditions of the homogenized model was not found. Therefore, establishing the boundary conditions of the homogenized model is critical and is the goal of this section which also extends [65].

To this end, the wave equation is written under the form of a first order formulation and the modulated two-scale transform W^ε_k , which is defined as in [35], is applied to the solution U^ε. Similarly to the homogenization based on the second order formulation, the homogenized model is also derived for a set of pairs of fibers I^k defined in (1.3) which allows to derive the expected boundary conditions. The weak limit of $\sum_{\sigma \in I^k} W^\varepsilon_\sigma U^\varepsilon$ includes low and high frequency waves, the former being solution of the homogenized model derived in [33], [60] and the latter are associated to Bloch wave expansions. Numerical results comparing solutions of the wave equation with solution of the two-scale model for fixed ε and k are reported in the forthcoming Section 3.6. The calculations are less detailed than the model in Section 3.2 only the main results and the proof principles are given.

3.5.1 Reformulation of the wave equation under the first order formulation

Similar to Section 2.4 in Chapter 2 for the case of the spectral problem, we start by setting,

$$U^\varepsilon := (\sqrt{a^\varepsilon}\partial_x u^\varepsilon, \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon),\ A^\varepsilon = \begin{pmatrix} 0 & \sqrt{a^\varepsilon}\partial_x\left(\frac{1}{\sqrt{\rho^\varepsilon}}.\right) \\ \frac{1}{\sqrt{\rho^\varepsilon}}\partial_x\left(\sqrt{a^\varepsilon}.\right) & 0 \end{pmatrix},$$

$$U_0^\varepsilon := (\sqrt{a^\varepsilon}\partial_x u_0^\varepsilon, \sqrt{\rho^\varepsilon}v_0^\varepsilon) \text{ and } F^\varepsilon := (0, f^\varepsilon/\sqrt{\rho^\varepsilon}).$$

We reformulate the wave equation (3.9) as an equivalent system,

$$(\partial_t - A^\varepsilon)\,U^\varepsilon = F^\varepsilon \text{ in } I\times\Omega,\ U^\varepsilon\,(t=0) = U_0^\varepsilon \text{ in } \Omega \text{ and } U_2^\varepsilon = 0 \text{ on } I\times\partial\Omega,$$

where U_2^ε is the second component of U^ε. From now on, this system will be referred to as the physical problem and taken in the distributional sense,

$$\int_{I\times\Omega} F^\varepsilon\cdot\Psi + U^\varepsilon\cdot(\partial_t - A^\varepsilon)\ \Psi dtdx + \int_\Omega U_0^\varepsilon\cdot\Psi\,(t=0)\ dx = 0, \tag{3.144}$$

for all the admissible test functions $\Psi\in H^1(I\times\Omega)^2$ such that $\Psi\,(t,.)\in D\,(A^\varepsilon)$ for a.e. $t\in I$ where the domain

$$D(A^\varepsilon) := \{(\varphi,\phi)\in L^2\,(\Omega)^2 | \sqrt{a^\varepsilon}\varphi\in H^1\,(\Omega)\,, \phi/\rho\in H_0^1\,(\Omega)\}.$$

As proved in [35], the operator iA^ε with the domain $D(A^\varepsilon)$ is self-adjoint on $L^2(\Omega)^2$. According to the assumption (3.10), U^ε is uniformly bounded in $L^2\,(I\times\Omega)$, see detail of proof of Theorem 3 in [35].

3.5.2 Homogenized results and proofs

The Bloch wave spectral problem $\mathcal{P}(k)$ in (1.5) is also reformulated under the first order formulation as in (2.54) in Section 2.4 in Chapter 2. For each $k\in Y^*$, $n\in\mathbb{Z}^*$, we also pose

$$M_n^k := \{m\in\mathbb{Z}^* | \lambda_{|m|}^k = \lambda_{|n|}^k \text{ and } sign\,(m) = sign\,(n)\}$$

and introduce the HF-macroscopic model coefficients

$$c(k,p,q) = i\frac{sign\,(n)}{2\sqrt{\lambda_{|p|}^k}}\int_Y \phi_{|p|}^k\cdot a\partial_y\phi_{|q|}^k - a\partial_y\phi_{|p|}^k\cdot\phi_{|q|}^k dy \text{ and } b(k,p,q) = \int_Y \rho\phi_{|p|}^k\cdot\phi_{|q|}^k dy$$

for any $p,q\in M_n^k$. For any $k\in Y^*$, we introduce the operator $W_k^\varepsilon : L^2\,(I\times\Omega)^2 \to L^2\,(I\times\Lambda\times\Omega\times Y)^2$ acting in all time and space variables,

$$W_k^\varepsilon := \chi_0\,(k)\left(1 - \sum_{n\in\mathbb{Z}^*}\Pi_n^0\right)S_0^\varepsilon + \sum_{\sigma\in I^k, n\in\mathbb{Z}^*} T^{\varepsilon\alpha_{|n|}^\sigma}\Pi_n^\sigma S_\sigma^\varepsilon \tag{3.145}$$

where $\alpha_{|n|}^\sigma$, the time and space two-scale transforms $T^{\varepsilon\alpha_{|n|}^\sigma}$ and S_σ^ε are defined in (1.7) and (1.15), the one-dimensional L^2-orthogonal projector Π_n^σ onto e_n^σ are defined in [35]. Thanks to the boundness of $T^{\varepsilon\alpha_{|n|}^k}$ and S_k^ε, it is proved that,

$$\|W_k^\varepsilon u\|_{L^2(I\times\Lambda\times\Omega\times Y)}^2 = \|u\|_{L^2(I\times\Omega)}^2\,. \tag{3.146}$$

For a function $v(t,\tau,x,y)$ defined in $I\times\Lambda\times\Omega\times Y$, we observe that

$$A^\varepsilon \mathfrak{B}_n^k v = \mathfrak{B}_n^k\left(\left(\frac{A_k}{\varepsilon}+B\right)v\right) \text{ and } \partial_t\left(\mathfrak{B}_n^k v\right) = \mathfrak{B}_n^k\left(\left(\frac{\partial_\tau}{\varepsilon\alpha_{|n|}^k}+\partial_t\right)v\right), \qquad (3.147)$$

where the operators A_k and B are defined in (2.52) and (2.57).

After extraction of a subsequence, we introduce the weak limits of the relevant projections along e_n^k for any $n\in\mathbb{Z}^*$,

$$F_n^k := \lim_{\varepsilon\to 0}\int_{\Lambda\times Y} T^{\varepsilon\alpha_{|n|}^k} S_k^\varepsilon F^\varepsilon\cdot e^{sign(n)2i\pi\tau} e_n^k dy d\tau \text{ and } U_{0,n}^k := \lim_{\varepsilon\to 0}\int_Y S_k^\varepsilon U_0^\varepsilon\cdot e_n^k dy. \qquad (3.148)$$

The low frequency part U_H^0 relates to the weak limit in $L^2(I\times\Omega\times Y)^2$ of the kernel part of S_k^ε in 3.145. It has been treated completely, in [33, 35]. Here, we focus on the non-kernel part of S_k^ε, it relates to the high frequency waves and microscopic and macroscopic scales. In order to obtain the solution of the model, we analyze the asymptotic behaviour of each mode through $T^{\varepsilon\alpha_{|n|}^k} S_k^\varepsilon$ as in Lemma 42 and Lemma 43 stated below. Then the full solution is the sum of all modes. The main Theorem states as follows.

Theorem 41 *For a given $k\in Y^*$, let U^ε be a solution of (3.144) bounded in $L^2(I\times\Omega)$, for $\varepsilon\in E_k$, as in (1.39, 1.40), the limit G_k of any weakly converging extracted subsequence of $\sum_{\sigma\in I^k} W_\sigma^\varepsilon U^\varepsilon$ in $L^2(I\times\Lambda\times\Omega\times Y)^2$ can be decomposed as*

$$G^k(t,\tau,x,y) = \chi_0(k)\,U_H^0(t,x,y) + \sum_{\sigma\in I^k, n\in\mathbb{Z}^*} u_n^\sigma(t,x)\, e^{sign(n)2i\pi\tau} e_n^\sigma(y) \qquad (3.149)$$

where $(u_n^\sigma)_{n,\sigma}$ are solutions of the HF-macroscopic equation (3.153)-(3.155) stated in Lemma 43.

Thus, it follows from (3.149) that the physical solution U^ε is approximated by two-scale modes

$$U^\varepsilon(t,x) \simeq \chi_0(k)\,U_H^0\left(t,x,\frac{x}{\varepsilon}\right) + \sum_{\sigma\in I^k, n\in\mathbb{Z}^*} u_n^\sigma(t,x)\, e^{sign(n)i\sqrt{\lambda_{|n|}^\sigma}t/\varepsilon} e_n^\sigma\left(\frac{x}{\varepsilon}\right). \qquad (3.150)$$

Proof. [Proof of Theorem 41] For a given $k\in Y^*$, let U^ε be solution of (3.144) which is bounded in $L^2(I\times\Omega)$, the property (3.146) yields the boundness of $\|W_\sigma^\varepsilon U^\varepsilon\|_{L^2(I\times\Lambda\times\Omega\times Y)}$ for $\sigma\in I^k$. So there exists $G^k\in L^2(I\times\Lambda\times\Omega\times Y)^2$ such that, up to the extraction of a subsequence, $\sum_{\sigma\in I^k} W_\sigma^\varepsilon U^\varepsilon$ tends weakly to

$$G^k = \chi_0(k)\,U_H^0 + \sum_{\sigma\in I^k, n\in\mathbb{Z}^*} U_n^\sigma \text{ in } L^2(I\times\Lambda\times\Omega\times Y)^2.$$

The high frequency part is based on the below decomposition (3.152) and Lemma 43. ■

The next lemmas state the HF-microscopic equation for each mode and the corresponding HF-macroscopic equation.

Lemma 42 *For $k \in Y^*$ and $n \in \mathbb{Z}^*$, let U^ε be a bounded solution of (3.144), there exists at least a subsequence of $T^{\varepsilon\alpha^k_{|n|}} S^\varepsilon_k U^\varepsilon$ converging weakly towards a limit U^k_n in $L^2(I \times \Lambda \times \Omega \times Y)^2$ when ε tends to zero. Then U^k_n is a solution of the weak formulation of the HF-microscopic equation*

$$\left(\frac{\partial_\tau}{\alpha^k_{|n|}} - A_k\right) U^k_n = 0 \text{ in } I \times \Lambda \times \Omega \times Y \tag{3.151}$$

and is periodic in τ and $k-$quasi-periodic in y. Moreover, it can be decomposed as

$$U^k_n(t,\tau,x,y) = \sum_{p \in M^k_n} u^k_p(t,x)\, e^{sign(p) 2i\pi\tau} e^k_p(y) \text{ with } u^k_p \in L^2(I \times \Omega). \tag{3.152}$$

Proof. [Proof of Lemma 42] The test functions of the weak formulation (3.144) are chosen as $\Psi^\varepsilon = \mathfrak{B}^k_n \Psi(t,x)$ for $k \in Y^*$, $n \in \mathbb{Z}^*$ where $\Psi \in C^\infty(I \times \Lambda \times \Omega \times Y)^2$ is periodic in τ and $k-$quasi-periodic in y. From (3.147) multiplied by ε, since $\left(\frac{\partial_\tau}{\alpha^k_{|n|}} - A_k\right)\Psi$ is periodic in τ and $k-$quasi-periodic in y and $T^{\varepsilon\alpha^k_{|n|}} S^\varepsilon_k U^\varepsilon \to \overset{k}{U}_n$ in $L^2(I \times \Lambda \times \Omega \times Y)^2$ weakly, Lemma 8 allows to pass to the limit in the weak formulation,

$$\int_{I \times \Lambda \times \Omega \times Y} U^k_n \cdot \left(\frac{\partial_\tau}{\alpha^k_{|n|}} - A_k\right) \Psi dt d\tau dx dy = 0.$$

Using the assumption $U^k_n \in D(A_k) \cap L^2(I \times \Omega \times Y; H^1(\Lambda))$ and applying an integration by parts,

$$\int_{I \times \Lambda \times \Omega \times Y} \left(-\frac{\partial_\tau}{\alpha^k_{|n|}} + A_k\right) U^k_n \cdot \Psi dt d\tau dx dy + \int_{I \times \partial\Lambda \times \Omega \times Y} U^k_n \cdot \Psi dt d\tau dx dy$$
$$- \int_{I \times \Lambda \times \Omega \times \partial Y} U^k_n \cdot n_{A_k} \Psi dt d\tau dx dy = 0.$$

Then, choosing $\Psi \in L^2(I \times \Omega; H^1_0(\Lambda \times Y))$ comes the strong form (3.151). Since the product of a periodic function by a $k-$quasi-periodic function is $k-$quasi-periodic then $n_{A_k}\Psi$ is $k-$quasi-periodic in y. Therefore, U^k_n is periodic in τ and $k-$quasi-periodic in y. Moreover, (3.152) is obtained, by projection. ∎

Lemma 43 *For $k \in Y^*$, $n \in \mathbb{Z}^*$ and $\varepsilon \in E_k$ as in Assumption 14, let U^ε be a bounded solution of (3.144) such that the weak limit U^σ_n of $T^{\varepsilon\alpha^\sigma_{|n|}} S^\varepsilon_k U^\varepsilon$ satisfies (3.152). For each $\sigma \in I^k$ and $q \in M^\sigma_n$, if $u^\sigma_p \in H^1(I \times \Omega)$ then u^σ_p is solution of the HF-macroscopic equation stated by*

$$\begin{aligned} &\sum_{p \in M^\sigma_n} b(\sigma,p,q)\, \partial_t u^\sigma_p - \sum_{p \in M^\sigma_n} c(\sigma,p,q)\, \partial_x u^\sigma_p = F^\sigma_q \text{ in } I \times \Omega, \\ &\sum_{p \in M^\sigma_n} b(\sigma,p,q)\, u^\sigma_p(t=0) = U^\sigma_{0,q} \text{ in } \Omega, \end{aligned} \tag{3.153}$$

with the boundary conditions in the case of $k \neq -\frac{1}{2}$ where there exist $p \in M^k_n$ such that $c(k,p,q) \neq 0$ and $\phi^k_{|p|}(0) \neq 0$

$$\sum_{\sigma \in I^k, p \in M^\sigma_n} u^\sigma_p \phi^\sigma_{|p|}(0)\, e^{sign(\sigma) 2i\pi \frac{l^k x}{\alpha}} = 0 \text{ on } I \times \partial\Omega \tag{3.154}$$

and in the case of $k=-\frac{1}{2}$ *where* $M_n^k=\{n,n'\}$

$$\begin{aligned}&\left(c\left(k,n,n\right)\phi^k_{|n'|}\left(0\right)-c\left(k,n',n\right)\phi^k_{|n|}\left(0\right)\right)u^k_n\\+&\left(c\left(k,n,n'\right)\phi^k_{|n'|}\left(0\right)-c\left(k,n',n'\right)\phi^k_{|n|}\left(0\right)\right)u^k_{n'}=0 \text{ on } I\times\partial\Omega.\end{aligned}\tag{3.155}$$

Proof. [Proof of Lemma 43] For $k\in Y^*$, let $\left(\lambda^\sigma_{|p|},e^\sigma_p\right)_{p\in M^\sigma_n,\sigma\in I^k}$ be the Bloch eigenmodes of the spectral equation $\mathcal{Q}\left(\sigma\right)$ corresponding to the eigenvalue $\lambda^k_{|n|}$. We pose

$$\Psi^\varepsilon\left(t,x\right)=\sum_{\sigma\in I^k}\mathfrak{B}^\sigma_n\Psi^\sigma_\varepsilon\in H^1\left(I\times\Omega\right)^2$$

as a test function in the weak formulation (3.144) with each

$$\Psi^\sigma_\varepsilon\left(t,\tau,x,y\right)=\sum_{q\in M^k_n}\varphi^{\sigma,\varepsilon}_q\left(t,x\right)e^{sign(q)2i\pi\tau}e^\sigma_q\left(y\right)$$

where $\varphi^{\sigma,\varepsilon}_q\in H^1\left(I\times\Omega\right)$ and satisfies the boundary conditions

$$\sum_{\sigma\in I^k,q\in M^\sigma_n}e^{sign(q)2i\pi t/(\varepsilon\alpha^\sigma_{|q|})}\varphi^{\sigma,\varepsilon}_q\left(t,x\right)\phi^\sigma_{|q|}\left(\frac{x}{\varepsilon}\right)=O\left(\varepsilon\right)\ \text{ on } I\times\partial\Omega.$$

Note that this condition is related to the second component of Ψ^ε only. Since $\alpha^\sigma_{|q|}=\alpha^k_{|n|}$ and $sign\left(q\right)=sign(n)$ for all $q\in M^\sigma_n$ and $\sigma\in I^k$, so $e^{sign(q)2i\pi t/(\varepsilon\alpha^\sigma_{|q|})}\neq 0$ can be eliminated. Extracting a subsequence $\varepsilon\in E_k$, using the $\sigma-$quasi-periodicity of $\phi^\sigma_{|q|}$ and (1.39,1.40), $\varphi^{\sigma,\varepsilon}_q$ converges strongly to some φ^σ_q in $H^1\left(I\times\Omega\right)$ as in Lemma 37, then the boundary conditions are

$$\sum_{\sigma\in I^k,q\in M^\sigma_n}\varphi^\sigma_q\left(t,x\right)\phi^\sigma_{|q|}\left(0\right)e^{sign(\sigma)2i\pi\frac{l^k x}{\alpha}}=0 \text{ on } I\times\partial\Omega.\tag{3.156}$$

Applying (3.147) and since $\left(\frac{\partial_\tau}{\alpha^\sigma_{|n|}}-A_\sigma\right)\Psi^\sigma=0$ for any $\sigma\in I^k$, then in the weak formulation it remains,

$$\sum_{\sigma\in I^k}\int_{I\times\Omega}F^\varepsilon\cdot\mathfrak{B}^\sigma_n\Psi^\sigma_\varepsilon+U^\varepsilon\cdot\mathfrak{B}^\sigma_n(\partial_t-B)\,\Psi^\sigma_\varepsilon dtdx-\int_\Omega U^\varepsilon_0\cdot\mathfrak{B}^\sigma_n\Psi^\sigma_\varepsilon\left(t=0\right)dx=0.$$

Since $\left(\partial_t-B\right)\Psi^\sigma_\varepsilon$ is $\sigma-$quasi-periodic, so passing to the limit thanks to Lemma 8, after using (3.148) and replacing the decomposition of U^σ_n,

$$\begin{aligned}\sum_{\sigma\in I^k,\{p,q\}\in M^\sigma_n}&\left(\int_{I\times\Omega}b\left(\sigma,p,q\right)u^\sigma_p\cdot\partial_t\varphi^\sigma_q-c\left(\sigma,p,q\right)u^\sigma_p\cdot\partial_x\varphi^\sigma_q-F^\sigma_q\cdot\varphi^\sigma_q\,dtdx\right.\\&\left.-\int_\Omega U^\sigma_{0,q}\cdot\varphi^\sigma_q\left(t=0\right)dx\right)=0 \text{ for all } \varphi^\sigma_q\in H^1\left(I\times\Omega\right)\ \text{ fulfilling (3.156).}\end{aligned}$$

Moreover, if $u^\sigma_q\in H^1\left(I\times\Omega\right)$ then it satisfies the strong form of the internal equations (3.27) for each $\sigma\in I^k$, $q\in M^\sigma_n$ and the boundary conditions

$$\sum_{\sigma,p,q}c\left(\sigma,p,q\right)u^\sigma_p\overline{\varphi^\sigma_q}=0 \text{ on } I\times\partial\Omega \text{ for } \varphi^\sigma_q \text{ satisfies (3.156).}\tag{3.157}$$

In order to find the boundary conditions of $\left(u_p^\sigma\right)_{\sigma,p}$, we distinguish between the three cases $k \neq 0$, $k = 0$ and $k = -\frac{1}{2}$.

First, for $k \neq 0$, $\lambda_{|n|}^k$ is simple so $M_n^k = \{n\}$. Introducing $C = diag\left(c\left(\sigma, n, n\right)\right)_\sigma$, $B = diag\left(b\left(\sigma, n, n\right)\right)_\sigma$, $U = \left(u_n^\sigma\right)_\sigma$, $F = \left(F_n^\sigma\right)_\sigma$, $U_0 = \left(U_{0,n}^\sigma\right)_\sigma$, $\Psi = \left(\varphi_n^\sigma\right)_\sigma$, $\Phi = \left(\phi_{|n|}^\sigma\left(0\right)e^{sign(\sigma)2i\pi l^k x/\alpha}\right)_\sigma$, Equation (3.27) states under matrix form

$$B\partial_t U + C\partial_x U = F \text{ in } I \times \Omega \text{ and } BU\left(t = 0\right) = U_0 \text{ in } \Omega, \tag{3.158}$$

which boundary condition (3.157) is rewritten as

$$CU\left(t, x\right).\overline{\Psi}\left(t, x\right) = 0 \text{ on } I \times \partial\Omega \text{ for all } \Psi \text{ such that } \overline{\Phi}(x).\overline{\Psi}(t, x) = 0 \text{ on } I \times \partial\Omega.$$

Equivalently, $CU\left(t, x\right)$ is collinear with $\overline{\Phi}(x)$ yielding the boundary condition

$$u_n^k\phi_{|n|}^k\left(0\right)e^{2i\pi\frac{l^k x}{\alpha}} + u_n^{-k}\phi_{|n|}^{-k}\left(0\right)e^{-2i\pi\frac{l^k x}{\alpha}} = 0 \text{ on } \; I \times \partial\Omega$$

after remarking that $c\left(k, n, n\right) \neq 0$ and $c\left(k, n, n\right) = -c\left(-k, n, n\right)$.

Second, for $k = 0$, $\lambda_{|n|}^0$ is double $\lambda_{|n|}^0 = \lambda_{|m|}^0$ so $M_n^k = \{n, m\}$. With $C = \left(c\left(0, p, q\right)\right)_{p,q}$, $B = \left(b\left(0, p, q\right)\right)_{p,q}$, $U = \left(u_p^0\right)_p$, $F = \left(F_q^0\right)_q$, $U_0 = \left(U_{0,q}^0\right)_q$, $\Psi = \left(\varphi_q^0\right)_q$, $\Phi = \left(\phi_{|q|}^0\left(0\right)\right)_q$, the matrix form is still stated as (3.158). Here, the eigenvectors are chosen as real functions then $c\left(0, p, p\right) = 0$. Since $c\left(0, n, m\right) \neq 0$, so the boundary condition is

$$u_n^0\phi_{|n|}^0\left(0\right) + u_m^0\phi_{|m|}^0\left(0\right) = 0 \text{ on } I \times \partial\Omega.$$

Finally, for $k = -\frac{1}{2}$, $\lambda_{|n|}^k$ is double $\lambda_{|n|}^k = \lambda_{|m|}^k$ so $M_n^k = \{n, m\}$. By using the same way for $k = 0$, the boundary condition (3.155) is obtained. ■

Remark 44 *This method allows to complete the homogenized model of the wave equation in [35] for the one-dimensional case for any $K \in \mathbb{N}^*$ by choosing a sequence $\varepsilon \in E_{1/K}$ as in Assumption 14. For any $k \in L_K^*$, defined in (1.2), we denote $p_k = kK \in \mathbb{N}$, so $\frac{\alpha p_k}{\varepsilon K} = p_k\left[\frac{\alpha}{\varepsilon K}\right] + p_k l_\varepsilon^1$ and $p_k l_\varepsilon^1 \to l^k := p_k l^1$ when $\varepsilon \to 0$ with the same sequence of $\varepsilon \in E_{1/K}$.*

3.6 Numerical examples

We report simulations regarding comparison of physical solution and its approximation for the homogenized model under the first order formulation for $I = (0, 1)$, $\Omega = (0, 1)$, $\rho = 1$, $a = \frac{1}{3}\left(\sin\left(2\pi y\right) + 2\right)$, $f^\varepsilon = 0$, $v_0^\varepsilon = 0$, $\varepsilon = \frac{1}{10}$ and $k = 0.16$. Since $k \neq 0$, so the approximation (3.150) comes

$$U^\varepsilon\left(t, x\right) \simeq \sum_{\sigma \in I^k, n \in \mathbb{Z}^*} u_n^\sigma\left(t, x\right)e^{isign(n)\sqrt{\lambda_{|n|}^\sigma}t/\varepsilon}e_n^\sigma\left(\frac{x}{\varepsilon}\right). \tag{3.159}$$

The validation of the approximation is based on the modal decomposition of any solution $U^\varepsilon = \sum_{l \in \mathbb{Z}^*} R_l^\varepsilon\left(t\right)V_l^\varepsilon\left(x\right)$ where the modes V_l^ε are built from the solutions v_l^ε of the spectral problem $\partial_x\left(a^\varepsilon\partial_x v_l^\varepsilon\right) = \lambda_l^\varepsilon v_l^\varepsilon$ in Ω with $v_l^\varepsilon = 0$ on $\partial\Omega$. Moreover, in Chapter 2, two-scale approximations of modes have been derived on the form of linear combinations $\sum_{\sigma \in I^k} \theta_n^\sigma (x)\, \phi_{|n|}^\sigma \left(\frac{x}{\varepsilon}\right)$ of Bloch modes, so the initial conditions of the physical problem are taken on the form

$$u_0^\varepsilon (x) = \sum_{n \in \mathbb{N}^*, \sigma \in I^k} \theta_n^\sigma (x)\, \phi_n^\sigma \left(\frac{x}{\varepsilon}\right). \tag{3.160}$$

Two simulations are reported, one for an initial condition u_0^ε spanned by the pair of Bloch modes corresponding to $n = 2$ when the other is spanned by three pairs $n \in \{2, 3, 4\}$. In the first case, the first component of U_0^ε approximates the first component of a single eigenvector V_l^ε approximated by (3.159) where all coefficients $u_n^\sigma = 0$ for $n \neq \pm 2$. Figure 3.1 (a) shows the initial condition u_0^ε, when Figure 3.1 (b) presents the real (dash line) and imaginary (dashed-dotted line) part of the initial condition $u_{0,n}^k$, and the real (dot line) and imaginary (solid line) part of the initial condition $u_{0,n}^{-k}$ of HF-macroscopic equation. Figure 3.2 (a, b) report the HF-macroscopic solutions $\left(u_n^k, u_n^{-k}\right)$ at $t = 0.466$ and $x = 0.699$ respectively for the real (dash line) and imaginary (dashed-dotted line) part of u_n^k, and the real (dot line) and imaginary (solid line) part of u_n^{-k}. In Figure 3.3 (a, b) the first component U_1^ε of physical solution and the relative error vector of U_1^ε with its approximation are plotted which $L^2(I)-$norm is equal to 8e-3 at $x = 0.699$. Moreover, Figure 3.4 (a, b) focus on the real part of the first component U_1^ε of physical solution and the relative error vector of U_1^ε with its approximation which $L^2(\Omega)-$norm is equal to 7e-3 at $t = 0.466$. For the second case where $u_n^\sigma = 0$ for $n \notin \{\pm 2, \pm 3, \pm 4\}$, the first component U_1^ε and the relative error vector of U_1^ε with its approximation at $t = 0.466$ which $L^2(\Omega)$-norm is 3.8e-3 are plotted in Figure 3.5 (a, b). Finally, the first component U_1^ε and the relative error vector of U_1^ε with its approximation are provided in Figure 3.6 (a, b) which the $L^2(I)$-relative errors at $x = 0.699$ on the first component is 3.5e-3.

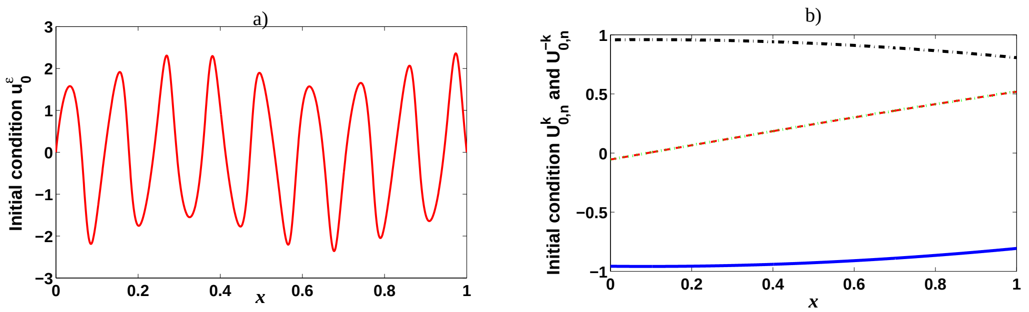

Figure 3.1: (a) Initial condition u_0^ε. (b) Initial conditions of HF-macroscopic equation $u_{0,n}^k$ and $u_{0,n}^{-k}$.

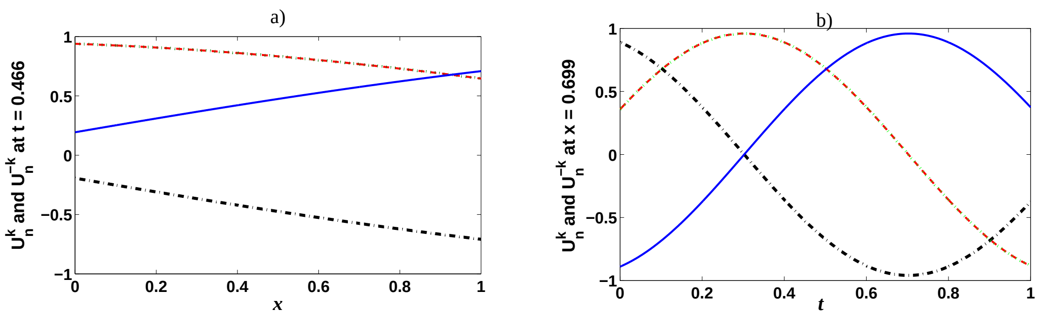

Figure 3.2: (a) HF-macroscopic solutions u_n^k and u_n^{-k} at $t = 0.466$. (b) HF-Macroscopic solutions u_n^k and u_n^{-k} at $x = 0.699$.

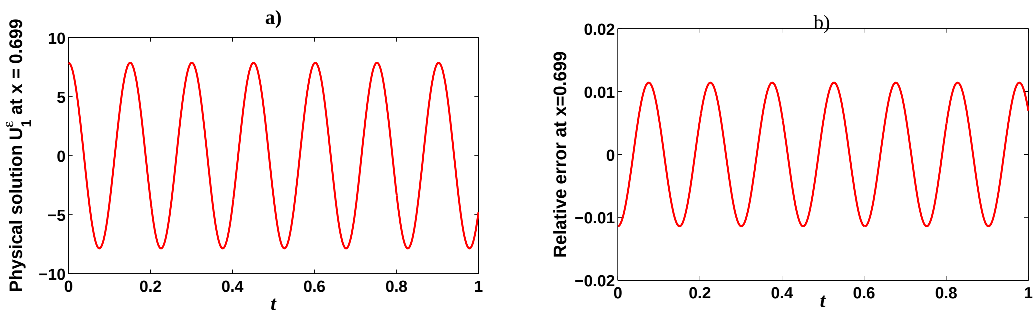

Figure 3.3: (a) Physical solution U_1^ε at $x = 0.699$. (b) Relative error vector between U_1^ε and its approximation in $L^2(I)$-norm is 8e-3.

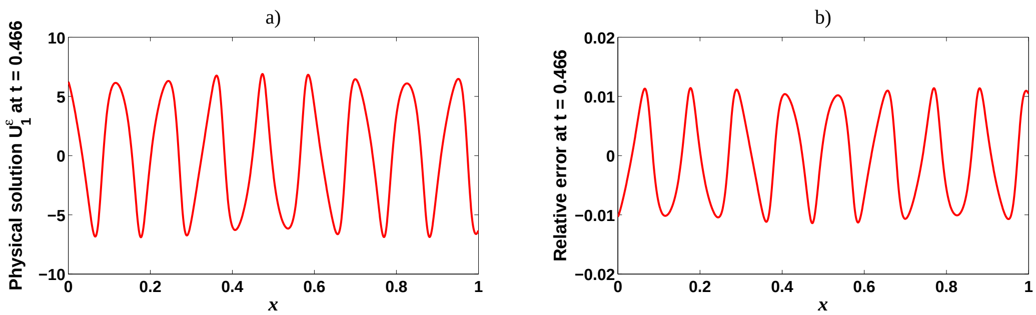

Figure 3.4: (a) Physical solution U_1^ε at $t = 0.466$. (b) Relative error vector between U_1^ε and its approximation in $L^2(\Omega)$-norm is 7e-3.

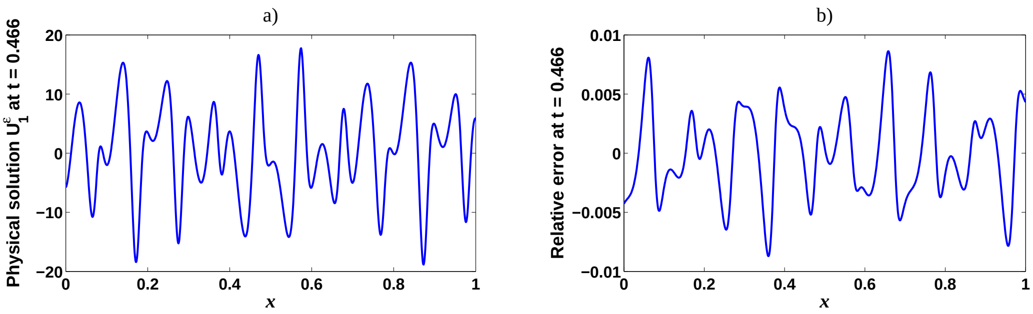

Figure 3.5: (a) Physical solution U_1^ε at $t = 0.466$. (b) Relative error vector between U_1^ε and its approximation in $L^2(\Omega)$-norm is 3.8e-3.

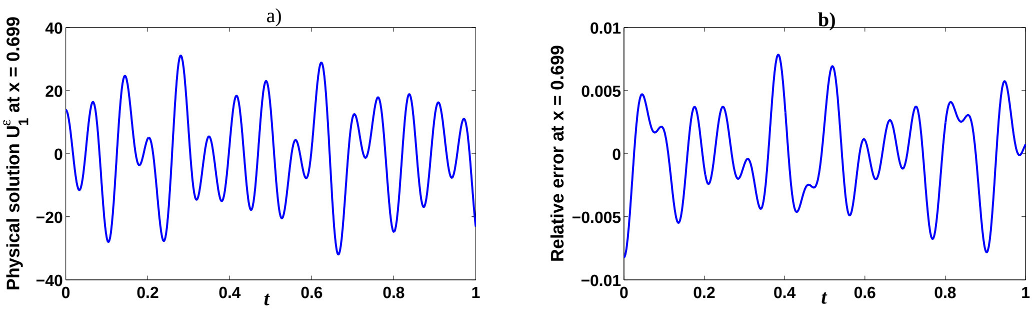

Figure 3.6: (a) Physical solution U_1^ε at $x = 0.699$. (b) Relative error vector between U_1^ε and its approximation in $L^2(I)$-norm is 3.5e-3.

Chapter 4

Homogenization of the spectral problem in a two dimensional strip including boundary layer effects

Contents

Abstract In this chapter, we present a result for periodic homogenization of the spectral problem in an open bounded strip $\Omega = (0, \alpha) \times (0, \varepsilon) \subset \mathbb{R}^2$. The results focus on the high frequency part of the spectrum and corresponding eigenvectors, which is addressed by a method of Bloch wave homogenization, including boundary layer effects. The oscillations are occurring at the microscopic scale and their amplitudes are governed by a system of first order boundary value problems and by a boundary layer equation.

4.1 Introduction

This chapter is concerned with the study of periodic homogenization of the spectral problem

$$-\text{div}\,(a^\varepsilon \nabla w^\varepsilon) = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon$$

posed in an open bounded strip $\Omega = \omega_1 \times (0, \varepsilon) \subset \mathbb{R}^2$ with $\omega_1 = (0, \alpha) \subset \mathbb{R}^+$ and the boundary conditions

$$w^\varepsilon = 0 \text{ on } \partial\omega_1 \times (0, \varepsilon) \;\text{ and }\; a^\varepsilon \nabla_x w^\varepsilon . n_x = 0 \text{ on } \omega_1 \times \{0, \varepsilon\}\,.$$

An asymptotic analysis of this problem is carried out where $\varepsilon > 0$ is a parameter tending to zero and the coefficients are $\varepsilon-$periodic, namely $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$ where $a(y)$ and $\rho(y)$ are $Y-$periodic with respect to a lattice of reference cell $Y \subset \mathbb{R}^2$.

In this work, we search eigenvalues λ^ε satisfying the expansion

$$\varepsilon^2 \lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \varepsilon O(\varepsilon). \tag{4.1}$$

It comes that λ^0 is equal to an eigenvalue λ_n^k solution of the Bloch wave spectral problem (1.25) for $k \in Y^* = [-\frac{1}{2}, \frac{1}{2})$ and $n \in \mathbb{N}^*$.

The physical eigenvector w^ε is approximated by a sum of Bloch waves and boundary layer terms,

$$w^\varepsilon(x) \approx \sum_{\sigma\in\{-k,k\}} \sum_m u_m^\sigma(x_1)\,\phi_m^\sigma\left(\frac{x}{\varepsilon}\right) + w_{b,k}^0\left(\frac{x}{\varepsilon}\right) + w_{b,k}^\alpha\left(\frac{\alpha - x}{\varepsilon}\right) \text{ if } k \notin \left\{0, -\frac{1}{2}\right\}$$

$$\text{and } w^\varepsilon(x) \approx \sum_m u_m^k(x_1)\,\phi_m^k\left(\frac{x}{\varepsilon}\right) + w_{b,k}^0\left(\frac{x}{\varepsilon}\right) + w_{b,k}^\alpha\left(\frac{\alpha - x}{\varepsilon}\right) \text{ otherwise} \tag{4.2}$$

where the sum $\sum\limits_m$ runs over all modes ϕ_m^σ with the same eigenvalue λ_n^k. The Bloch wave amplitudes $(u_m^k)_m$ are solution of a first order system of differential equations constituting the high frequency macroscopic problem. In particular, for $k \notin \left\{0, -\frac{1}{2}\right\}$ and for each n, the high frequency macroscopic model has the following form, where $\sigma \in \{-k, k\}$ and $l \in \mathbb{N}^*$ such that $\lambda_l^\sigma = \lambda_n^k$,

$$\sum_m c(\sigma, m, l)\,\partial_{x_1} u_m^\sigma - \lambda^1 b(\sigma, m, l)\,u_m^\sigma = 0 \text{ in } \omega_1, \tag{4.3}$$

with boundary conditions

$$\begin{aligned} \sum_{\sigma\in\{k,-k\},m} sign(\sigma)\,e_0(\sigma, m, l)\,u_m^\sigma(0)\,\widehat{\phi}_l^{\sigma,0} &= 0 \\ \text{and} \sum_{\sigma\in\{k,-k\},m} sign(\sigma)\,e_\alpha(\sigma, m, l)\,u_m^\sigma(\alpha)\,\widehat{\phi}_l^{\sigma,\alpha} &= 0. \end{aligned} \tag{4.4}$$

We observe that the two partial differential equations in (4.3) are not coupled by k and $-k$, the coupling is due to the boundary conditions only. Moreover, $w_{b,k}^0$ and $w_{b,k}^\alpha$ are solution to the boundary layer equation stated as

$$-\operatorname{div}_y\left(a\nabla_y w_{b,k}^{x_1}\right) - \lambda^0 \rho w_{b,k}^{x_1} = 0 \text{ in } \mathbb{R}^+ \times (0,1), \tag{4.5}$$

$$w_{b,k}^{x_1} = -\sum_{\sigma\in\{-k,k\}} \sum_m u_m^\sigma(x_1)\,\phi_m^\sigma \text{ in } \{0\} \times (0,1) \text{ and } a\nabla_y w_{b,k}^{x_1}.n_y = 0 \text{ on } \mathbb{R}^+ \times \{0,1\}$$

$$\text{and } w_{b,k}^{x_1} \text{ is exponentially decaying when } y_1 \to \infty.$$

for $x_1 \in \{0, \alpha\}$.

This chapter is organized as follows. Section 4.2 is devoted to the statements of the models and of the main result. Section 4.3 includes the model derivation.

4.2 Statement of the results

We consider an open bounded domain $\Omega = \omega_1 \times \omega_2$ with $\omega_1 = (0, \alpha) \subset \mathbb{R}^+$ and $\omega_2 = (0, \varepsilon)$ with ends $\Gamma_{end} = \partial\omega_1 \times \omega_2$ and lateral boundary $\Gamma_{lat} = \omega_1 \times \partial\omega_2$. As usual in homogenization papers, $\varepsilon > 0$ denotes a small parameter intended to go to zero. A 2×2 matrix a^ε and a real function ρ^ε are assumed to obey a prescribed profile,

$$a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \text{ and } \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right), \tag{4.6}$$

where $\rho \in L^\infty(\mathbb{R}^2)$ and $a \in W^{1,\infty}(\mathbb{R}^2)^{2\times 2}$ is symmetric. They are both $Y-$periodic with respect to the reference cell $Y \subset \mathbb{R}^2$. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions,

$$\rho^0 \leq \rho \leq \rho^1 \text{ and } a^0 \|\xi\|^2 \leq \xi^T a \xi \leq a^1 \|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^2 \tag{4.7}$$

for some given strictly positive numbers ρ^0, ρ^1, a^0 and a^1. We consider $(\lambda^\varepsilon, w^\varepsilon)$ solution to the spectral problem

$$\begin{aligned} -\operatorname{div}_x(a^\varepsilon \nabla_x w^\varepsilon) &= \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \text{ in } \Omega \text{ with } w^\varepsilon = 0 \text{ on } \Gamma_{end}, \\ a^\varepsilon \nabla_x w^\varepsilon . n_x &= 0 \text{ on } \Gamma_{lat} \text{ and } \|w^\varepsilon\|_{L^2(\Omega)} = \sqrt{\varepsilon}. \end{aligned} \tag{4.8}$$

We set $H^1_\Gamma(\Omega) = H^1_{end}(\Omega) \cap H^1_{lat}(\Omega)$ where

$$\begin{aligned} H^1_{end}(\Omega) &:= \left\{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_{end}\right\}, \\ \text{and } H^1_{lat}(\Omega) &:= \left\{v \in H^1(\Omega) \mid a^\varepsilon \nabla_x v . n_x = 0 \text{ on } \Gamma_{lat}\right\}. \end{aligned}$$

Then the eigenvectors w^ε belong to $H^2(\Omega) \cap H^1_\Gamma(\Omega)$ and we search the eigenvalues such that,

$$\varepsilon^2 \lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \varepsilon^2\lambda^2 + \varepsilon^2 O(\varepsilon). \tag{4.9}$$

The weak formulation of the spectral problem (4.8) is: find $w^\varepsilon \in H^1_\Gamma(\Omega)$ such that

$$\int_\Omega a^\varepsilon \nabla_x w^\varepsilon \cdot \nabla_x v \, dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon \cdot v \, dx \text{ for all } v \in H^1_{end}(\Omega). \tag{4.10}$$

Posing $v = w^\varepsilon$,

$$\varepsilon^{-1} \int_\Omega a^\varepsilon |\varepsilon \nabla_x w^\varepsilon|^2 \, dx = \varepsilon^2 \lambda^\varepsilon \|\rho^\varepsilon\|_{L^\infty(\Omega)} \varepsilon^{-1} \|w^\varepsilon\|^2_{L^2(\Omega)}.$$

Since $\varepsilon^2\lambda^\varepsilon$ is bounded, then

$$\varepsilon^{-1} \int_\Omega a^\varepsilon |\varepsilon \nabla_x w^\varepsilon|^2 \, dx \leq \varepsilon^2 \lambda^\varepsilon \|\rho^\varepsilon\|_{L^\infty(\Omega)} \leq c,$$

so the uniform estimates

$$\frac{1}{\sqrt{\varepsilon}} \|\varepsilon \nabla_x w^\varepsilon\|_{L^2(\Omega)} \leq c \text{ and } \frac{1}{\sqrt{\varepsilon}} \|w^\varepsilon\|_{L^2(\Omega)} \leq c \tag{4.11}$$

hold.

4.2.1 Assumptions

For $k \in Y^*$, let w^ε be a sequence satisfying the uniform bound (4.11), we consider a subsequence $S_k^\varepsilon w^\varepsilon$ converging weakly to w^k in $L^2(\omega_1 \times Y)$, i.e,

$$S_k^\varepsilon w^\varepsilon (x_1, y) = w^k (x_1, y) + O(\varepsilon) \text{ in the } L^2(\omega_1 \times Y) \text{ weak sense.} \tag{4.12}$$

Then, we pose

$$w_{b,k}^\varepsilon (x) := w^\varepsilon (x) - \sum_{\sigma \in I^k} (\mathfrak{R}^\sigma w^\sigma)(x), \tag{4.13}$$

and assume that there exists at least a subsequence of $S_b^\vartheta w_{b,k}^\varepsilon$ converging weakly towards a function $w_{b,k}^\vartheta$ in $L^2(Y_\infty^+)$ when ε tends to zero for any $\vartheta \in \{0, \alpha\}$, i.e,

$$\left(S_b^\vartheta w_{b,k}^\varepsilon\right)(y) = w_{b,k}^\vartheta (y) + O(\varepsilon) \text{ in the } L^2\left(Y_\infty^+\right) \text{ weak sense.} \tag{4.14}$$

4.2.2 The model

For $k \in Y^*$, $n \in \mathbb{N}^*$, a given Bloch eigenvalue $\lambda^0 = \lambda_n^k$ and $\vartheta \in \{0, \alpha\}$, the boundary layer equation is stated as an Helmholtz equation

$$\begin{gathered} -\operatorname{div}_y\left(a\nabla_y w_{b,k}^\vartheta\right) - \lambda^0 \rho w_{b,k}^\vartheta = 0 \text{ in } Y_\infty^+, \qquad (4.15)\\ w_{b,k}^\vartheta = -\sum_{\sigma \in I^k} w^\sigma (x_1 = \vartheta) \text{ on } \gamma_{\infty,end}^+ \text{ and } a\nabla_y w_{b,k}^\vartheta . n_y = 0 \text{ on } \gamma_{\infty,lat}^+, \\ \text{and } w_{b,k}^\vartheta \text{ is exponentially decaying when } y_1 \to \infty. \end{gathered}$$

Remark 45 *For the moment the Helmholtz equation with an exponentially decay has not been analyzed for time reason.*

The solution $w_{b,k}^\vartheta$ is called the boundary layer term. In the scope of this work, we assume that this solution is unique. Hence, we can define the linear operator

$$\begin{array}{rccc} \mathcal{L}: & H^{1/2}\left(\gamma_{end}^+\right) & \to & H^1\left(Y_\infty^+\right) \\ & g & \mapsto & v = \mathcal{L}(g) \end{array} \tag{4.16}$$

such that v is the solution of (4.15) with $v(0, y_2) = g(y_2)$ in Y_2. We introduce

$$\widehat{\phi}_n^{k,0} = \int_{Y_2} \phi_n^k (0, y_2)\, dy_2 \text{ and } \widehat{\phi}_n^{k,\alpha} = e^{sign(k) 2 i \pi l^k} \int_{Y_2} \phi_n^k (0, y_2)\, dy_2, \tag{4.17}$$

and define the set

$$M_n^k = \left\{m \in \mathbb{N}^* \text{ such that } \lambda_m^k = \lambda_n^k\right\}. \tag{4.18}$$

For $p, q \in M_n^k$, the HF-macroscopic model coefficients are

$$\begin{aligned} c(k,p,q) &= \int_Y a\nabla_y \phi_q^k \cdot \phi_p^k - \phi_q^k \cdot a\nabla_y \phi_p^k \, dy,\ b(k,p,q) = \int_Y \rho \phi_p^k \cdot \phi_q^k \, dy, && (4.19)\\ d(k,p,q) &= \int_{Y_2} a\nabla_y \left(\phi_p^k - \mathcal{L}\left(\phi_p^k (0, y_2)\right)\right) . n_y \cdot \phi_q^k (y_1 = 0)\, dy_2, && (4.20)\\ e_0(k,p,q) &= c(k,p,q) - d(k,p,q),\ e_\alpha(k,p,q) = -c(k,p,q) - d(k,p,q), && (4.21) \end{aligned}$$

and observe that,

$$c(k,p,q) = \overline{c(-k,p,q)},\ c(k,q,p) = -\overline{c(k,p,q)},\ c(k,p,q) = -c(-k,q,p),$$

$$b(k,p,q) = \overline{b(k,q,p)},\ \ b(k,p,q) = \overline{b(-k,p,q)},\ \ b(k,p,p) > 0,$$

$$d(k,p,q) = \overline{d(-k,p,q)}, e_0(k,p,q) = \overline{e_0(-k,p,q)} \text{ and } e_\alpha(k,p,q) = \overline{e_\alpha(-k,p,q)}.$$

In particular, for $k = 0$, for the real eigenvector, $c(0,p,p) = 0$.

For any $k \in Y^*$ and $n \in \mathbb{N}^*$, the HF-macroscopic model is introduced corresponding to a Bloch eigenvalue λ^0. For $k \in Y^* \backslash \left\{0, -\frac{1}{2}\right\}$, the solutions of the HF-macroscopic model are the family of pairs $\left(u_p^k, u_p^{-k}\right)_{p\in M_n^\sigma, n\in\mathbb{N}^*}$ solution to the system of equations where $\sigma \in I^k$ and $q \in M_n^\sigma$,

$$\sum_{p\in M_n^\sigma} c(\sigma,p,q)\,\partial_{x_1} u_p^\sigma - \lambda^1 b(\sigma,p,q)\, u_p^\sigma = 0 \text{ in } \omega_1, \tag{4.22}$$

with the boundary conditions

$$\begin{aligned} \sum_{\sigma\in I^k, p\in M_n^\sigma} sign(\sigma)\, e_0(\sigma,p,q)\, u_p^\sigma(0)\, \widehat{\phi}_q^{\sigma,0} &= 0 \\ \text{and} \sum_{\sigma\in I^k, p\in M_n^\sigma} sign(\sigma)\, e_\alpha(\sigma,p,q)\, u_p^\sigma(\alpha)\, \widehat{\phi}_q^{\sigma,\alpha} &= 0. \end{aligned} \tag{4.23}$$

For $k \in \left\{0, -\frac{1}{2}\right\}$, the family $\left(u_p^k\right)_{p\in M_n^k, n\in\mathbb{N}^*}$ is solution to the system of first order problems where $q \in M_n^k$,

$$\sum_{p\in M_n^\sigma} c(k,p,q)\,\partial_{x_1} u_p^k - \lambda^1 b(k,p,q)\, u_p^k = 0 \text{ in } \omega_1. \tag{4.24}$$

4.2.3 Two-scale asymptotic behaviour

Theorem 46 *For a given $k \in Y^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be a solution of the weak formulation (4.10) satisfying the uniform bound (4.11) then $S_k^\varepsilon w^\varepsilon$ is bounded in $L^2(\omega_1 \times Y)$. Take a subsequence of w^ε that the weak limit of $S_k^\varepsilon w^\varepsilon$ in $L^2(\omega_1 \times Y)$ is non-vanishing and that the renormalized sequence $\varepsilon^2\lambda^\varepsilon$ satisfies a decomposition as (4.9), there exists $n \in \mathbb{N}^*$ such that $\lambda^0 = \lambda_n^k$ with λ_n^k an eigenvalue in the Bloch wave spectrum and the limit g_k of the weakly converging extracted subsequence of $\sum_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ in $L^2(\omega_1 \times Y)$ can be decomposed on the Bloch modes,*

$$g_k = \sum_{\sigma\in I^k, p\in M_n^\sigma} u_p^\sigma(x_1)\, \phi_p^\sigma(y) \tag{4.25}$$

where $u_p^k \in L^2(\omega_1)$. Moreover, for $\sigma \in I^k$ and $\vartheta \in \{0, \alpha\}$, we assume that the assumption (4.14) is satisfied, and that w^σ and $w_{b,k}^\vartheta$ are sufficiently regular solution to get the strong form then $w_{b,k}^\vartheta$ is solution of the boundary layer equation (4.15). Finally, for $\varepsilon \in E_k$ as in Assumption 14, if $u_p^k \in H^1(\omega_1)$ then $\left(\lambda^1, u_p^k\right)$ is a solution of the HF-macroscopic models (4.22), (4.24) with boundary condition (4.23) for $k \notin \left\{0, -\frac{1}{2}\right\}$.

From Theorem 46 and the expansion (4.13), w^ε can be approximated by two-scale modes

$$w^\varepsilon(x) \approx \sum_{\sigma\in I^k, p\in M_n^\sigma} u_p^\sigma(x_1)\,\phi_p^\sigma\left(\frac{x}{\varepsilon}\right) + w_{b,k}^0\left(\frac{x}{\varepsilon}\right) + w_{b,k}^\alpha\left(\frac{\alpha - x}{\varepsilon}\right). \tag{4.26}$$

Remark 47 *(i) If $c(k,p,q) = 0$ for all p,q varying in M_n^k, the macroscopic equations (4.22) and (4.24) are $\lambda^1 = 0$ or $u = \left(u_p^\sigma\right)_{p,\sigma} = 0$ with the boundary condition at $x_1 \in \{0,\alpha\}$ for all $q \in M_n^k$*

$$\sum_{\sigma\in I^k, p\in M_n^\sigma} sign(\sigma)\, d(\sigma,p,q)\, u_p^\sigma(x_1)\,\widehat{\phi}_q^{\sigma,x_1} = 0 \text{ for } k \notin \left\{0, -\frac{1}{2}\right\}, \tag{4.27}$$

If $\lambda^1 = 0$ then this model does not provide any equation for u_p^σ satisfying (4.27).

(ii) For $k \neq 0$, if $\phi_m^k(0,y_2) = 0$ then $\phi_m^k(1,y_2) = 0$ and ϕ_m^k is a periodic solution in y_1 that is a solution of $k = 0$. So, we consider always that $\phi_m^k(0,y_2) \neq 0$ for the case $k \neq 0$. Moreover, for all $q \in M_n^k$, if $\int_{Y_2} \phi_q^k(0,y_2)\,dy = 0$ then the boundary conditions of the macroscopic equation vanishes.

iii) For $k \notin \left\{0, -\frac{1}{2}\right\}$, we observe that the matrix

$$C = \begin{pmatrix} (c(k,p,q))_{p,q} & 0 \\ 0 & (c(-k,p,q))_{p,q} \end{pmatrix}$$

is skew-symmetric with even-dimension, as we know that its eigenvalues always come in pairs $\pm\lambda$. From the spectral theorem, for a real skew-symmetric matrix C, the nonzero eigenvalues are all pure imaginary and thus are of the form $i\lambda_1$, $-i\lambda_1$, $i\lambda_2$, $-i\lambda_2$, $i\lambda_3$, $-i\lambda_3$,.... where each of the λ_n are real. Hence, the boundary condition (4.23) is found based on the properties of anti-symmetric matrix iC and the relation between eigenelements $\left(\lambda_n^k, \phi_n^k\right)$ and $\left(\lambda_n^{-k}, \phi_n^{-k}\right)$. However, this does not apply in the cases $k = 0$ and $k = -\frac{1}{2}$ since the size of matrix $C = (c(k,p,q))_{p,q}$ can be even or odd.

Remark 48 *Here we focus on the Bloch spectrum while the boundary layer spectrum is not mentioned. To avoid eigenmodes related to the boundary spectrum, according to Proposition 7.7 in [8] we assume that the weak limit of $S_k^\varepsilon w^\varepsilon$ in $L^2(\Omega; H^1(Y))$ is not vanishing. Moreover, we observe that the weak limit g_k of subsequence of $\sum_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ in $L^2(\omega_1 \times Y)$ has the same form in one dimension in [95]. In fact, the processes and methods are extended trivially from the one-dimensional case, except what refers to the HF-macroscopic boundary conditions which need to applied the boundary layer term. However, this boundary layer term is not related to boundary layer spectrum and also not to the HF-microscopic equation. It plays a role as a corrector in the asymptotic behaviour of the macroscopic eigenvectors.*

Remark 49 *The analysis of the Helmholtz equation (4.15) for the boundary layer problem has not yet been carried out. In particular, it remains to exhibit a family of exponentially decaying solutions.*

Remark 50 *The case $\lambda^0 = \lambda_n^k = \lambda_{n'}^{k'}$ is not considered as a special case, two different models corresponding to k and k'.*

4.3 Model derivation

In order to prove the main result, we introduce some preliminary homogenized results and their proofs are reported in Section 4.3.1, 4.3.2 and 4.3.3. Finally, Theorem 46 is proved in Section 4.3.4.

4.3.1 Derivation of the HF-microscopic equation

The next lemma states the HF-microscopic equation for each $k \in Y^*$ and $n \in \mathbb{N}^*$.

Lemma 51 *For a fixed $k \in Y^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be solution of the weak formulation (4.10), and satisfy (4.9) and (4.11), there exists at least a subsequence of $S_k^\varepsilon w^\varepsilon$ converging weakly towards a non-vanishing function w^k in $L^2(\omega_1 \times Y)$, when ε tends to zero, which is a solution of the very weak formulation of the HF-microscopic boundary value problem where $\lambda^0 = \lambda_n^k$ for an $n \in \mathbb{N}^*$,*

$$-\operatorname{div}_y\left(a\nabla_y w^k\right) = \lambda^0 \rho w^k \text{ in } \omega_1 \times Y, \tag{4.28}$$
$$w^k \text{ is } k\text{-quasi-periodic in } y_1,\ (a\nabla_y w^k).n_y \text{ is } k\text{-anti-quasi-periodic in } y_1,$$
$$\text{and } a\nabla_y w^k.n_y = 0 \text{ on } \omega_1 \times \gamma_{lat}.$$

Moroever, assuming that w^k is sufficiently regular solution then it admits the modal decomposition,

$$w^k(x_1, y) = \sum_{m \in M_n^k} u_m^k(x_1)\,\phi_m^k(y) \text{ for } u_m^k \in L^2(\omega_1) \tag{4.29}$$

with conjugate u_m^k and u_m^{-k}.

Proof. [Proof of Lemma 51]
For a given $k \in Y^*$, for $v(x_1, y)$ a $k-$quasi-periodic function in y_1 such that

$$a\nabla_y v.n_y = 0 \text{ on } \omega_1 \times \gamma_{lat}, \tag{4.30}$$

we choose test functions as

$$v^\varepsilon = \left(\mathfrak{R}^k v\right)(x) \in H^1_{end}(\Omega) \cap H^2(\Omega) \tag{4.31}$$

in the weak formulation (4.10) of the spectral problem. Applying the Green formula so that to put all derivative terms on test functions,

$$-\int_\Omega w^\varepsilon \cdot \operatorname{div}_x(a^\varepsilon \nabla_x v^\varepsilon)\,dx + \int_{\partial\Omega} w^\varepsilon \cdot a^\varepsilon \nabla_x v^\varepsilon.n_x\,dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon \cdot v^\varepsilon\,dx.$$

Using the definition $P^\varepsilon = -\operatorname{div}_x(a^\varepsilon \nabla_x .)$ and since $w^\varepsilon = 0$ on Γ_{end}, so

$$\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon \rho^\varepsilon)\,\mathfrak{R}^k v\,dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot a^\varepsilon \nabla_x \mathfrak{R}^k v.n_x\,dx = 0. \tag{4.32}$$

Observing that,

$$n_x.a^\varepsilon \nabla_x \mathfrak{R}^k v = \mathfrak{R}^k(a\partial_{x_1} v.n_y) + \frac{1}{\varepsilon}\mathfrak{R}^k(a\nabla_y v.n_y)$$

and using the condition (4.30), it remains,

$$n_x.a^\varepsilon \nabla_x \mathfrak{R}^k v = \mathfrak{R}^k \left(a\partial_{x_1} v.n_y \right). \tag{4.33}$$

Applying (4.33), (4.9) and (1.38), so in Equation (4.32) it yields,

$$\int_\Omega w^\varepsilon \cdot \sum_{n=0}^{2} \varepsilon^{-n} \mathfrak{R}^k \left(P^n v - \lambda^{2-n} \rho v \right) \, dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot \mathfrak{R}^k \left(a\partial_{x_1} v.n_y \right) \, dx = O\left(\varepsilon\right).$$

Multiply by ε^2,

$$\int_\Omega w^\varepsilon \cdot \mathfrak{R}^k \left(P^2 v - \lambda^0 \rho v \right) \, dx + \varepsilon \int_\Omega w^\varepsilon \cdot \mathfrak{R}^k \left(P^1 v - \lambda^1 \rho v \right) \, dx = \varepsilon O\left(\varepsilon\right).$$

Since $P^2 v - \lambda^0 \rho v$ is $k-$quasi-periodic in y_1, from the approximation (1.31) of $\mathfrak{R}^k$ by $S_k^{\varepsilon *}$,

$$\int_\Omega w^\varepsilon \cdot S_k^{\varepsilon *} \left(P^2 v - \lambda^0 \rho v \right) \, dx + \varepsilon \int_\Omega w^\varepsilon \cdot S_k^{\varepsilon *} \left(P^1 v - \lambda^1 \rho v \right) \, dx = \varepsilon O\left(\varepsilon\right).$$

Or equivalently,

$$\int_{\omega_1 \times Y} S_k^\varepsilon w^\varepsilon \cdot \left(P^2 v - \lambda^0 \rho v \right) \, dx_1 dy = O\left(\varepsilon\right).$$

Since $S_k^\varepsilon w^\varepsilon \to w^k$ in $L^2\left(\omega_1 \times Y\right)$ weakly when ε tends to 0, passing to the limit,

$$\int_{\omega_1 \times Y} w^k \cdot \left(P^2 v - \lambda^0 \rho v \right) \, dx_1 dy = 0,$$

or equivalently,

$$-\int_{\omega_1 \times Y} w^k \cdot \operatorname{div}_y \left(a \nabla_y v \right) + w^k \cdot \lambda^0 \rho v \, dx_1 dy = 0.$$

Assume that $w^k \in L^2\left(\omega_1; H^2\left(Y\right)\right)$ and take the integrations by parts,

$$\int_{\omega_1 \times Y} -\operatorname{div}_y \left(a \nabla_y w^k \right) \cdot v - \lambda^0 w^k \cdot \rho v \, dx_1 dy + \int_{\omega_1 \times \partial Y} -w^k \cdot a \nabla_y v.n_y + a \nabla_y w^k.n_y \cdot v \, dx_1 dy = 0.$$

Hence, choosing test functions $v \in L^2\left(\omega_1; H_0^2\left(Y\right)\right)$ yields the strong form,

$$-\operatorname{div}_y \left(a \nabla_y w^k \right) - \lambda^0 \rho w^k = 0 \text{ in } \omega_1 \times Y.$$

So, the boundary term remains,

$$\int_{\omega_1 \times \partial Y} w^k \cdot a \nabla_y v.n_y - a \nabla_y w^k \cdot v.n_y \, dx_1 dy = 0$$

for general test functions v a $k-$quasi-periodic function in y_1 satisfying (4.30). Equivalently,

$$\int_{\omega_1 \times \gamma_{end}} w^k \cdot a \nabla_y v.n_y - a \nabla_y w^k.n_y \cdot v \, dx_1 dy + \int_{\omega_1 \times \gamma_{lat}} w^k \cdot a \nabla_y v.n_y - a \nabla_y w^k.n_y \cdot v \, dx_1 dy = 0.$$

This implies that w^k and $a\nabla_y w^k.n_y$ are respectively $k-$quasi-periodic and $k-$anti-quasi-periodic in the variable y_1. Furthermore, since $a\nabla_y v.n_y = 0$ on $\omega_1 \times \gamma_{lat}$, then

$$a\nabla_y w^k.n_y = 0 \text{ on } \omega_1 \times \gamma_{lat}.$$

From the positive self-adjoint of operator P_k^2 (P^2), we know that λ^0 is an eigenvalue λ_n^k of the Bloch wave spectrum, then w^k is a Bloch eigenvector and is decomposed as

$$w^k(x,y) = \sum_{m \in M_n^k} u_m^k(x_1)\, \phi_m^k(y) \;\text{ with } u_m^k \in L^2(\omega_1)$$

where

$$u_m^k(x) = \int_Y w^k(x_1,y) \cdot \phi_m^k(y) dy.$$

Moreover, for $k \neq 0$, the property $S_k^\varepsilon w^\varepsilon = \overline{S_{-k}^\varepsilon w^\varepsilon}$ for any positive ε is conserved to the limit $w^k = \overline{w^{-k}}$. Finally, u_m^k and u_m^{-k} are conjugate i.e. $u_m^k = \overline{u_m^{-k}}$ since $\phi_m^k = \overline{\phi_m^{-k}}$. ■

Remark 52 *There is an alternative:*

1. *If λ^0 is a Bloch eigenvalue then $S_k^\varepsilon w^\varepsilon$ converges weakly to a solution w^k such that the partial function $y \mapsto w^k(.,y)$ is an internal Bloch mode.*
2. *Otherwise, λ^0 is not a Bloch eigenvalue and there exists no solution of the above problem and so the weak limit $w^k = 0$.*

4.3.2 Derivation of the boundary layer equation

The next lemma establishes the boundary layer equation (4.15) where we introduce the notation

$$w_\vartheta^k(y) := w^k(\vartheta, y) \tag{4.34}$$

extended by quasi-periodicity to Y_∞^+.

Lemma 53 *For $(\lambda^\varepsilon, w^\varepsilon)$ solution of the weak formulation (4.10) satisfying (4.9) and (4.11), let (λ^0, w^k) be solution of the very weak formulation of (4.28), if the assumption (4.14) is fulfilled for $\vartheta \in \{0, \alpha\}$, then the boundary layer term $w_{b,k}^\vartheta$ is a solution of the very weak formulation of the boundary layer equation (4.15),*

$$-\sum_{\vartheta \in \{0,\alpha\}} \int_{Y_\infty^+} \left(\sum_{\sigma \in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta \right) \cdot \left(\operatorname{div}_y \left(a\nabla_y v_b^\vartheta \right) + \lambda^0 \rho v_b^\vartheta \right) \; dy = 0.$$

for all $v_b^\vartheta \in H^1_{\gamma^+_{\infty,end}}(Y_\infty^+) \cap H^2(Y_\infty^+)$ such that $a\nabla_y v_b^s.n_y = 0$ on $\gamma^+_{\infty,lat}$. Moreover, if $w_\vartheta^\sigma \in H^2(Y_\infty^+)$ and $w_{b,k}^\vartheta \in H^2(Y_\infty^+)$ then it is a solution of the equation (4.15).

In order to prove Lemma 53, we start by proving the next one stating the relation between the

Lemma 54 *For any w^k in $L^2(\omega_1 \times Y)$,*

$$S_b^\vartheta \left(\mathfrak{R}^k w^k \right)(y) = w_\vartheta^k(y) + O(\varepsilon) \;\text{ in } L^2\left(Y_\infty^+\right).$$

Proof. According to the definition (1.32) of the boundary layer two-scale transform S_b^0, we get

$$S_b^0\left(\mathfrak{R}^k w^k\right)(y) = \left(\mathfrak{R}^k w^k\right)(\varepsilon y)\,\chi_{(0,\alpha/\varepsilon)}(y_1).$$

Since $\left(\mathfrak{R}^k w^k\right)(x) = w^k\left(x_1, \frac{x}{\varepsilon}\right)$, so

$$\left(\mathfrak{R}^k w^k\right)(\varepsilon y) = w^k\left(\varepsilon y_1, y\right),$$

hence,

$$S_b^0\left(\mathfrak{R}^k w^k\right)(y) = w^k\left(\varepsilon y_1, y\right)\chi_{(0,\alpha/\varepsilon)}(y_1).$$

Moreover, $w^k\left(\varepsilon y_1, y\right)\chi_{(0,\alpha/\varepsilon)}(y_1) \to w^k(0,y)\,\chi_{\mathbb{R}^+}(y_1)$ in $L^2\left(Y_\infty^+\right)$ when $\varepsilon \to 0$, then,

$$S_b^0\left(\mathfrak{R}^k w^k\right)(y) = w^k(0,y) + O(\varepsilon) \text{ in the } L^2\left(Y_\infty^+\right).$$

Similarly, the definition (1.33) of the boundary layer two-scale transform S_b^α implies that,

$$S_b^\alpha\left(\mathfrak{R}^k w^k\right)(y) = \left(\mathfrak{R}^k w^k\right)(-\varepsilon y_1 + \alpha, \varepsilon y_2)\,\chi_{(0,\alpha/\varepsilon)}(y_1).$$

Since $\left(\mathfrak{R}^k w^k\right)(x) = w^k\left(x_1, \frac{x}{\varepsilon}\right)$, so

$$\left(\mathfrak{R}^k w^k\right)(-\varepsilon y_1 + \alpha, \varepsilon y_2) = w^k\left(-\varepsilon y_1 + \alpha, \frac{-\varepsilon y_1 + \alpha}{\varepsilon}, y_2\right),$$

then,

$$S_b^\alpha\left(\mathfrak{R}^k w^k\right)(y) = w^k\left(-\varepsilon y_1 + \alpha, \frac{-\varepsilon y_1 + \alpha}{\varepsilon}, y_2\right)\chi_{(0,\alpha/\varepsilon)}(y_1).$$

Since $w^k\left(-\varepsilon y_1 + \alpha, \frac{-\varepsilon y_1+\alpha}{\varepsilon}, y_2\right)\chi_{(0,\alpha/\varepsilon)}(y_1) \to w^k(\alpha, y)\,\chi_{\mathbb{R}^+}(y_1)$ in $L^2\left(Y_\infty^+\right)$ when $\varepsilon \to 0$, therefore,

$$S_b^\alpha\left(\mathfrak{R}^k w^k\right)(y) = w^k(\alpha, y) + O(\varepsilon) \text{ in } L^2\left(Y_\infty^+\right).$$

■

Assuming that for each $\sigma \in I^k$, w^σ is sufficiently regular so that

$$\left(P^2 - \lambda^0\rho\right) w^\sigma = 0 \text{ in } Y \text{ for all } x_1 > 0 \tag{4.35}$$

implies the equality at the boundaries

$$\left(P^2 - \lambda^0\rho\right) w^\sigma = 0 \text{ in } Y \text{ at } x_1 \in \{0, \alpha\}. \tag{4.36}$$

Applying to Lemma 54,

$$\left(P^2 - \lambda^0\rho\right) w_\vartheta^\sigma = 0 \text{ in } Y_\infty^+ \text{ for } \sigma \in I^k \text{ and } \vartheta \in \{0, \alpha\} \tag{4.37}$$

in the very weak sense with test functions such their value and their derivatives vanish on the boundary. By periodicity, the boundary condition on the lateral boundary is

$$a\nabla_y w_\vartheta^\sigma . n_y = 0 \text{ on } \gamma_{\infty,lat}^+. \tag{4.38}$$

In addition, from (4.13), the eigenmode w^ε is rewritten by

$$w^\varepsilon(x) = \sum_{\sigma \in I^k}\left(\mathfrak{R}^\sigma w^\sigma\right)(x) + w_{b,k}^\varepsilon(x). \tag{4.39}$$

According to the assumption (4.14),

$$S_b^\vartheta w^\varepsilon(y) = \sum_{\sigma\in I^k} w_b^{\sigma,\vartheta}(y) + w_b^\vartheta(y) + O(\varepsilon) \text{ in the } L^2\left(Y_\infty^+\right) \text{ weak sense.} \tag{4.40}$$

Proof. [Proof of Lemma 53] For each $\vartheta \in \{0,\alpha\}$, let $v_b^\vartheta \in H^1_{\gamma^+_{\infty,end}}(Y_\infty^+) \cap H^2(Y_\infty^+)$ such that

$$a\nabla_y v_b^s.n_y = 0 \text{ on } \gamma^+_{\infty,lat}, \tag{4.41}$$

we choose $v^\varepsilon := \sum_{\vartheta\in\{0,\alpha\}} \mathfrak{R}_b^\vartheta v_b^\vartheta \in H^1_{end}(\Omega) \cap H^2(\Omega)$ as a test function of the weak formulation (4.10). Applying the Green formula so that to put all derivative terms on the test functions,

$$-\int_\Omega w^\varepsilon \cdot \mathrm{div}_x\left(a^\varepsilon\nabla_x v^\varepsilon\right)\, dx + \int_{\partial\Omega} w^\varepsilon \cdot a^\varepsilon\nabla_x v^\varepsilon.n_x\, dx = \lambda^\varepsilon\int_\Omega \rho^\varepsilon w^\varepsilon \cdot v^\varepsilon\, dx.$$

Since $w^\varepsilon = 0$ on Γ_{end}, so,

$$-\int_\Omega w^\varepsilon \cdot \mathrm{div}_x\left(a^\varepsilon\nabla_x v^\varepsilon\right)\, dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot a^\varepsilon\nabla_x v^\varepsilon.n_x\, dx = \lambda^\varepsilon\int_\Omega \rho^\varepsilon w^\varepsilon \cdot v^\varepsilon\, dx.$$

Equivalently,

$$\sum_{\vartheta\in\{0,\alpha\}}\left[\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon\rho^\varepsilon)\,\mathfrak{R}_b^\vartheta v_b^\vartheta\, dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot a^\varepsilon\nabla_x\mathfrak{R}_b^\vartheta v_b^\vartheta.n_x\, dx\right] = 0. \tag{4.42}$$

Since v is independent on x,

$$n_x.a^\varepsilon\nabla_x\mathfrak{R}_b^\vartheta v_b^\vartheta = n_\vartheta\frac{1}{\varepsilon}\mathfrak{R}_b^\vartheta\left(a\nabla_y v_b^\vartheta.n_y\right) \tag{4.43}$$

with $n_\vartheta = -1$ for $\vartheta = 0$ and $n_\vartheta = 1$ for $\vartheta = \alpha$. Applying (4.43), (4.9) and (1.38), so in Equation (4.42) yields,

$$\sum_{\vartheta\in\{0,\alpha\}}\left[\int_\Omega w^\varepsilon \cdot \varepsilon^{-2}\mathfrak{R}_b^\vartheta\left(P^2 v_b^\vartheta - \lambda^0\rho v_b^\vartheta\right)\, dx - \int_\Omega w^\varepsilon \cdot \varepsilon^{-1}\lambda^1\mathfrak{R}_b^\vartheta\left(\rho v_b^\vartheta\right)\, dx\right.$$
$$\left. - \int_\Omega w^\varepsilon \cdot \lambda^2\mathfrak{R}_b^\vartheta\left(\rho v_b^\vartheta\right)\, dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot n_\vartheta\varepsilon^{-1}\mathfrak{R}_b^\vartheta\left(a\nabla_y v_b^\vartheta.n_y\right)\, dx\right] = O(\varepsilon).$$

According to Lemma 13 stating the equality $\mathfrak{R}_b^\vartheta = S_b^{\vartheta*}$,

$$\sum_{\vartheta\in\{0,\alpha\}}\left[\int_\Omega w^\varepsilon \cdot \varepsilon^{-2}S_b^{\vartheta*}\left(P^2 v_b^\vartheta - \lambda^0\rho v_b^\vartheta\right)\, dx - \int_\Omega w^\varepsilon \cdot \varepsilon^{-1}\lambda^1 S_b^{\vartheta*}\left(\rho v_b^\vartheta\right)\, dx\right.$$
$$\left. - \int_\Omega w^\varepsilon \cdot \lambda^2 S_b^{\vartheta*}\left(\rho v_b^\vartheta\right)\, dx + \int_{\Gamma_{lat}} w^\varepsilon \cdot n_\vartheta\varepsilon^{-1}S_b^{\vartheta*}\left(a\nabla_y v_b^\vartheta.n_y\right)\, dx\right] = O(\varepsilon).$$

Using the definition of the adjoint operator $S_b^{\vartheta*}$,

$$\sum_{\vartheta\in\{0,\alpha\}}\int_{Y_\infty^+} S_b^\vartheta w^\varepsilon \cdot \left(P^2 v_b^\vartheta - \lambda^0\rho v_b^\vartheta\right)\, dy = O(\varepsilon).$$

Using (4.40) and passing to the limit,

$$\sum_{\vartheta\in\{0,\alpha\}}\int_{Y_\infty^+}\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot\left(P^2 v_b^\vartheta - \lambda^0\rho v_b^\vartheta\right)\,dy = 0,$$

where w_ϑ^σ and $w_{b,k}^\vartheta$ are defined in (4.34) and (4.14). Or equivalently,

$$-\sum_{\vartheta\in\{0,\alpha\}}\int_{Y_\infty^+}\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot\left(\operatorname{div}_y\left(a\nabla_y v_b^\vartheta\right) + \lambda^0\rho v_b^\vartheta\right)\,dy = 0.$$

Assuming that $w_{b,k}^\vartheta \in H^2\left(Y_\infty^+\right)$ and taking the integrations by parts,

$$\sum_{\vartheta\in\{0,\alpha\}}\left[\int_{Y_\infty^+} -\operatorname{div}_y\left(a\nabla_y\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\right)\cdot v_b^\vartheta - \lambda^0\rho\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot v_b^\vartheta\,dy\right.$$
$$\left.-\int_{\partial Y_\infty^+}\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot a\nabla_y v_b^\vartheta.n_y dy + \int_{\partial Y_\infty^+} a\nabla_y\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right).n_y\cdot v_b^\vartheta dy\right] = 0.$$

However,

$$-\operatorname{div}_y a\nabla_y\left(w_\vartheta^\sigma\right) - \lambda^0\rho w_\vartheta^\sigma = 0 \text{ in } Y_\infty^+,$$
$$a\nabla_y v_b^\vartheta.n_y = 0 \text{ on } \gamma_{\infty,lat}^+,\ v_b^\vartheta = 0 \text{ on } \gamma_{\infty,end}^+, \text{ and } v_b^\vartheta \to 0 \text{ as } y_1\to\infty,$$

so,

$$\sum_{\vartheta\in\{0,\alpha\}}[\int_{Y_\infty^+} -\operatorname{div}_y\left(a\nabla_y w_{b,k}^\vartheta\right)\cdot v_b^\vartheta - \lambda^0\rho w_{b,k}^\vartheta\cdot v_b^\vartheta\,dy$$
$$-\int_{\gamma_{\infty,end}^+}\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot a\nabla_y v_b^\vartheta.n_y dy + \int_{\gamma_{\infty,lat}^+} a\nabla_y\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right).n_y\cdot v_b^\vartheta dy] = 0.$$

So the internal equation of each $w_{b,k}^\vartheta$ follows,

$$-\operatorname{div}_y\left(a\nabla_y w_{b,k}^\vartheta\right) - \lambda^0\rho w_{b,k}^\vartheta = 0 \text{ in } Y_\infty^+ \text{ for any } \vartheta\in\{0,\alpha\}\,, \tag{4.44}$$

as well as the boundary term,

$$\sum_{\vartheta\in\{0,\alpha\}}\left[\int_{\gamma_{\infty,end}^+}\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right)\cdot a\nabla_y v_b^\vartheta.n_y dy - \int_{\gamma_{\infty,lat}^+} a\nabla_y\left(\sum_{\sigma\in I^k} w_\vartheta^\sigma + w_{b,k}^\vartheta\right).n_y\cdot v_b^\vartheta dy\right] = 0.$$

Therefore, $w_{b,k}^\vartheta = -\sum_{\sigma\in I^k} w_\vartheta^\sigma$ on $\gamma_{\infty,end}^+$. Moreover $a\nabla_y w_\vartheta^\sigma.n_y = 0$ on $\gamma_{\infty,lat}^+$, so $a\nabla_y w_{b,k}^\vartheta.n_y = 0$ on $\gamma_{\infty,lat}^+$ for each ϑ. ■

4.3.3 Derivation of the macroscopic equation

Before to study the HF-macroscopic equation, we provide some necessary calculations and notations. For $k\in Y^*$ and $n\in\mathbb{N}^*$, we pose $m = \left|M_n^k\right|$ and recall that $\widehat{\phi}_q^{k,\vartheta} =$

$\int_{Y_2} \phi_q^k (0, y_2)\, dy_2 e^{2i\pi l^k \chi_\alpha(\vartheta)}$ where the characteristic function $\chi_\alpha(\vartheta) = 1$ if $\vartheta = \alpha$ and $= 0$ otherwise. For $k \notin \{0, -\frac{1}{2}\}$, we denote,

$$M_n^\sigma = \{q_1, .., q_m\}, \; \Phi^\vartheta = \left(\widehat{\phi}_{q_1}^{k,\vartheta}, ..., \widehat{\phi}_{q_m}^{k,\vartheta}, \widehat{\phi}_{q_1}^{-k,\vartheta}, ..., \widehat{\phi}_{q_m}^{-k,\vartheta}\right) \text{ and the vector } F_i^\vartheta = (I_i)^T \Phi^\vartheta,$$

and I_i are $2m \times 2m$ diagonal matrices defined by $(I_i)_{qq} = 1$ if $q = i$ or $q = m + i$ and $= 0$ otherwise for $i \in \{1, ..., m\}$. The vector F_i^ϑ can be rewritten as,

$$F_i^\vartheta = \left(0, ..., 0, \widehat{\phi}_{q_i}^{k,\vartheta}, 0, ..., 0, \widehat{\phi}_{q_i}^{-k,\vartheta}, 0, ..., 0\right),$$

and $\{F_i^\vartheta\}_{i\in\{1,..,m\}}$ generate the subspace L^ϑ of the vector space $\mathbb{C}^{2m}$,

$$L^\vartheta = \operatorname{span}\{F_1^\vartheta, ..., F_m^\vartheta\}.$$

Since $\dim(L^\vartheta) = m$ then $\dim(L^{\vartheta\perp}) = m$ and the orthogonal vector space $L^{\vartheta\perp} \subset \mathbb{C}^{2m}$ with a basis denoted by $\{X_i^\vartheta\}_{i\in\{1,..,m\}} \in \mathbb{C}^{2m}$. Now we shall find a basis $\{X_i^\vartheta\}_{i\in\{1,..,m\}}$ of the orthogonal vector space $L^{\vartheta\perp}$. We denote $\Xi_i = (\xi_i^j)_{j\in\{1,...,2m\}}$ the canonical basis of $\mathbb{C}^{2m}$ ie with $\xi_i^j = 0$ for $i \neq j$ and $= 1$ otherwise for $i \in \{1, ..., 2m\}$. Let $Z \in L^{\vartheta\perp}$ so $Z = \sum_{\ell=1}^{2m} z_\ell \Xi_\ell$ and satisfies $\langle Z, F_i^\vartheta\rangle = 0$ for all $i \in \{1, .., m\}$. Equivalently,

$$z_i \widehat{\phi}_{q_i}^{k,\vartheta} + z_{m+i} \widehat{\phi}_{q_i}^{-k,\vartheta} = 0 \text{ or } z_{m+i} = -\frac{\widehat{\phi}_{q_i}^{k,\vartheta}}{\widehat{\phi}_{q_i}^{-k,\vartheta}} z_i \text{ for any } i \in \{1, .., m\}.$$

Thus,

$$Z = \left(-\frac{\widehat{\phi}_1^{k,\vartheta}}{\widehat{\phi}_1^{-k,\vartheta}} \Xi_{m+1} + \Xi_1\right) z_1 + ... + \left(-\frac{\widehat{\phi}_{q_i}^{k,\vartheta}}{\widehat{\phi}_{q_i}^{-k,\vartheta}} \Xi_{m+i} + \Xi_i\right) z_i + ... + \left(-\frac{\widehat{\phi}_m^{k,\vartheta}}{\widehat{\phi}_m^{-k,\vartheta}} \Xi_{2m} + \Xi_m\right) z_m.$$

So the family

$$\begin{aligned} X_i^\vartheta \quad : &= \Big(-\frac{\widehat{\phi}_{q_i}^{k,\vartheta}}{\widehat{\phi}_{q_i}^{-k,\vartheta}} \Xi_{m+i} + \Xi_i\Big) \Big/ \sqrt{1 + \left(\frac{\widehat{\phi}_{q_i}^{k,\vartheta}}{\widehat{\phi}_{q_i}^{-k,\vartheta}}\right)^2} \quad \text{for } i \in \{1, .., m\} \\ &= \left(0, ..., 0, \widehat{\phi}_{q_i}^{-k,\vartheta}, 0, ..., 0, -\widehat{\phi}_{q_i}^{k,\vartheta}, 0, ..., 0\right) \Big/ \sqrt{(\widehat{\phi}_{q_i}^{-k,\vartheta})^2 + (\widehat{\phi}_{q_i}^{k,\vartheta})^2} \qquad (4.45) \end{aligned}$$

constitutes an orthornormal basis of $L^{\vartheta\perp}$. The HF-macroscopic equation (4.22)-(4.24) is built for each k and n in the next lemma.

Lemma 55 *For $k \in Y^*$, $n \in \mathbb{N}^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be solution of the weak formulation (4.10), and be satisfying (4.9) and (4.11), so there exists at least a subsequence of $S_k^\varepsilon w^\varepsilon$ converging weakly towards a non-vanishing function w^k in $L^2(\omega_1 \times Y)$ when ε tends to zero such that $\lambda^0 = \lambda_n^k$ and (λ^0, w^k) is solution of the HF-microscopic equation (4.28). For $\varepsilon \in E_k$ as in Assumption 14, $\sigma \in I^k$ and $p \in M_n^k$, if w^σ and $w_{b,k}^\vartheta$ are sufficiently regular solution and $u_p^\sigma \in H^1(\omega_1)$ then (λ^1, u_p^σ) is a solution of the HF-macroscopic equation (4.22)-(4.24).*

Proof. [Proof of Lemma 55] The proof distinguishes between the two cases $k \notin \left\{0, -\frac{1}{2}\right\}$ and $k \in \left\{0, -\frac{1}{2}\right\}$.

i) **Case** $k \notin \left\{0, -\frac{1}{2}\right\}$. We take $v^{\sigma,\varepsilon} \in H^2(\omega_1 \times Y)$ $\sigma-$quasi-periodic functions in y_1 such that they are decomposed as a linear combination of Bloch modes

$$v^{\sigma,\varepsilon}(x_1, y) = \sum_{q \in M_n^\sigma} v_q^{\sigma,\varepsilon}(x_1)\, \phi_q^\sigma(y), \tag{4.46}$$

satisfying the conditions $v^{\sigma,\varepsilon} = 0$ on γ_{lat} and the end conditions in average,

$$\sum_{\sigma \in I^k} v_q^{\sigma,\varepsilon}(x_1) \left(\int_{Y_2} \phi_q^\sigma (y_1 = 0)\, dy_2 \right) e^{2i\pi\sigma\frac{x}{\varepsilon}} = 0 \text{ at } x_1 \in \partial\omega_1 \text{ for all } q \in M_n^\sigma. \tag{4.47}$$

We also choose functions a boundary layer test function $v_b^\vartheta \in H^2(Y_\infty^+)$ for $\vartheta \in \{0, \alpha\}$ such that

$$a\nabla_y \left(e^{-\eta y_1} v_b^\vartheta\right).n_y = 0 \text{ on } \gamma_{\infty,lat}^+$$

with $\eta > 0$. We pose

$$v^\varepsilon(x) = \sum_{\sigma \in I^k} \left(\mathfrak{R}^k v^{\sigma,\varepsilon}\right)(x),\ v^{b,\varepsilon}(x) = e^{-\eta x_1/\varepsilon}\left(\mathfrak{R}_b^0 v_b^0\right)(x) + e^{-\eta(\alpha - x_1)/\varepsilon}\left(\mathfrak{R}_b^\alpha v_b^\alpha\right)(x), \tag{4.48}$$

and choose

$$\psi^\varepsilon = v^\varepsilon + v^{b,\varepsilon},$$

which satisfies the boundary conditions $\psi^\varepsilon = 0$ at the ends ie

$$\sum_{\sigma \in I^k} \left(\mathfrak{R}^k v^{\sigma,\varepsilon}\right)(x) + e^{-\eta x_1/\varepsilon}\left(\mathfrak{R}_b^0 v_b^0\right)(x) + e^{-\eta(\alpha - x_1)/\varepsilon}\left(\mathfrak{R}_b^\alpha v_b^\alpha\right)(x) = 0 \text{ on } \Gamma_{end}. \tag{4.49}$$

In addition,

$$\psi^\varepsilon \in H^1_{end}(\Omega) \cap H^2(\Omega) \text{ such that } n_x.a^\varepsilon \nabla_x \psi^\varepsilon = 0 \text{ on } \Gamma_{lat}, \tag{4.50}$$

as test functions of the weak formulation (4.10) of the spectral problem. From (4.46) and (4.49), the boundary conditions on Γ_{end} of test functions ψ^ε are equivalent to,

$$\sum_{\sigma \in I^k, q \in M_n^\sigma} v_q^{\sigma,\varepsilon}(0)\, \phi_q^\sigma\left(0, \frac{x_2}{\varepsilon}\right) + v_b^0\left(0, \frac{x_2}{\varepsilon}\right) + e^{-\eta\alpha/\varepsilon} v_b^\alpha\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = 0, \tag{4.51}$$

$$\text{and} \sum_{\sigma \in I^k, q \in M_n^\sigma} v_q^{\sigma,\varepsilon}(\alpha)\, \phi_q^\sigma\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + e^{-\eta\alpha/\varepsilon} v_b^0\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + v_b^\alpha\left(0, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = \alpha,$$

for all $x_2 \in \omega_2$. Using the $\sigma-$quasi-periodicity of ϕ_q^σ in the variable y_1, the second condition becomes,

$$\sum_{\sigma \in I^k, q \in M_n^\sigma} v_q^{\sigma,\varepsilon}(\alpha)\, \phi_q^\sigma\left(0, \frac{x_2}{\varepsilon}\right) e^{2i\pi\sigma\frac{\alpha}{\varepsilon}} + e^{-\eta\alpha/\varepsilon} v_b^0\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + v_b^\alpha\left(0, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = \alpha. \tag{4.52}$$

For $\varepsilon \in E_k$, according to Assumption 14 with remarking that $e^{sign(\sigma)2i\pi h_\varepsilon^k \frac{x_1}{\alpha}} = 1$ for all $x_1 \in \partial\omega_1$ we build the test function $v_q^{\sigma,\varepsilon}$ as in the case of the wave equation so that

$v_q^{\sigma,\varepsilon} \to v_q^\sigma$ in $H^2(\omega_1)$. Passing to the limit, then (4.46), (4.47), (4.51) and (4.52) imply the form of v^σ as

$$v^\sigma(x_1, y) = \sum_{q \in M_n^\sigma} v_q^\sigma(x_1)\,\phi_q^\sigma(y) \tag{4.53}$$

and satisfying the boundary conditions

$$\sum_{\sigma \in I^k} v_q^\sigma(x_1)\left(\int_{Y_2} \phi_q^\sigma(0, y_2)\,dy_2\right) e^{sign(\sigma)2i\pi l^k x_1/\alpha} = 0 \text{ at } x_1 \in \partial\omega_1, \tag{4.54}$$

$$\sum_{\sigma \in I^k, q \in M_n^\sigma} v_q^\sigma(0)\,\phi_q^\sigma(0, y_2) + v_b^0(0, y_2) = 0 \text{ at } x_1 = 0, \tag{4.55}$$

$$\text{and} \quad \sum_{\sigma \in I^k, q \in M_n^\sigma} v_q^\sigma(\alpha)\,\phi_q^\sigma(0, y_2)\,e^{sign(\sigma)2i\pi l^k} + v_b^\alpha(0, y_2) = 0 \text{ at } x_1 = \alpha.$$

Applying the Green formula so that to put all derivative terms on the test functions,

$$-\int_\Omega w^\varepsilon \cdot \nabla_x\left(a^\varepsilon \nabla_x \psi^\varepsilon\right)\,dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon \cdot \psi^\varepsilon\,dx.$$

Equivalently,

$$\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon \rho^\varepsilon)\left(\left(\sum_{\sigma \in I^k} \mathfrak{R}^\sigma v^{\sigma,\varepsilon}\right) + e^{-\eta x_1/\varepsilon}\left(\mathfrak{R}_b^0 v_b^0\right) + e^{-\eta(\alpha - x_1)/\varepsilon}\left(\mathfrak{R}_b^\alpha v_b^\alpha\right)\right)\,dx = 0,$$

or

$$\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon \rho^\varepsilon)\left(\left(\sum_{\sigma \in I^k} \mathfrak{R}^\sigma v^{\sigma,\varepsilon}\right) + \mathfrak{R}_b^0\left(e^{-\eta y_1} v_b^0\right) + \mathfrak{R}_b^\alpha\left(e^{-\eta y_1} v_b^\alpha\right)\right)\,dx = 0.$$

Using the decomposition (1.38) of P^ε,

$$\sum_{\sigma \in I^k} \int_\Omega w^\varepsilon \cdot \mathfrak{R}^\sigma\left(\sum_{l=0}^2 \varepsilon^{-l}\left(P^l - \lambda^{2-l}\rho\right) v^{\sigma,\varepsilon}\right) dx + \sum_{\vartheta \in \{0,\alpha\}}\left[\frac{1}{\varepsilon^2}\int_\Omega w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(P^2 - \lambda^0\rho\right)\left(e^{-\eta y_1} v_b^\vartheta\right)\right.$$
$$\left. -\frac{1}{\varepsilon} w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(\lambda^1 \rho e^{-\eta y_1} v_b^\vartheta\right) - w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(\lambda^2 \rho e^{-\eta y_1} v_b^\vartheta\right) dx\right] = O(\varepsilon).$$

From the special form (4.46) of the test function $v^{\sigma,\varepsilon}$ then $P^2 v^{\sigma,\varepsilon} - \lambda^0 \rho v^{\sigma,\varepsilon} = 0$,

$$\sum_{\sigma \in I^k} \int_\Omega w^\varepsilon \cdot \frac{1}{\varepsilon}\mathfrak{R}^\sigma\left(P^1 v^{\sigma,\varepsilon} - \lambda^1 \rho v^{\sigma,\varepsilon}\right) + w^\varepsilon \cdot \mathfrak{R}^\sigma\left(P^0 v^{\sigma,\varepsilon} - \lambda^2 \rho v^{\sigma,\varepsilon}\right)\,dx$$
$$+ \sum_{\vartheta \in \{0,\alpha\}}\left[\frac{1}{\varepsilon^2}\int_\Omega w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(P^2 - \lambda^0\rho\right)\left(e^{-\eta y_1} v_b^\vartheta\right)\right.$$
$$\left. -\frac{1}{\varepsilon}\lambda^1 w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(\rho e^{-\eta y_1} v_b^\vartheta\right) - \lambda^2 w^\varepsilon \cdot \mathfrak{R}_b^\vartheta\left(\rho e^{-\eta y_1} v_b^\vartheta\right) dx\right] = O(\varepsilon).$$

Multiplying by ε^2, using the approximations (1.31) and (1.37) of $\mathfrak{R}^\sigma$ by $S_\sigma^{\varepsilon*}$ and $\mathfrak{R}_b^\vartheta = S_b^{\vartheta*}$,

$$\sum_{\sigma\in I^k}\left[\varepsilon\int_\Omega w^\varepsilon\cdot S_\sigma^{\varepsilon*}\left(P^1v^{\sigma,\varepsilon}-\lambda^1\rho v^{\sigma,\varepsilon}\right)+w^\varepsilon\cdot S_\sigma^{\varepsilon*}\left(P^0v^{\sigma,\varepsilon}-\lambda^2\rho v^{\sigma,\varepsilon}\right)\,dx\right]$$
$$+\sum_{\vartheta\in\{0,\alpha\}}[\int_\Omega w^\varepsilon\cdot S_b^{\vartheta*}\left(P^2-\lambda^0\rho\right)\left(e^{-\eta y_1}v_b^\vartheta\right)$$
$$-\varepsilon\lambda^1w^\varepsilon\cdot S_b^{\vartheta*}\left(\rho e^{-\eta y_1}v_b^\vartheta\right)-\varepsilon^2\lambda^2w^\varepsilon\cdot S_b^{\vartheta*}\left(\rho e^{-\eta y_1}v_b^\vartheta\right)dx\big]=\varepsilon^2O\left(\varepsilon\right).$$

From the definitions (1.10) and (1.35) of the adjoint operators $S_\sigma^{\varepsilon*}$ and $S_b^{\vartheta*}$, the boundness of $S_\sigma^\varepsilon w^\varepsilon$ and $S_b^\vartheta w^\varepsilon$, and dividing by ε^2,

$$\sum_{\sigma\in I^k}\int_{\omega_1\times Y}S_\sigma^\varepsilon w^\varepsilon\cdot\left(P^1v^{\sigma,\varepsilon}-\lambda^1\rho v^{\sigma,\varepsilon}\right)\,dx_1dy$$
$$+\sum_{\vartheta\in\{0,\alpha\}}\int_{Y_\infty^+}S_b^\vartheta w^\varepsilon\cdot\left(P^2-\lambda^0\rho\right)\left(e^{-\eta y_1}v_b^\vartheta\right)\,dy=O\left(\varepsilon\right).$$

Since $v^{\sigma,\varepsilon}$ converges to v^σ strongly in $H^2\left(\omega_1\right)$, $S_\sigma^\varepsilon w^\varepsilon$ converges to w^σ weakly in $L^2\left(\omega_1\times Y\right)$ and the convergence (4.40) of $S_b^\vartheta w^\varepsilon$,

$$\sum_{\sigma\in I^k}\int_{\omega_1\times Y}w^\sigma\cdot\left(P^1v^\sigma-\lambda^1\rho v^\sigma\right)dx_1dy$$
$$+\sum_{\vartheta\in\{0,\alpha\}}\int_{Y_\infty^+}\left(\sum_{\sigma\in I^k}w_\vartheta^\sigma+w_{b,k}^\vartheta\right)\cdot\left(P^2-\lambda^0\rho\right)\left(e^{-\eta y_1}v_b^\vartheta\right)dy=0.$$

If w^σ and $w_{b,k}^\vartheta$ are sufficiently regular, applying the Green formula,

$$\sum_{\sigma\in I^k}\left[\int_{\omega_1\times Y}-\sum_{j=1}^2\partial_{y_j}\left(a_{1j}\partial_{x_1}w^\sigma\right)\cdot v^\sigma-\partial_{x_1}\left(\sum_{i=1}^2a_{i1}\partial_{y_i}w^\sigma\right)\cdot v^\sigma-\lambda^1\rho w^\sigma\cdot v^\sigma\,dx_1dy\right.$$
$$+\int_Y\left[-w^\sigma\cdot\sum_{j=1}^2a_{1j}\partial_{y_j}v^\sigma+\sum_{i=1}^2a_{i1}\partial_{y_i}w^\sigma\cdot v^\sigma\right]_{x_1=0}^{x_1=\alpha}dy$$
$$\left.+\int_{\omega_1\times\partial Y}\partial_{x_1}w^\sigma\cdot\sum_{i,j=1}^2a_{1j}v^\sigma.n_{y_j}-w^\sigma\cdot\sum_{i=1}^2a_{i1}\partial_{x_1}v^\sigma.n_{y_i}dx_1dy\right]$$

$$+\sum_{\vartheta\in\{0,\alpha\}}[\int_{Y_\infty^+}\sum_{\sigma\in I^k}\left(-\operatorname{div}_y\left(a\nabla_yw_\vartheta^\sigma\right)-\lambda^0\rho w_\vartheta^\sigma\right)\cdot\left(e^{-\eta y_1}v_b^\vartheta\right)$$
$$+\left(-\operatorname{div}_y\left(a\nabla_yw_{b,k}^\vartheta\right)-\lambda^0\rho w_{b,k}^\vartheta\right)\cdot\left(e^{-\eta y_1}v_b^\vartheta\right)\,dy$$
$$+\int_{\partial Y_\infty^+}-\left(\sum_{\sigma\in I^k}w_\vartheta^\sigma+w_{b,k}^\vartheta\right)\cdot a\nabla_y\left(e^{-\eta y_1}v_b^\vartheta\right).n_y$$
$$+a\nabla_y\left(\sum_{\sigma\in I^k}w_\vartheta^\sigma+w_{b,k}^\vartheta\right).n_y\cdot\left(e^{-\eta y_1}v_b^\vartheta\right)dy]=0.$$

From the $\sigma-$quasi-periodicity of w^σ and v^σ, so,

$$\int_{\omega_1\times\gamma_{end}}\sum_{i,j=1}^{2}\left(\partial_{x_1}w^\sigma\cdot a_{1j}v^\sigma n_{y_j}-w^\sigma\cdot a_{i1}\partial_{x_1}v^\sigma n_{y_i}\right)dx_1dy=0.$$

Moreover, since $v^\sigma=0$ on γ_{lat}, $a\nabla_y w^\sigma_\vartheta.n_y=0$ and $a\nabla_y\left(e^{-\eta y_1}v^\vartheta_b\right).n_y=0$ on $\gamma^+_{\infty,lat}$ and using (4.37) and (4.15), thus the equation reads,

$$\begin{aligned}\sum_{\sigma\in I^k}\Bigg[\int_{\omega_1\times Y}&-\sum_{i,j=1}^{2}\left(\partial_{y_j}\left(a_{1j}\partial_{x_1}w^\sigma\right)\cdot v^\sigma-\partial_{x_1}\left(a_{i1}\partial_{y_i}w^\sigma\right)\cdot v^\sigma\right)-\lambda^1\rho w^\sigma\cdot v^\sigma\ dx_1dy\\&+\sum_{i,j=1}^{2}\int_Y\left[-w^\sigma\cdot a_{1j}\partial_{y_j}v^\sigma+a_{i1}\partial_{y_i}w^\sigma\cdot v^\sigma\right]_{x_1=0}^{x_1=\alpha}\ dy\Bigg]\\&+\sum_{\vartheta\in\{0,\alpha\}}\int_{\gamma^+_{\infty,end}}a\nabla_y\left(\sum_{\sigma\in I^k}w^\sigma_\vartheta+w^\vartheta_{b,k}\right).n_y\cdot\left(e^{-\eta y_1}v^\vartheta_b\right)dy=0.\end{aligned}\tag{4.56}$$

From (4.15) and (4.16), we get that $w^\vartheta_{b,k}(y)$ is the linear function of $-\sum\limits_{\sigma\in I^k}w^\sigma_\vartheta(0,y_2)$, it means that,

$$w^\vartheta_{b,k}(y)=\mathcal{L}\left(-\sum_{\sigma\in I^k}w^\sigma_\vartheta(0,y_2)\right).\tag{4.57}$$

Hence, in Equation (4.56) it is equivalent to,

$$\begin{aligned}\sum_{\sigma\in I^k}\Bigg[\int_{\omega_1\times Y}&\sum_{i,j=1}^{2}\left(-\partial_{y_j}\left(a_{1j}\partial_{x_1}w^\sigma\right)\cdot v^\sigma-\partial_{x_1}\left(a_{i1}\partial_{y_i}w^\sigma\right)\cdot v^\sigma\right)-\lambda^1\rho w^\sigma\cdot v^\sigma\ dx_1dy\\&+\sum_{i,j=1}^{2}\int_Y\left[-w^\sigma\cdot a_{1j}\partial_{y_j}v^\sigma+a_{i1}\partial_{y_i}w^\sigma\cdot v^\sigma\right]_{x_1=0}^{x_1=\alpha}\ dy\Bigg]\\&+\sum_{\vartheta\in\{0,\alpha\}}\int_{\gamma^+_{\infty,end}}a\nabla_y\left(\sum_{\sigma\in I^k}w^\sigma_\vartheta+\mathcal{L}\left(-\sum_{\sigma\in I^k}w^\sigma_\vartheta\right)\right).n_y\cdot\left(e^{-\eta y_1}v^\vartheta_b\right)dy=0.\end{aligned}$$

Thanks to (4.34), the decompositions (4.29) and (4.53) of w^σ and v^σ, as well as the linearity of $\mathcal{L}$, we get

$$w^0_b(y)=\mathcal{L}\left(-\sum_{\sigma\in I^k,p\in M^\sigma_n}u^\sigma_p(0)\,\phi^\sigma_p(0,y_2)\right)=-\sum_{\sigma\in I^k,p\in M^\sigma_n}u^\sigma_p(0)\,\mathcal{L}\left(\phi^\sigma_p(0,y_2)\right),$$

$$w^\alpha_b(y)=\mathcal{L}\left(-\sum_{\sigma\in I^k,p\in M^\sigma_n}u^\sigma_p(\alpha)\,\phi^\sigma_p(0,y_2)\right)=-\sum_{\sigma\in I^k,p\in M^\sigma_n}u^\sigma_p(\alpha)\,\mathcal{L}\left(\phi^\sigma_p(0,y_2)\right),$$

and the equation yields,

$$\sum_{\sigma\in I^k,p,q\in M_n^\sigma}\int_{\omega_1}\sum_{i,j=1}^{2}\left(\int_Y -\partial_{y_j}\left(a_{1j}\phi_p^\sigma\right)\cdot\phi_q^\sigma - a_{i1}\partial_{y_i}\phi_p^\sigma\cdot\phi_q^\sigma dy\right)\partial_{x_1}u_p^\sigma\cdot v_q^\sigma \tag{4.58}$$

$$-\lambda^1\left(\int_Y \rho\phi_p^\sigma\cdot\phi_q^\sigma dy\right)u_p^\sigma\cdot v_q^\sigma\ dx_1$$

$$+\sum_{i,j=1}^{2}\left[\left(\int_Y -\phi_p^\sigma\cdot a_{1j}\partial_{y_j}\phi_q^\sigma + a_{i1}\partial_{y_i}\phi_p^\sigma\cdot\phi_q^\sigma dy\right)u_p^\sigma\cdot v_q^\sigma\right]_{x_1=0}^{x_1=\alpha}$$

$$-\sum_{\sigma\in I^k,p\in M_n^\sigma}\int_{Y_2}\left[a\nabla_y\left(u_p^\sigma(0)\,\phi_p^\sigma - u_p^\sigma(0)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^0\,(y_1=0)\right.$$

$$\left.+a\nabla_y\left(u_p^\sigma(\alpha)\,\phi_p^\sigma - u_p^\sigma(\alpha)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^\alpha\,(y_1=0)\right]dy_2 = 0.$$

We observe that,

$$\sum_{i,j=1}^{2}\int_Y -\partial_{y_j}\left(a_{1j}\phi_p^\sigma\right)\cdot\phi_q^\sigma - a_{i1}\partial_{y_i}\phi_p^\sigma\cdot\phi_q^\sigma dy = \sum_{i,j=1}^{2}\int_Y \phi_p^\sigma\cdot a_{1j}\partial_{y_j}\phi_q^\sigma - a_{i1}\partial_{y_i}\phi_p^\sigma\cdot\phi_q^\sigma dy.$$

Using (4.19) the definition of coefficients $c(.,.,.)$ and $b(.,.,.)$, the equation (4.58) becomes,

$$\sum_{\sigma\in I^k,p,q\in M_n^\sigma}\left[\int_{\omega_1} c(\sigma,p,q)\,\partial_{x_1}u_p^\sigma\cdot v_q^\sigma - b(\sigma,p,q)\,\lambda^1 u_p^\sigma\cdot v_q^\sigma\ dx_1 - \left[c(\sigma,p,q)\,u_p^\sigma\cdot v_q^\sigma\right]_{x_1=0}^{x_1=\alpha}\right]$$

$$-\sum_{\sigma\in I^k,p\in M_n^\sigma}\int_{Y_2}\left[a\nabla_y\left(u_p^\sigma(0)\,\phi_p^\sigma - u_p^\sigma(0)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^0\,(y_1=0)\right.$$

$$\left.+a\nabla_y\left(u_p^\sigma(\alpha)\,\phi_p^\sigma - u_p^\sigma(\alpha)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^\alpha\,(y_1=0)\right]dy_2 = 0.$$

Choosing the test functions $v^\sigma\in L^2\left(H_0^1(\omega_1);Y\right)\cap L^2\left(\Omega;H_0^1(Y)\right)$ proves the equations of u_p^σ:

$$\sum_{p\in M_n^\sigma} c(\sigma,p,q)\,\partial_{x_1}u_p^\sigma - b(\sigma,p,q)\,\lambda^1 u_p^\sigma = 0 \text{ for each } q\in M_n^\sigma \text{ and } \sigma\in I^k.$$

Thus, the boundary term remains,

$$-\sum_{\sigma\in I^{k_1},p,q\in M_n^\sigma}\left[c(\sigma,p,q)\,u_p^\sigma\cdot v_q^\sigma\right]_{x_1=0}^{x_1=\alpha} \tag{4.59}$$

$$-\sum_{\sigma\in I^k,p\in M_n^\sigma}\int_{Y_2} a\nabla_y\left(u_p^\sigma(0)\,\phi_p^\sigma - u_p^\sigma(0)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^0\,(y_1=0)$$

$$+a\nabla_y\left(u_p^\sigma(\alpha)\,\phi_p^\sigma - u_p^\sigma(\alpha)\,\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot v_b^\alpha\,(y_1=0)\,dy_2 = 0.$$

From the relation (4.55) between v^σ and v_b^ϑ at $y_1=0$,

$$v_b^0(0,y_2) = -\sum_{\sigma\in I^k} v^\sigma(0,0,y_2) = -\sum_{\sigma\in I^k,q\in M_n^\sigma} v_q^\sigma(0)\,\phi_q^\sigma(0,y_2)$$

and $v_b^\alpha(0,y_2) = -\sum_{\sigma\in I^k} v^\sigma(\alpha,0,y_2)\,e^{sign(\sigma)2i\pi l^k} = -\sum_{\sigma\in I^k,q\in M_n^\sigma} v_q^\sigma(\alpha)\,\phi_q^\sigma(0,y_2)\,e^{sign(\sigma)2i\pi l^k}$, hence, the expression (4.59) reads

$$\sum_{\sigma\in I^k,p,q\in M_n^\sigma}\left[\left(c(\sigma,p,q)-\int_{Y_2}a\nabla_y\left(\phi_p^\sigma-\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot\phi_q^\sigma(y_1=0)\,dy_2\right)u_p^\sigma(0)\overline{v_q^\sigma}(0)\right.$$
$$\left.-\left(c(\sigma,p,q)+\int_{Y_2}a\nabla_y\left(\phi_p^\sigma-\mathcal{L}\left(\phi_p^\sigma(0,y_2)\right)\right)\cdot\phi_q^\sigma(y_1=0)\,dy_2\right)u_p^\sigma(\alpha)\overline{v_q^\sigma}(\alpha)\right]=0.$$

Using (4.20) and (4.21) the definition of coefficients e_0 and e_α, the boundary conditions are

$$\sum_{\sigma\in I^k,p,q\in M_n^\sigma}e_0(\sigma,p,q)\,u_p^\sigma(0)\,\overline{v_q^\sigma}(0)=0\text{ and }\sum_{\sigma\in I^k,p,q\in M_n^\sigma}e_\alpha(\sigma,p,q)\,u_p^\sigma(\alpha)\,\overline{v_q^\sigma}(\alpha)=0.$$

Introducing the matrices $C_{qp}^\sigma=c(\sigma,p,q)$, $B_{qp}^\sigma=b(\sigma,p,q)$, $U_p^\sigma=u_p^\sigma$, $V_q^\sigma=\overline{v_q^\sigma}$, $E_{qp}^{\vartheta,\sigma}=e_\vartheta(\sigma,p,q)$ and $\Phi_q^{\vartheta,\sigma}=\widehat{\phi}_q^{\sigma,\vartheta}$ leads to the matrix form

$$C_{qp}^\sigma\partial_x U_p^\sigma+\lambda^1B_{qp}^\sigma U_p^\sigma=0_q\text{ for each }\sigma\text{ and }q,$$

with the boundary conditions

$$\sum_{\sigma\in I^k}\left(V^\sigma\right)^TE^{\vartheta,\sigma}U^\sigma=0\text{ at }x_1=\vartheta,$$

or in short with block vectors and matrices $U=\left(U^\sigma\right)_\sigma$, $V=\left(V^\sigma\right)_\sigma$ and $E^\vartheta=\begin{pmatrix}E^{\vartheta,k}&0\\0&E^{\vartheta,-k}\end{pmatrix}$,

$$V^TE^\vartheta U=0\text{ at }x_1=\vartheta$$

for all V such that,

$$\left(I_jV\right)^T\Phi^\vartheta=0\text{ at }x_1=\vartheta\text{ for all }j\in\{1,...,m\}$$

where the matrix I_j, defined in the beginning of the section, is considered here as a 2×2 block matrix of $m\times m$ submatrices. The boundary conditions are equivalent to

$$V(x_1=\vartheta)\perp E^\vartheta U(x_1=\vartheta)\ \text{ for all }V$$
$$\text{such that }V(\vartheta)\perp\left(I_j\right)^T\Phi^\vartheta\text{ at }\vartheta\in\{0,\alpha\}\ \text{ and with }j\in\{1,...,m\}.$$

Since $V(x_1=\vartheta)\perp\left(I_j\right)^T\Phi^\vartheta$ at $\vartheta\in\{0,\alpha\}$ for all $j\in\{1,...,m\}$, therefore, $V(x_1=\vartheta)\in L^{\vartheta\perp}$. Moreover, since $E^\vartheta U(x_1=\vartheta)\perp V(x_1=\vartheta)$, so

$$\left\langle E^\vartheta U,X_j^\vartheta\right\rangle=0\text{ for all }\ \vartheta\in\{0,\alpha\}\ \text{ and }j\in\{1,...,m\},$$

where X_j^ϑ is defined in (4.45). It is equivalent to

$$\sum_{p\in M_n^k}e_\vartheta(k,p,q)\,u_p^k(\vartheta)\,\overline{\widehat{\phi}_q^{-k,\vartheta}}-e_\vartheta(-k,p,q)\,u_p^{-k}(s)\,\overline{\widehat{\phi}_q^{k,\vartheta}}=0\text{ for all }q\in M_n^k.$$

Finally, the boundary conditions of the macroscopic equation are

$$\begin{aligned}\sum_{p\in M_n^k}e_0(k,p,q)\,u_p^k(0)\,\widehat{\phi}_q^{k,0}-e^0(-k,p,q)\,u_p^{-k}(0)\,\widehat{\phi}_q^{-k,0}&=0,\\\text{and }\sum_{p\in M_n^k}e_\alpha(k,p,q)\,u_p^k(\alpha)\,\widehat{\phi}_q^{k,\alpha}-e^\alpha(-k,p,q)\,u_p^{-k}(\alpha)\,\widehat{\phi}_q^{-k,\alpha}&=0,\end{aligned}$$

for all $q \in M_n^k$.

ii) **Case** $k \in \left\{0, -\frac{1}{2}\right\}$. The process is similar to the case of $k \notin \left\{0, -\frac{1}{2}\right\}$ but the final boundary condition for HF-macroscopic model are not found. We choose $v^k \in H^2(\omega_1 \times Y)$ a k-quasi-periodic function in y_1 such that it is decomposed by

$$v^k(x_1, y) = \sum_{q \in M_n^k} v_q^k(x_1)\, \phi_q^k(y) \tag{4.60}$$

and is satisfied the condition

$$\sum_{q \in M_n^k} v_q^k(x_1) \int_{Y_2} \phi_q^k(0, y_2)\, dy_2 = 0 \text{ at } x_1 \in \partial\omega_1. \tag{4.61}$$

We also take a function $v_b^s \in H^2(Y_\infty^+)$ where $\vartheta \in \{0, \alpha\}$ and $\eta > 0$ such that,

$$a\nabla_y \left(e^{-\eta y_1} v_b^{\vartheta}\right).n_y = 0 \text{ on } \gamma_{\infty,lat}^+,$$

and

$$\left(\mathfrak{R}v^k\right)(x) + e^{-\eta x_1/\varepsilon}\left(\mathfrak{R}^0 v_b^0\right)(x) + e^{-\eta(\alpha - x_1)/\varepsilon}\left(\mathfrak{R}^\alpha v_b^\alpha\right)(x) = 0 \text{ on } \Gamma_{end}. \tag{4.62}$$

We pose

$$v^\varepsilon(x) = \left(\mathfrak{R}v^k\right)(x), v^{b,\varepsilon}(x) = e^{-\eta x_1/\varepsilon}\left(\mathfrak{R}^0 v_b^0\right)(x) + e^{-\eta(\alpha - x_1)/\varepsilon}\left(\mathfrak{R}^\alpha v_b^\alpha\right)(x), \tag{4.63}$$

and choose $\psi^\varepsilon = v^\varepsilon + v^{b,\varepsilon}$ with

$$\psi^\varepsilon \in H_{end}^1(\Omega) \cap H^2(\Omega) \text{ such that } a^\varepsilon \nabla_x \psi^\varepsilon.n_x = 0 \text{ on } \Gamma_{lat}, \tag{4.64}$$

as a test function of the weak formulation (4.10) of the spectral problem. According to (4.60) and (4.62), the boundary conditions on Γ_{end} of test functions ψ^ε are rewritten by,

$$\sum_{q \in M_n^k} v_q^k(0)\, \phi_q^k\left(0, \frac{x_2}{\varepsilon}\right) + v_b^k\left(0, \frac{x_2}{\varepsilon}\right) + e^{-\eta\alpha/\varepsilon} v_b^\alpha\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = 0,$$

$$\text{and } \sum_{q \in M_n^k} v_q^k(\alpha)\, \phi_q^k\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + e^{-\eta\alpha/\varepsilon} v_b^0\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + v_b^\alpha\left(0, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = \alpha,$$

for all $x_2 \in \omega_2$. Using the periodicity or anti - periodicity of ϕ_q^k in the variable y_1, the second condition becomes,

$$\sum_{q \in M_n^k} v_q^k(\alpha)\, \phi_q^k\left(0, \frac{x_2}{\varepsilon}\right) e^{2i\pi k\alpha/\varepsilon} + e^{-\eta\alpha/\varepsilon} v_b^0\left(\frac{\alpha}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + v_b^\alpha\left(0, \frac{x_2}{\varepsilon}\right) = 0 \text{ at } x_1 = \alpha.$$

Since $\alpha/\varepsilon \in \mathbb{N}^*$, $e^{2i\pi k\alpha/\varepsilon} = 1$ if $k = 0$ and $= e^{-i\pi}$ otherwise. Passing to the limit, the boundary conditions of the test function are,

$$\sum_{q \in M_n^k} v_q^k(0)\, \phi_q^k(0, y_2) + v_b^0(0, y_2) = 0 \text{ at } x_1 = 0, \tag{4.65}$$

$$\text{and } \sum_{q \in M_n^k} v_q^k(\alpha)\, \phi_q^k(0, y_2)\, e^{2i\pi k} + v_b^\alpha(0, y_2) = 0 \text{ at } x_1 = \alpha.$$

The procedure of proof is the same in case $k \notin \left\{0, -\frac{1}{2}\right\}$ and we get the final equation as,

$$\sum_{p,q\in M_n^k} \int_{\omega_1} c(k,p,q)\,\partial_{x_1} u_p^k \cdot v_q^k - b(k,p,q)\,\lambda^1 u_p^k \cdot v_q^k \; dx_1 - \sum_{p,q\in M_n^k} \left[c(0,p,q)\,u_p^0 \cdot v_q^0\right]_{x_1=0}^{x_1=\alpha}$$
$$-d(k,p,q)\,u_p^k(0)\cdot v_q^k(0) - d(k,p,q)\,u_p^k(\alpha)\cdot v_q^k(\alpha) = 0.$$

Choosing the test functions such that $v^k \in L^2\left(H_0^1(\omega_1);Y\right) \cap L^2\left(\Omega; H_0^1(Y)\right)$, the internal equations are stated for each $q \in M_n^k$ by

$$\sum_{p\in M_n^k} c(k,p,q)\,\partial_{x_1} u_p^k - b(k,p,q)\,\lambda^1 u_p^k = 0.$$

So, the boundary term remains,

$$\begin{aligned} \sum_{p,q\in M_n^k} \left(c(k,p,q) - d(k,p,q)\right) u_p^k(0)\,\overline{v_q^k}(0) &= 0 \text{ at } x_1 = 0, \\ \text{and} \sum_{p,q\in M_n^k} \left(c(k,p,q) - d(k,p,q)\right) u_p^k(\alpha)\,\overline{v_q^k}(\alpha) &= 0 \text{ at } x_1 = \alpha. \end{aligned}$$

Or,

$$\sum_{p,q\in M_n^k} e_0(k,p,q)\,u_p^k(0)\,\overline{v_q^k}(0) = 0 \text{ and } \sum_{p,q\in M_n^k} e_\alpha(k,p,q)\,u_p^k(\alpha)\,\overline{v_q^k}(\alpha) = 0. \tag{4.66}$$

We introduce the matrices $C = \left(c(k,p,q)\right)_{p,q}$, $B = \left(b(k,p,q)\right)_{p,q}$, $U = \left(u_p^k\right)_p$, $V = \left(\overline{v_p^k}\right)_p$, $E^\vartheta = \left(e_\vartheta(k,p,q)\right)_{p,q}$, $\Phi^\vartheta = \left(\int_{Y_2} \widehat{\phi}_q^{k,\vartheta}(0,y_2)\,dy_2\right)_q$, then the matrix form is,

$$C\partial_x U + \lambda^1 BU = 0,$$

$$\text{with } V^T E^0 U = 0 \text{ at } x_1 = 0 \text{ and } V^T E^\alpha U = 0 \text{ at } x_1 = \alpha, \tag{4.67}$$
$$\text{for all } V \text{ such that } V^T(x_1 = \vartheta)\,\Phi^\vartheta = 0.$$

Finally, the internal of the HF-macroscopic equation (4.24) allows with unknown the boundary condition. ■

4.3.4 Proof of Theorem 46

For $k \in Y^*$, let $(\lambda^\varepsilon, w^\varepsilon)$ be solution of the weak formulation (4.10) and satisfies the uniform bound (4.11), the property (1.8) yields the uniform bound of $S_\sigma^\varepsilon w^\varepsilon$ in $L^2(\omega_1 \times Y)$ for any $\sigma \in I^k$. So there exist $w^\sigma \in L^2(\omega_1 \times Y)$ such that up the extraction of a subsequence $S_\sigma^\varepsilon w^\varepsilon \to w^\sigma$ in $L^2(\omega_1 \times Y)$ weakly. Hence, $\sum_{\sigma\in I^k} S_\sigma^\varepsilon w^\varepsilon$ converges to

$$g_k(x_1, y) = \sum_{\sigma\in I^k} w^\sigma(x_1, y).$$

According to Lemma 51, there exist $n \in \mathbb{N}^*$ such that $\lambda^0 = \lambda_n^k$ and w^σ is decomposed as in (4.29) based on $\left(\phi_p^\sigma\right)_{\sigma,p}$ the Bloch wave eigenmodes corresponding to Bloch eigenvalue λ^0, so

$$g_k\left(x,y\right) = \sum_{\sigma \in I^k, m \in M_n^\sigma} u_m^\sigma\left(x_1\right)\phi_m^\sigma\left(y\right)$$

for $u_m^\sigma \in L^2\left(\omega_1\right)$. Moreover, as in proof of Lemma 53, $w_{b,k}^\vartheta$ is solution of the boundary layer equation (4.15) for $\vartheta \in \{0,\alpha\}$. Finally, for $\varepsilon \in E_k$ as in Assumption 14, if $u_p^\sigma \in H^1\left(\omega_1\right)$ then u_p^σ is a solution of the HF-macroscopic models (4.22)-(4.24) as in the proof of Lemma 55.

Chapter 5

Conclusions and perspectives

The periodic homogenization has been studied for the spectral problem and the wave equation with periodic coefficients in a one-dimensional bounded domain. It has also been done for the spectral problem posed in a two-dimensional thin bounded strip. Applying our method, so-called Bloch wave homogenization, provides two-scale models including the expected high frequency parts and also a low frequency part for the wave equation. Our work focuses mainly on the high frequency part. It comprises so-called high-frequency microscopic and macroscopic equations, the first being a second order partial differential equation and the second a system of first order partial differential equations. In the strip case, a boundary layer occurs under the form of a second order partial differential equation. The boundary conditions have been found for the high frequency macroscopic equation. For the spectral problem, the asymptotic behaviors were addressed for both the eigenvalues and the corresponding eigenvectors. Numerical simulations are provided to corroborate the theory in the one-dimensional cases.

The same method might be extended to other cases. The homogenization of the wave equation posed in a bounded strip should be obtained by a combination of the results obtained in one-dimension to the boundary layer result of the spectral problem. However, the boundary layer equation should be a time-space wave equation posed in an infinite strip that might be using a time two-scale transform together with the boundary layer two-scale transform. In addition, the homogenization for both the spectral problem and the wave equation should be done also in the two dimensional open bounded domain by extending the approach. The boundary layers should be considered in both y_1 and y_2 directions with a specific problem to take into account a boundary layer effect at the corners.

Finally, we mention possible short-term research works.

1. Numerical simulation for the strip case.

2. Homogenization of the spectral and wave equations in two-dimensions or higher dimension including the boundary conditions.

3. Cases with non-homogeneous boundary conditions.

4. Extension to the system of elasticity equations.

4. Applications in optics or in mechanics to photonic crystals, phononic devices or other waveguides.

Appendix

In this Appendix, we report some mathematical proofs, supplementary results and remarks.

Proof. [Proof of Lemma 8] The proof is carried out in two steps. First the explicit expression of $T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}v$ is derived, then the approximations (1.21) and (1.22) are deduced.

(i) Let us prove that

$$\begin{gathered}\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}v\right)(t,x)\\ = \sum_{\theta_\varepsilon\in D,\omega_\varepsilon\in C}\frac{1}{\alpha_n^k\varepsilon^2}\int_{\theta_\varepsilon\times\omega_\varepsilon} v\left(z_t,\frac{t-\varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k},z_x,\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dz_t dz_x\chi_{\theta_\varepsilon}(t)\,\chi_{\omega_\varepsilon}(x)e^{2i\pi k l_{\omega_\varepsilon}}.\end{gathered}$$

From the definitions of the two-scale transforms $T^{\varepsilon\alpha_n^k}$ and S_k^ε with $r_t = \left(\varepsilon\alpha_n^k\right)l_{\theta_\varepsilon} + \left(\varepsilon\alpha_n^k\right)\tau\in\theta_\varepsilon$ and $r_x = \varepsilon l_{\omega_\varepsilon}+\varepsilon y\in\omega_\varepsilon$,

$$\begin{gathered}\int_{I\times\Lambda\times\Omega\times Y} v(t,\tau,x,y)\cdot\left(T^{\varepsilon\alpha_n^k}S_k^\varepsilon w\right)(t,\tau,x,y)\,dtd\tau dxdy\\ = \int_{I\times\Omega}\sum_{\theta_\varepsilon\in D,\omega_\varepsilon\in C}[\frac{1}{\alpha_n^k\varepsilon^2}\int_{\theta_\varepsilon\times\omega_\varepsilon} v\left(t,\frac{r_t-\left(\varepsilon\alpha_n^k\right)l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k},x,\frac{r_x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dtdx\cdot w(r_t,r_x)\\ \chi_{\theta_\varepsilon}(r_t)\,\chi_{\omega_\varepsilon}(r_x)e^{-2i\pi k l_{\omega_\varepsilon}}]\;dr_t dr_x.\end{gathered}$$

Changing the variable names and using the definitions of $S_k^{\varepsilon *}$ and $T^{\varepsilon\alpha_n^k *}$,

$$\begin{gathered}\int_{I\times\Omega}\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}v\right)(t,x)\cdot w(t,x)\;dtdx\\ = \int_{I\times\Omega}[\sum_{\theta_\varepsilon\in D,\omega_\varepsilon\in C}\frac{1}{\alpha_n^k\varepsilon^2}\int_{\theta_\varepsilon\times\omega_\varepsilon} v\left(z_t,\frac{t-\left(\varepsilon\alpha_n^k\right)l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k},z_x,\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dz_t dz_x\\ e^{2i\pi k l_{\omega_\varepsilon}}\cdot w(t,x)\,\chi_{\theta_\varepsilon}(t)\,\chi_{\omega_\varepsilon}(x)]\;dtdx.\end{gathered}$$

This establishes the explicit expression of $T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}$.

(ii) Let us derive the expected approximation for $v\in C^1(I\times\Lambda\times\Omega\times Y)$ a periodic function in τ and $k-$quasi-periodic function in y. The first order Taylor formula expresses (z_t,z_x) in terms of (t,x) as,

$$v(z_t,\tau,z_x,y) = v(t,\tau,x,y)+\partial_t v(x,y)(z_t-t)+\partial_x v(x,y)(z_x-x)+\varepsilon O(\varepsilon)$$

in $L^2(\theta_\varepsilon \times \omega_\varepsilon)$ for a.e. $\tau \in \Lambda$ and $y \in Y$. Hence,

$$\begin{aligned}
&(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v)\left(\left(\varepsilon\alpha_n^k\right) l_{\theta_\varepsilon} + \left(\varepsilon\alpha_n^k\right)\tau, \varepsilon l_{\omega_\varepsilon} + \varepsilon y\right) \\
&= \frac{1}{\alpha_n^k \varepsilon^2} \int_{\theta_\varepsilon \times \omega_\varepsilon} [v(t,\tau,x,y) + \partial_t v(t,\tau,x,y)(z_t - t) \\
&+ \partial_x v(t,\tau,x,y)(z_x - x) \quad + \varepsilon O(\varepsilon)] dz_t dz_x \; e^{2i\pi k l_{\omega_\varepsilon}}
\end{aligned}$$

for a.e. $(\tau, y) \in \Lambda \times Y$ and all $\theta_\varepsilon \in D$, $\omega_\varepsilon \in C$. Remarking that

$$z_t - t = \left(z_t - \varepsilon\alpha_n^k l_{\theta_\varepsilon}\right) + \left(\varepsilon\alpha_n^k l_{\theta_\varepsilon} - t\right) \text{ and } z_x - x = (z_x - \varepsilon l_{\omega_\varepsilon}) + (\varepsilon l_{\omega_\varepsilon} - x)$$

with

$$\int_{\theta_\varepsilon} \left(z_t - \varepsilon\alpha_n^k l_{\theta_\varepsilon}\right) dz_t = \frac{1}{2}\left(\varepsilon\alpha_n^k\right)^2 \text{ and } \int_{\omega_\varepsilon} (z_x - \varepsilon l_{\omega_\varepsilon})\, dz_x = \frac{1}{2}\varepsilon^2.$$

For a.e. $(\tau, y) \in \Lambda \times Y$, $(t, x) \in \theta_\varepsilon \times \omega_\varepsilon$ and all $\theta_\varepsilon \in D$, $\omega_\varepsilon \in C$, since $|\theta_\varepsilon| = \varepsilon\alpha_n^k$ and $|\omega_\varepsilon| = \varepsilon$, so

$$\begin{aligned}
&(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v)\left(\left(\varepsilon\alpha_n^k\right) l_{\theta_\varepsilon} + \left(\varepsilon\alpha_n^k\right)\tau, \varepsilon l_{\omega_\varepsilon} + \varepsilon y\right) \\
&= \left[v(t,\tau,x,y) - \varepsilon\alpha_n^k \left(\frac{t - \varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k} - \frac{1}{2}\right) \partial_t v(t,\tau,x,y) \right. \\
&\left. - \varepsilon \left(\frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon} - \frac{1}{2}\right) \partial_x v(t,\tau,x,y)\right] \; e^{2i\pi k l_{\omega_\varepsilon}} + \varepsilon O(\varepsilon).
\end{aligned}$$

From the explicit expressions of $T^{\varepsilon\alpha_n^k *}$ and $S_0^{\varepsilon *}$, also refer to Remark 56,

$$\frac{t - \varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k} - \frac{1}{2} = \frac{1}{\varepsilon\alpha_n^k} \int_{\theta_\varepsilon} \left(\frac{t - \varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k} - \frac{1}{2}\right) dz_t = \left(T^{\varepsilon\alpha_n^k *}\left(\tau - \frac{1}{2}\right)\right)\left(\varepsilon\alpha_n^k\right) l_{\theta_\varepsilon} + \left(\varepsilon\alpha_n^k\right)\tau,$$

$$\text{and } \frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon} - \frac{1}{2} = \frac{1}{\varepsilon} \int_{\omega_\varepsilon} \left(\frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon} - \frac{1}{2}\right) dz_x = \left(S_0^{\varepsilon *}\left(y - \frac{1}{2}\right)\right)(\varepsilon l_{\omega_\varepsilon} + \varepsilon y),$$

so,

$$\begin{aligned}
&= \left(v(t,\tau,x,y) - \varepsilon\alpha_n^k T^{\varepsilon\alpha_n^k *}\left(\tau - \frac{1}{2}\right)\partial_t v(t,\tau,x,y) - \varepsilon S_0^{\varepsilon *}\left(y - \frac{1}{2}\right)\partial_x v(t,\tau,x,y)\right) \\
&\qquad \chi_{\theta_\varepsilon}(t)\, \chi_{\omega_\varepsilon}(x)\, e^{2i\pi k l_{\omega_\varepsilon}} + \varepsilon O(\varepsilon)
\end{aligned}$$

in the $L^2(\theta_\varepsilon \times \Lambda \times \omega_\varepsilon \times Y)$ weak sense. Therefore,

$$\begin{aligned}
(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v)(t,x) &= \sum_{\theta_\varepsilon \in D, \omega_\varepsilon \in C} \left[v\left(t, \frac{t}{\varepsilon\alpha_n^k} - l_{\theta_\varepsilon}, x, \frac{x}{\varepsilon} - l_{\omega_\varepsilon}\right)\right. \\
&- \varepsilon\alpha_n^k T^{\varepsilon\alpha_n^k *}\left(\tau - \frac{1}{2}\right)\partial_t v\left(t, \frac{t}{\varepsilon\alpha_n^k} - l_{\theta_\varepsilon}, x, \frac{x}{\varepsilon} - l_{\omega_\varepsilon}\right) \\
&\left. - \varepsilon S_0^{\varepsilon *}\left(y - \frac{1}{2}\right)\partial_x v\left(t, \frac{t}{\varepsilon\alpha_n^k} - l_{\theta_\varepsilon}, x, \frac{x}{\varepsilon} - l_{\omega_\varepsilon}\right)\right] \chi_{\theta_\varepsilon}(t)\, \chi_{\omega_\varepsilon}(x)\, e^{2i\pi k l_{\omega_\varepsilon}} + \varepsilon O(\varepsilon).
\end{aligned}$$

Using the periodicity in τ and $k-$quasi-periodicity in y of function v,

$$\begin{aligned}
&= v\left(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}\right) - \varepsilon\alpha_n^k T^{\varepsilon\alpha_n^k *}\left(\tau - \frac{1}{2}\right)\partial_t v\left(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}\right) \qquad (1) \\
&- \varepsilon S_0^{\varepsilon *}\left(y - \frac{1}{2}\right)\partial_x v\left(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon}\right) + \varepsilon O(\varepsilon)
\end{aligned}$$

in the $L^2(I \times \Omega)$ weak sense. Hence the formula (1.21) follows.

From (1.20), Equation (1) is equivalent to

$$\left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v\right)(t,x) = \mathfrak{B}_n^k v - \varepsilon \alpha_n^k T^{\varepsilon\alpha_n^k *} \left(\tau - \frac{1}{2}\right) \mathfrak{B}_n^k \left(\partial_t v\right) - \varepsilon S_0^{\varepsilon *} \left(y - \frac{1}{2}\right) \mathfrak{B}_n^k \left(\partial_x v\right) + \varepsilon O\left(\varepsilon\right).$$

Applying the approximation (1.21) to $\partial_t v$ and $\partial_x v$ with any function $v \in C^2(I \times \Lambda \times \Omega \times Y)$,

$$= \mathfrak{B}_n^k v - \varepsilon \alpha_n^k T^{\varepsilon\alpha_n^k *} \left(\tau - \frac{1}{2}\right) T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\partial_t v\right) - \varepsilon S_0^{\varepsilon *} \left(y - \frac{1}{2}\right) T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\partial_x v\right) + \varepsilon O\left(\varepsilon\right).$$

Thanks to the explicit expression of $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v$, also refer to Remark 56, we get

$$\begin{aligned} \alpha_n^k T^{\varepsilon\alpha_n^k *} \left(\tau - \frac{1}{2}\right) T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\partial_t v\right) &= T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\alpha_n^k \left(\tau - \frac{1}{2}\right) \partial_t v\right), \\ \text{and } S_0^{\varepsilon *} \left(y - \frac{1}{2}\right) T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\partial_x v\right) &= T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\left(y - \frac{1}{2}\right) \partial_x v\right). \end{aligned} \tag{2}$$

We see more detail for (2) in (3), (4) and (5) of Remark 56. Hence,

$$= \mathfrak{B}_n^k v - \varepsilon T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\alpha_n^k \left(\tau - \frac{1}{2}\right) \partial_t v\right) - \varepsilon T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\left(y - \frac{1}{2}\right) \partial_x v\right) + \varepsilon O\left(\varepsilon\right).$$

Finally,

$$\mathfrak{B}_n^k v = T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v + \varepsilon T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} \left(\alpha_n^k \left(\tau - \frac{1}{2}\right) \partial_t v + \left(y - \frac{1}{2}\right) \partial_x v\right) + \varepsilon O\left(\varepsilon\right).$$

■

Remark 56 *For any $k \in Y^*$ and $n \in \mathbb{N}^*$, let $v \in L^2\left(I \times \Lambda \times \Omega \times Y\right)$ be a periodic function in τ and $k-$quasi-periodic function in y, then*

$$T^{\varepsilon\alpha_n^k *}\left(\tau\right) \left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v\right) = \left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\right) \left(\tau v\right), \tag{3}$$

and

$$S_0^{\varepsilon *}\left(y\right) \left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\left(v\right)\right) = T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\left(y v\right). \tag{4}$$

Consequently, for any $\mu_0 \in \mathbb{R}^*$,

$$\left(\left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\right) \left(\mu_0 v\right)\right)(t,x) = \mu_0 \left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v\right)(t,x). \tag{5}$$

Indeed, from the explicit expression of $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}$,

$$\left(\left(T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *}\right) \left(\tau v\right)\right)(t,x)$$

$$= \sum_{\theta_\varepsilon \in D, \omega_\varepsilon \in C} \frac{1}{\alpha_n^k \varepsilon^2} \int_{\theta_\varepsilon \times \omega_\varepsilon} \left(\frac{t - \varepsilon \alpha_n^k l_{\theta_\varepsilon}}{\varepsilon \alpha_n^k}\right) v\left(z_t, \frac{t - \varepsilon \alpha_n^k l_{\theta_\varepsilon}}{\varepsilon \alpha_n^k}, z_x, \frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dz_t dz_x \chi_{\theta_\varepsilon}\left(t\right) \chi_{\omega_\varepsilon}\left(x\right) e^{2i\pi k l_{\omega_\varepsilon}}$$

$$= \sum_{\theta_\varepsilon \in D, \omega_\varepsilon \in C} \left(\frac{t - \varepsilon \alpha_n^k l_{\theta_\varepsilon}}{\varepsilon \alpha_n^k}\right) \frac{1}{\alpha_n^k \varepsilon^2} \int_{\theta_\varepsilon \times \omega_\varepsilon} v\left(z_t, \frac{t - \varepsilon \alpha_n^k l_{\theta_\varepsilon}}{\varepsilon \alpha_n^k}, z_x, \frac{x - \varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right) dz_t dz_x \chi_{\theta_\varepsilon}\left(t\right) \chi_{\omega_\varepsilon}\left(x\right) e^{2i\pi k l_{\omega_\varepsilon}}.$$

We observe that

$$\frac{t-\varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k}\chi_{\theta_\varepsilon}(t)=\frac{1}{\alpha_n^k\varepsilon}\int_{\theta_\varepsilon}\left(\frac{t-\varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k}\right)dz_t\chi_{\theta_\varepsilon}(t)=T^{\varepsilon\alpha_n^k *}(\tau)\left(\left(\varepsilon\alpha_n^k\right)l_{\theta_\varepsilon}+\left(\varepsilon\alpha_n^k\right)\tau\right).$$

Therefore,

$$\left(\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}\right)(\tau v)\right)(t,x)=\left(T^{\varepsilon\alpha_n^k *}(\tau)\right)(t)\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}v\right)(t,x). \tag{6}$$

Similarly, we apply to the function yv the adjoint operator $T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}$

$$\left(\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}\right)(yv)\right)(t,x)$$

$$=\sum_{\theta_\varepsilon\in D,\omega_\varepsilon\in C}\frac{1}{\alpha_n^k\varepsilon^2}\int_{\theta_\varepsilon\times\omega_\varepsilon}\left(\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)v\left(z_t,\frac{t-\varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k},z_x,\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dz_t dz_x\chi_{\theta_\varepsilon}(t)\chi_{\omega_\varepsilon}(x)e^{2i\pi k l_{\omega_\varepsilon}}$$

$$=\sum_{\theta_\varepsilon\in D,\omega_\varepsilon\in C}[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}\left(\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dz_x\frac{1}{\alpha_n^k\varepsilon^2}\int_{\theta_\varepsilon\times\omega_\varepsilon}v\left(z_t,\frac{t-\varepsilon\alpha_n^k l_{\theta_\varepsilon}}{\varepsilon\alpha_n^k},z_x,\frac{x-\varepsilon l_{\omega_\varepsilon}}{\varepsilon}\right)dz_t dz_x$$

$$\chi_{\theta_\varepsilon}(t)\chi_{\omega_\varepsilon}(x)e^{2i\pi k l_{\omega_\varepsilon}}]=\left(S_0^{\varepsilon *}(y)\right)(x)\left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}v\right)(t,x). \tag{7}$$

Moreover, for any $\mu_0\in\mathbb{R}^*$, we get

$$S_0^{\varepsilon *}\mu_0=\mu_0 \text{ and } T^{\varepsilon\alpha_n^k *}\mu_0=\mu_0.$$

Finally, (5) is obtained thank to (6) and (7).

Proof. [Proof of Lemma 10] The proof is carried out in two steps. First the explicit expression of $S_k^{\varepsilon *}v$ is derived, then the approximation is deduced.

(i) Let us prove that

$$\left(S_k^{\varepsilon *}v\right)(x)=\sum_{j\in J}\int_{\omega_{1\varepsilon}^j}\varepsilon^{-1}v\left(z,\frac{x-\varepsilon l_{\omega_\varepsilon^j}}{\varepsilon}\right)dz\ \chi_{\omega_\varepsilon^j}(x)e^{2i\pi kj}.$$

From the definition of the modulated-two-scale transform with $r=\varepsilon l_{\omega_\varepsilon^j}+\varepsilon y\in\omega_\varepsilon^j$ and $dr=\varepsilon^2 dy$,

$$\int_{\omega_1\times Y}v(x_1,y)\cdot\left(S_k^\varepsilon w\right)(x_1,y)\,dx_1dy$$

$$=\sum_{j\in J}\int_{\omega_1\times\omega_\varepsilon^j}\varepsilon^{-2}v\left(x_1,\frac{r-\varepsilon l_{\omega_\varepsilon^j}}{\varepsilon}\right)\cdot w(r)\chi_{\omega_{1\varepsilon}^j}(x_1)e^{-2i\pi kj}\ dx_1dr$$

or equivalently,

$$=\int_\Omega\sum_{j\in J}\varepsilon^{-2}\int_{\omega_{1\varepsilon}^j}v\left(x_1,\frac{r-\varepsilon l_{\omega_\varepsilon^j}}{\varepsilon}\right)dx_1\cdot w(r)\chi_{\omega_\varepsilon^j}(r)e^{-2i\pi kj}\ dr.$$

Changing the variable names $z=x_1$, $x=r$ and using the definition of $S_k^{\varepsilon *}$,

$$\frac{1}{\varepsilon}\int_\Omega\left(S_k^{\varepsilon *}v\right)(x)\cdot w(x)\ dx=\int_\Omega\sum_j\varepsilon^{-2}\int_{\omega_{1\varepsilon}^j}v\left(z,\frac{x-\varepsilon l_{\omega_\varepsilon^j}}{\varepsilon}\right)dze^{2i\pi kj}\cdot w(x)\chi_{\omega_\varepsilon^j}(x)\ dx.$$

This establishes the explicit expression of $S_k^{\varepsilon*}$.

(ii) Let us derive the expected approximation for $v \in C^1(\omega_1 \times Y)$ and $k-$quasi-periodic in y_1. Since $\left|\omega_{1\varepsilon}^j\right| = \varepsilon$ and

$$v(z,y) = v(x_1,y) + \partial_{x_1} v(x_1,y)\,.\,(z-x_1) + \varepsilon O(\varepsilon) \text{ in } L^2(\omega_1) \text{ for a.e. } y \in Y,$$

then

$$\left(S_k^{\varepsilon*} v\right)\left(\varepsilon l_{\omega_\varepsilon^j} + \varepsilon y\right) = \left(\varepsilon^{-1} \int_{\omega_{1\varepsilon}^j} v(x_1,y) + \partial_{x_1} v(x_1,y)\,.\,(z-x_1)\ dz\right) e^{2i\pi kj} + O(\varepsilon),$$

for a.e. $y \in Y$ and all $j \in J$. Remarking that $z - x_1 = (z - \varepsilon j) + (\varepsilon j - x_1)$ and

$$\int_{\omega_{1\varepsilon}^j} (z - \varepsilon j)\, dz = \frac{1}{2}\varepsilon O(\varepsilon).$$

So for all $\omega_{1\varepsilon}^j$ and $y \in Y$,

$$\varepsilon e^{-2i\pi kj} \left(S_k^{\varepsilon*} v\right)\left(\varepsilon l_{\omega_\varepsilon^j} + \varepsilon y\right) = \left|\omega_{1\varepsilon}^j\right| v(x_1,y) + (\frac{1}{2}\varepsilon O(\varepsilon) + \left(\varepsilon^2 y\right)).\partial_{x_1} v(x_1,y) + \varepsilon O(\varepsilon).$$

Therefore,

$$\left(S_k^{\varepsilon*} v\right)(x) = \sum_{j \in J} v\left(x_1, \frac{x}{\varepsilon} - l_{\omega_\varepsilon^j}\right) \chi_{\omega_\varepsilon^j}(x)\, e^{2i\pi kj} + \varepsilon O(\varepsilon).$$

Using the $k-$quasi-periodic of v in y_1,

$$\left(S_k^{\varepsilon*} v\right)(x) = \sum_{j \in J} v\left(x_1, \frac{x}{\varepsilon}\right) \chi_{\omega_\varepsilon^j}(x) + \varepsilon O(\varepsilon),$$

in $L^2(\Omega)$, hence the formula (1.31) follows. ■

Proof. [Proof of Lemma 12] For $u \in L^2(\Omega)$ such that u is bounded in $L^2(\Omega)$, for $\vartheta = 0$, the definition (1.32) of $S_b^0 u$ gives

$$\int_{Y_\infty^+} \left|S_b^0 u\right|^2(y)\, dy = \int_{Y_\infty^+} |u|^2(\varepsilon y)\, \chi_{(0,1/\varepsilon)}(y_1)\, dy = \int_{Y_2} \int_{(0,\alpha/\varepsilon)} |u|^2(\varepsilon y_1, \varepsilon y_2)\, dy_1 dy_2$$

By changing variable $x = \varepsilon y$, so $dy = dx/\varepsilon^2$ and

$$= \varepsilon^{-2} \int_{\omega_2} \int_{\omega_1} |u|^2(x_1,x_2)\, dx_1 dx_2 = \varepsilon^{-2} \int_\Omega |u|^2(x)\, dx.$$

Similarly, for $x_1 = \alpha$, the definition (1.33) of $S_b^\alpha u$ implies that

$$\begin{aligned} \int_{Y_\infty^+} \left|S_b^\alpha u\right|^2(y)\, dy &= \int_{Y_\infty^+} |u|^2(-\varepsilon y_1 + \alpha, \varepsilon y_2)\, \chi_{(0,\alpha/\varepsilon)}(y_1)\, dy_1 dy_2 \\ &= \int_{Y_2} \int_{(0,\alpha/\varepsilon)} |u|^2(-\varepsilon y_1 + \alpha, \varepsilon y_2)\, dy_1 dy_2 \end{aligned}$$

By changing variables $x_1 = -\varepsilon y_1 + \alpha$ and $x_2 = \varepsilon y_2$, so $dy = -dx/\varepsilon^2$ and

$$\begin{aligned} &= -\varepsilon^{-2} \int_{\omega_2} \int_\alpha^0 |u|^2(x_1,x_2)\, dx_1 dx_2 \\ &= \varepsilon^{-2} \int_{\omega_2} \int_{\omega_1} |u|^2(x_1,x_2)\, dx_1 dx_2 = \varepsilon^{-2} \int_\Omega |u|^2(x)\, dx. \end{aligned}$$

■

Proof. [Proof of Lemma 13] For $v(y) \in C^1(Y_\infty^+)$, we prove that

$$\left(S_b^{0*}v\right)(x) = v\left(\frac{x}{\varepsilon}\right) \text{ and } \left(S_b^{\alpha *}v\right)(x) = v\left(\frac{\alpha - x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right). \tag{8}$$

First, for $\vartheta = 0$, let $w \in L^2(\Omega)$, from the definition (1.32) of S_b^0,

$$\begin{aligned}\int_{Y_\infty^+} v(y) \cdot \left(S_b^0 w\right)(y)\, dy &= \int_{Y_\infty^+} v(y) \cdot w(\varepsilon y)\, \chi_{(0,\alpha/\varepsilon)}(y_1)\, dy \\ &= \int_{Y_2}\int_{(0,\alpha/\varepsilon)} v(y_1, y_2) \cdot w(\varepsilon y_1, \varepsilon y_2)\, dy_1 dy_2\end{aligned}$$

Using the definition (1.35) of the adjoint operator S_b^{0*},

$$\frac{1}{\varepsilon}\int_\Omega \left(S_b^{0*}v\right)(x) \cdot w(x)\, dx = \varepsilon \int_{Y_2}\int_{(0,\alpha/\varepsilon)} v(y_1, y_2) \cdot w(\varepsilon y_1, \varepsilon y_2) dy_1 dy_2$$

and changing the variable names $x_1 = \varepsilon y_1$ and $x_2 = \varepsilon y_2$,

$$= \varepsilon^{-1}\int_{\omega_2}\int_{\omega_1} v\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \cdot w(x_1, x_2)\, dx_1 dx_2.$$

Therefore,

$$\int_\Omega \left(S_b^{0*}v\right)(x) \cdot w(x)\, dx = \int_\Omega v\left(\frac{x}{\varepsilon}\right) \cdot w(x)\, dx. \tag{9}$$

Second, at $\vartheta = \alpha$, let $w \in L^2(\Omega)$, similarly to the case of $\vartheta = 0$, we get

$$\begin{aligned}\int_{Y_\infty^+} v(y) \cdot \left(S_b^\alpha w\right)(y)\, dy &= \int_{Y_\infty^+} v(y) \cdot w(-\varepsilon y_1 + \alpha, \varepsilon y_2)\, \chi_{(0,\alpha/\varepsilon)}(y_1)\, dy \\ &= \int_{Y_2}\int_{(0,\alpha/\varepsilon)} v(y_1, y_2) \cdot w(-\varepsilon y_1 + \alpha, \varepsilon y_2)\, dy_1 dy_2.\end{aligned}$$

So, the definition (1.35) of the adjoint operator $S_b^{\alpha *}$ implies,

$$\frac{1}{\varepsilon}\int_\Omega \left(S_b^{\alpha *}v\right)(x) \cdot w(x)\, dx = \varepsilon \int_{Y_2}\int_{(0,\alpha/\varepsilon)} v(y_1, y_2) \cdot w(-\varepsilon y_1 + \alpha, \varepsilon y_2)\, dy_1 dy_2.$$

By changing the variable names $x_1 = -\varepsilon y_1 + \alpha$ and $x_2 = \varepsilon y_2$, it remains,

$$= \varepsilon^{-1}\int_{\omega_2}\int_{\omega_1} v\left(\frac{\alpha - x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \cdot w(x_1, x_2)\, dx_1 dx_2.$$

Therefore,

$$\int_\Omega \left(S_b^{\alpha *}v\right)(x) \cdot w(x)\, dx = \int_\Omega v\left(\frac{\alpha - x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \cdot w(x_1, x_2)\, dx_1 dx_2. \tag{10}$$

Then, from (9) and (10), the formula (8) follows. Finally, (1.37) is deduced from (8) and (1.36). ■

Remark 57 *Here we explain the the reason why we defined the (n,k)-model two-scale approximation (3.14) of u^ε instead of using $T^{\varepsilon\alpha_n^k}S_k^\varepsilon$ directly as in [36] and [94]. For a given $k \in Y^*$ and $n \in \mathbb{N}^*$, we restart with the very weak formulation (3.62) in the proof of Lemma 34 by choosing test functions as in (3.61, 3.59, 3.60) but such that $w_0 = 0$ in $I \times \Omega$ and $w_1 \in L^2\left(C_c^\infty\left(I \times \Omega\right);\Lambda \times Y\right)$. Multiplying by ε^2, the equation (3.63) is equivalent to,*

$$\int_{I\times\Omega}[u^\varepsilon \cdot \mathfrak{B}_n^k\left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1 + \varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1 + \varepsilon^3 Q^0 w_1 - \varepsilon^3 P^0 w_1\right)$$
$$-\varepsilon^3 f^\varepsilon \cdot \mathfrak{B}_n^k w_1]dtdx = 0.$$

Equivalently,

$$\int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k\left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1 + \varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1\right) dtdx = \varepsilon^2 O\left(\varepsilon\right).$$

According to the relation between $\mathfrak{B}_n^k$ and $T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}$ in Lemma 8, it remains,

$$\int_{I\times\Omega} u^\varepsilon \cdot \left(T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *} + O\left(\varepsilon\right)\right)\left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1 + \varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1\right) dtdx = \varepsilon^2 O\left(\varepsilon\right)$$

Or equivalently,

$$\int_{I\times\Omega} u^\varepsilon \cdot T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}\left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1 + \varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1\right) dtdx = \varepsilon O\left(\varepsilon\right).$$

Then,

$$\int_{I\times\Lambda\times\Omega\times Y} T^{\varepsilon\alpha_n^k}S_k^\varepsilon u^\varepsilon \cdot \left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1 + \varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1\right) dtd\tau dxdy = \varepsilon O\left(\varepsilon\right).$$

Using the decomposition (3.52) of $T^{\varepsilon\alpha_n^k}S_k^\varepsilon u^\varepsilon$, the equation becomes,

$$\int_{I\times\Lambda\times\Omega\times Y}\left(\chi_0\left(k\right)u_n^{0,k} + \varepsilon\overline{u}_n^{1,k}\right)\cdot(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1$$
$$+\varepsilon^2\frac{1}{\alpha_n^k}Q^1 w_1 - \varepsilon^2 P^1 w_1)\, dtd\tau dxdy = \varepsilon O\left(\varepsilon\right).$$

Using (3.82), the equation yields

$$\int_{I\times\Lambda\times\Omega\times Y}\chi_0\left(k\right)u_n^{0,k}\cdot\varepsilon^2 P^1 w_1 + \varepsilon\overline{u}_n^{1,k}\cdot\left(\varepsilon\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - \varepsilon P^2 w_1\right) dtdx = \varepsilon O\left(\varepsilon\right).$$

Finally, dividing by ε^2, we get the equation

$$\int_{I\times\Lambda\times\Omega\times Y}\chi_0\left(k\right)u_n^{0,k}\cdot P^1 w_1 + \overline{u}_n^{1,k}\cdot\left(\left(\frac{1}{\alpha_n^k}\right)^2 Q^2 w_1 - P^2 w_1\right) dtdx = \frac{O\left(\varepsilon\right)}{\varepsilon}$$

but we can not pass to the limit of $\frac{O(\varepsilon)}{\varepsilon}$ when $\varepsilon \to 0$. Therefore, we can not obtain the HF-microscopic equation by applying $T^{\varepsilon\alpha_n^k}S_k^\varepsilon$ to u^ε directly.

Here we also bring the similar result to Lemma 37 about the strong convergence of test function in the case of Neumann Boundary condition. For $k \in Y^*/\left\{0, -\frac{1}{2}\right\}$, $n \in \mathbb{N}^*$ and $\sigma \in I^k$, we consider the two functions $\varphi_n^k(t,x)$, $\varphi_n^{-k}(t,x) \in H^2(I \times \Omega)$ such that

$$\varphi_n^k(t,x)\,\partial_y \phi_n^k(0)\, e^{2i\pi l^k \frac{x}{\alpha}} + \varphi_n^{-k}(t,x)\,\partial_y \phi_n^{-k}(0)\, e^{-2i\pi l^k \frac{x}{\alpha}} = 0 \text{ on } I \times \partial\Omega \tag{11}$$

where l^k is defined in (1.40).

Lemma 58 *For $k \in Y^*/\left\{0, -\frac{1}{2}\right\}$, let $\varepsilon \in E_k$, there exist $\varphi_n^{k,\varepsilon}$, $\varphi_n^{-k,\varepsilon} \in H^2(I \times \Omega)$ satisfying*

i) the boundary conditions

$$\sum_{\sigma \in I^k} \partial_x \varphi_n^{\sigma,\varepsilon}(t,x)\, \phi_n^\sigma(0)\, e^{2i\pi\sigma\frac{x}{\varepsilon}} + \frac{1}{\varepsilon}\varphi_n^{\sigma,\varepsilon}(t,x)\, \partial_y \phi_n^\sigma(0)\, e^{2i\pi\sigma\frac{x}{\varepsilon}} = 0 \ \textit{on}\ I \times \partial\Omega, \tag{12}$$

ii) and the strong convergence

$$\varphi_n^{\sigma,\varepsilon} \to \varphi_n^\sigma \ \textit{in}\ H^2(I \times \Omega) \ \textit{when}\ \varepsilon \to 0 \ \textit{for}\ \sigma \in I^k. \tag{13}$$

Before starting the proof, we denote

$$\zeta^\varepsilon(t,x) = -\varepsilon \sum_{\sigma \in I^k} \partial_x \varphi_n^{\sigma,\varepsilon}(t,x)\, \phi_n^\sigma\left(\frac{x}{\varepsilon}\right) \text{ on } I \times \partial\Omega$$

and remark that $\zeta^\varepsilon(t,x)$ converges to 0 in $H^2(I)$ when ε tends to 0 at $x \in \partial\Omega$. Similarly to the case of Dirichlet boundary condition, to avoid the case that boundary conditions are vanishing, we assume that $\partial_y \phi_n^{-k}(0) \neq 0$.

Proof. For any $\varepsilon \in E_k$ and let the two functions $\varphi_n^k(t,x)$, $\varphi_n^{-k}(t,x) \in H^2(I \times \Omega)$ satisfy (11), we choose

$$\begin{aligned} \varphi_n^{k,\varepsilon}(t,x) &= \varphi_n^k(t,x) \in H^2(I \times \Omega) \\ \text{and } \varphi_n^{-k,\varepsilon}(t,x) &= \varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x) \ \text{ where } \mu^\varepsilon(t,x) \in H^2(I \times \Omega). \end{aligned} \tag{14}$$

i) Let us prove that

$$\mu^\varepsilon(t,x) = -\left(\varphi_n^{-k}(t,\alpha)\left(1 - e^{4i\pi\left(l_k^\varepsilon - l^k\right)}\right) + \frac{\zeta^\varepsilon(t,\alpha)\, e^{2i\pi l_k^\varepsilon}}{\partial_y \phi_n^{-k}(0)}\right)\frac{x}{\alpha} + \frac{\zeta^\varepsilon(t,0)}{\partial_y \phi_n^{-k}(0)}$$

where l_ε^k and l^k is defined in (1.39) and (1.40).

Replacing (14) in (12), the boundary conditions are

$$\varphi_n^k(t,x)\,\partial_y\phi_n^k(0)\, e^{2i\pi k\frac{x}{\varepsilon}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\partial_y \phi_n^{-k}(0)\, e^{-2i\pi k\frac{x}{\varepsilon}} = \zeta^\varepsilon(t,x) \ \text{ on } I \times \partial\Omega.$$

Using (1.39) and (1.40) with remarking that $e^{2i\pi h_k^\varepsilon \frac{x}{\alpha}} = 1$ at $x \in \partial\Omega$,

$$\varphi_n^k(t,x)\,\partial_y\phi_n^k(0)\, e^{2i\pi l_k^\varepsilon\frac{x}{\alpha}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\partial_y \phi_n^{-k}(0)\, e^{-2i\pi l_k^\varepsilon\frac{x}{\alpha}} = \zeta^\varepsilon(t,x) \ \text{ on } I \times \partial\Omega.$$

Or equivalently,

$$\varphi_n^k(t,x)\,\partial_y\phi_n^k(0)\, e^{2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} + \left(\varphi_n^{-k}(t,x) + \mu^\varepsilon(t,x)\right)\partial_y \phi_n^{-k}(0)\, e^{-2i\pi\left(l^k + l_\varepsilon^k - l^k\right)\frac{x}{\alpha}} = \zeta^\varepsilon(t,x)$$

on $I \times \partial\Omega$. From (11),

$$\varphi_n^k(t,x)\,\partial_y\phi_n^k(0)\,e^{2i\pi l^k\frac{x}{\alpha}} = -\varphi_n^{-k}(t,x)\,\partial_y\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}} \text{ on } I \times \partial\Omega.$$

After replacement, the equation remains,

$$\begin{aligned}&\varphi_n^{-k}(t,x)\,\partial_y\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}\left(e^{-2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}} - e^{2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}}\right)\\ &+\mu^\varepsilon(t,x)\,\partial_y\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}e^{-2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}} = \zeta^\varepsilon(t,x) \text{ on } I \times \partial\Omega.\end{aligned}$$

Since $\partial_y\phi_n^{-k}(0) \neq 0$ and $e^{-2i\pi l^k\frac{x}{\alpha}} \neq 0$ at $x \in \partial\Omega$, then the function μ^ε is defined at $x \in \partial\Omega$ as,

$$\begin{aligned}\mu^\varepsilon(t,x) = &-\varphi_n^{-k}(t,x)\left(e^{-2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}} - e^{2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}}\right)e^{2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}}\\ &+\frac{\zeta^\varepsilon(t,x)}{\partial_y\phi_n^{-k}(0)\,e^{-2i\pi l^k\frac{x}{\alpha}}e^{-2i\pi\left(l_k^\varepsilon-l^k\right)\frac{x}{\alpha}}} \text{ on } I \times \partial\Omega,\end{aligned}$$

i.e.,

$$\mu^\varepsilon(t,0) = \frac{\zeta^\varepsilon(t,0)}{\partial_y\phi_n^{-k}(0)} \text{ and } \mu^\varepsilon(t,\alpha) = -\varphi_n^{-k}(t,\alpha)\left(1 - e^{4i\pi\left(l_k^\varepsilon-l^k\right)}\right) + \frac{\zeta^\varepsilon(t,\alpha)\,e^{2i\pi l_k^\varepsilon}}{\partial_y\phi_n^{-k}(0)}.$$

Finally, we choose the function $\mu^\varepsilon \in H^2(I \times \Omega)$ by

$$\mu^\varepsilon(t,x) = -\left(\varphi_n^{-k}(t,\alpha)\left(1 - e^{4i\pi\left(l_k^\varepsilon-l^k\right)}\right) + \frac{\zeta^\varepsilon(t,\alpha)\,e^{2i\pi l_k^\varepsilon}}{\partial_y\phi_n^{-k}(0)}\right)\frac{x}{\alpha} + \frac{\zeta^\varepsilon(t,0)}{\partial_y\phi_n^{-k}(0)}.$$

ii) For $\sigma = k$, the strong convergence is true since $\varphi_n^{k,\varepsilon}$ is independent on ε. For $\sigma = -k$, the strong convergence of $\mu^\varepsilon(t,x)$ in $H^2(I \times \Omega)$ is trivial, i.e. $\mu^\varepsilon(t,x) \to 0$ in $H^2(I \times \Omega)$ strongly when $\varepsilon \to 0$. Therefore, $\varphi_n^{-k,\varepsilon} \to \varphi_n^{-k}$ in $H^2(I \times \Omega)$ strongly when $\varepsilon \to 0$. ■

Finally, an example is provided of a sequence ε corresponding to Assumption 14.

Example 59 *i) For a given $\varepsilon_0 \in \mathbb{R}^+$, according to (1.39), $\frac{\alpha k}{\varepsilon_0}$ is decomposed as,*

$$\frac{\alpha k}{\varepsilon_0} = h_{\varepsilon_0}^k + l_{\varepsilon_0}^k \text{ with } h_{\varepsilon_0}^k = \left[\frac{\alpha k}{\varepsilon_0}\right] \text{ and } l_{\varepsilon_0}^k \in [0,1). \tag{15}$$

For a subsequence ε_n, we can decompose

$$\frac{\alpha k}{\varepsilon_n} = h_{\varepsilon_n}^k + l_{\varepsilon_n}^k$$

Here we need to choose a subsequence ε_n such that $l_{\varepsilon_n}^k = l_{\varepsilon_0}^k + O(\varepsilon_n)$. Choosing a subsequence $\varepsilon_n = \frac{\varepsilon_0}{n}$ for $n \in \mathbb{N}^*$,

$$\frac{\alpha k}{\varepsilon_n} = \frac{\alpha k}{\frac{\varepsilon_0}{n}} = nh_{\varepsilon_0}^k + nl_{\varepsilon_0}^k,$$

hence, a sequence n satisfies

$$nl_{\varepsilon_0}^k = n' + l_{\varepsilon_0}^k + O(\varepsilon_0) \text{ or } (n-1)\,l_{\varepsilon_0}^k = n' + O(\varepsilon_0) \text{ with } n' \in \mathbb{N}.$$

We approximate $l_{\varepsilon_0}^k$ *by a fraction* $\frac{r}{s} \in \mathbb{Q}$, $\frac{r}{s} \geq 0$, *i.e.* $l_{\varepsilon_0}^k = \frac{r}{s} + O(\varepsilon_0)$. *It is equivalent to,*

$$(n-1)\left(\frac{r}{s} + O(\varepsilon_0)\right) = n' + O(\varepsilon_0) \text{ with } n' \in \mathbb{N}.$$

If $l_{\varepsilon_0}^k = \frac{r}{s}$ *then*

$$(n-1)\frac{r}{s} = n' + O(\varepsilon_0) \text{ with } n' \in \mathbb{N},$$

therefore,

$$n = n'\frac{s}{r} + 1 + \frac{s}{r}O(\varepsilon_0) \text{ with } n' \in \mathbb{N}.$$

Since $n \in \mathbb{N}^*$ *so* $n'\frac{s}{r} + 1 + \frac{s}{r}O(\varepsilon_0) \in \mathbb{N}^*$ *then* $n'\frac{s}{r} \approx n_0 \in \mathbb{N}$.

If $l_{\varepsilon_0}^k = \frac{r}{s} + O(\varepsilon_0)$ *then*

$$n = \frac{n' + O(\varepsilon_0)}{\frac{r}{s} + O(\varepsilon_0)} + 1 = \frac{n'}{\frac{r}{s} + O(\varepsilon_0)} + 1 + O(\varepsilon_0) \text{ with } n' \in \mathbb{N}.$$

Since $n \in \mathbb{N}^*$ *so* $\frac{n'}{\frac{r}{s}+O(\varepsilon_0)} + 1 \approx n_0 \in \mathbb{N}$ *with* $n' \in \mathbb{N}$.

ii) For a given ε_0, *a given* $k \in Y^*$ *and*

$$\frac{\alpha k}{\varepsilon_0} = p(\varepsilon_0) + l(\varepsilon_0) \text{ with } p(\varepsilon_0) = \left[\frac{\alpha k}{\varepsilon_0}\right] \text{ and } l(\varepsilon_0) = 0.2 = \frac{1}{5},$$

then a sequence n *is chosen such that*

$$n = 5n' + 1 + 5O(\varepsilon_0) \text{ and } 5n' \approx n_0 \in \mathbb{N} \text{ with } n' \in \mathbb{N}.$$

So,

$$n' = 0 \text{ so } n = 1.$$

$$n' = 1 \text{ so } n = 6.$$

$$n' = 2 \text{ so } n = 11.$$

$$....$$

Finally, the sequence ε_n *can be chosen by* $\varepsilon_n = \frac{\varepsilon_0}{n}$ *with* $n = 1 + 5n'$ *for* $n' \in \mathbb{N}$.

Bibliography

[1] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[2] Grégoire Allaire. Two-scale convergence and homogenization of periodic structures. *School on homogenization, ICTP, Trieste*, 1993.

[3] Grégoire Allaire. Dispersive limits in the homogenization of the wave equation. In *Annales de la Faculté des Sciences de Toulouse*, volume 12, pages 415–431. Université Paul Sabatier, 2003.

[4] Grégoire Allaire and Micol Amar. Boundary layer tails in periodic homogenization. *ESAIM: Control, Optimisation and Calculus of Variations*, 4:209–243, 1999.

[5] Grégoire Allaire and Guillaume Bal. Homogenization of the criticality spectral equation in neutron transport. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(04):721–746, 1999.

[6] Grégoire Allaire, Yves Capdeboscq, Marjolaine Puel, et al. Homogenization of a one-dimensional spectral problem for a singularly perturbed elliptic operator with neumann boundary conditions. *DCDS-B*, 2012.

[7] Grégoire Allaire and Carlos Conca. Bloch-wave homogenization for a spectral problem in fluid-solid structures. *Archive for rational mechanics and analysis*, 135(3):197–257, 1996.

[8] Grégoire Allaire and Carlos Conca. Bloch wave homogenization and spectral asymptotic analysis. *Journal de mathématiques pures et appliquées*, 77(2):153–208, 1998.

[9] Grégoire Allaire and Carlos Conca. Boundary layers in the homogenization of a spectral problem in fluid–solid structures. *SIAM Journal on Mathematical Analysis*, 29(2):343–379, 1998.

[10] Grégoire Allaire, Mariapia Palombaro, and Jeffrey Rauch. Diffractive behavior of the wave equation in periodic media: weak convergence analysis. *Annali di Matematica Pura ed Applicata*, 188(4):561–589, 2009.

[11] Igor V Andrianov, Vladimir I Bolshakov, Vladyslav V Danishevs' kyy, and Dieter Weichert. Higher order asymptotic homogenization and wave propagation in periodic composite materials. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 464(2093):1181–1201, 2008.

[12] Andrei Andryieuski, Sangwoo Ha, Andrey A Sukhorukov, Yuri S Kivshar, and Andrei V Lavrinenko. Bloch-mode analysis for retrieving effective parameters of metamaterials. *Physical Review B*, 86(3):035127, 2012.

[13] T Antonakakis, RV Craster, and S Guenneau. Asymptotics for metamaterials and photonic crystals. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 469(2152):20120533, 2013.

[14] Ivo Babuska. Solution of problem with interfaces and singularities. *Mathematical aspects of finite elements in partial differential equations*, pages 213–277, 1974.

[15] Ivo Babuška. Solution of interface problems by homogenization. i. *SIAM Journal on Mathematical Analysis*, 7(5):603–634, 1976.

[16] Ivo Babuška. Solution of interface problems by homogenization. ii. *SIAM Journal on Mathematical Analysis*, 7(5):635–645, 1976.

[17] Ivo Babuška. Solution of interface problems by homogenization. iii. *SIAM Journal on Mathematical Analysis*, 8(6):923–937, 1977.

[18] L Baffico, C Conca, and M Rajesh. Homogenization of a class of nonlinear eigenvalue problems. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 136(01):7–22, 2006.

[19] Nikolaï Bakhvalov and G Panasenko. *Homogenisation averaging processes in periodic media*. Springer, 1989.

[20] NS Bakhvalov and ME Eglit. Equations of higher order of accuracy describing the vibrations of thin plates. *Journal of applied mathematics and mechanics*, 69(4):593–610, 2005.

[21] Guillaume Bal. Homogenization of a spectral equation with drift in linear transport. *ESAIM: Control, Optimisation and Calculus of Variations*, 6:613–627, 2001.

[22] VD Belov and SA Rybak. Applicability of transport-equation on one-dimensional wave-propagation problem. *Soviet Physics Acoustics-Ussr*, 21(2):110–114, 1975.

[23] VD Belov, SA Rybak, and BD Tartakovskii. Propagation of vibrational energy in absorbing structures. *Soviet Physics Acoustics-Ussr*, 23(2):115–119, 1977.

[24] Henri Berestycki, François Hamel, and Nikolai Nadirashvili. Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Communications in mathematical physics*, 253(2):451–480, 2005.

[25] YANG Bin, Walid Belkhir, and Michel Lenczner. Computer-aided derivation of multiscale models: A rewriting framework. *International Journal for Multiscale Computational Engineering*, 12(2), 2014.

[26] M Sh Birman and TA Suslina. On homogenization procedure for periodic operators near the edge of an internal gap. *St petersburg Mathematical journal c/c of algebra I analiz*, 15(4):507–514, 2004.

[27] M Sh Birman and TA Suslina. Homogenization of a multidimensional periodic elliptic operator in a neighborhood of the edge of an internal gap. *Journal of Mathematical Sciences*, 136(2):3682–3690, 2006.

[28] F Blanc and SA Nazarov. Asymptotics of solutions to the poisson problem in a perforated domain with corners. *Journal de mathématiques pures et appliquées*, 76(10):893–911, 1997.

[29] Felix Bloch. Über die quantenmechanik der elektronen in kristallgittern. *Zeitschrift für physik*, 52(7-8):555–600, 1929.

[30] OM Bouthier and RJ Bernhard. Simple models of the energetics of transversely vibrating plates. *Journal of Sound and Vibration*, 182(1):149–164, 1995.

[31] C Boutin. Microstructural influence on heat conduction. *International journal of heat and mass transfer*, 38(17):3181–3195, 1995.

[32] C Boutin and JL Auriault. Rayleigh scattering in elastic composite materials. *International journal of engineering science*, 31(12):1669–1689, 1993.

[33] S Brahim-Otsmane, GA Francfort, and F Murat. Correctors for the homogenization of the wave and heat equations. *Journal de mathématiques pures et appliquées*, 71(3):197–231, 1992.

[34] Andrea Braides, Giuseppe Riey, and Margherita Solci. Homogenization of penrose tilings. *Comptes Rendus Mathematique*, 347(11):697–700, 2009.

[35] Matthieu Brassart and Michel Lenczner. A two-scale model for the wave equation with oscillating coefficients and data. *Comptes Rendus Mathematique*, 347(23):1439–1442, 2009.

[36] Matthieu Brassart and Michel Lenczner. A two-scale model for the periodic homogenization of the wave equation. *Journal de mathématiques pures et appliquées*, 93(5):474–517, 2010.

[37] J Casado-Diaz. Two-scale convergence for nonlinear dirichlet problems in perforated domains. In *Proceedings Section A: Mathematics-Royal Society of Edinburgh*, volume 130, pages 249–276. Royal Society of Edinburgh, 2000.

[38] Juan Casado-Díaz, Julio Couce-Calvo, Faustino Maestre, and José D Martín-Gómez. Homogenization and corrector for the wave equation with discontinuous coefficients in time. *Journal of Mathematical Analysis and Applications*, 379(2):664–681, 2011.

[39] Juan Casado-Díaz, Julio Couce-Calvo, Faustino Maestre, and José d Martín Gómez. Homogenization and correctors for the wave equation with periodic coefficients. *Mathematical Models and Methods in Applied Sciences*, pages 1–46, 2013.

[40] Juan Casado-Díaz, Julio Couce-Calvo, Faustino Maestre, and José D Martín-Gómez. A corrector for a wave problem with periodic coefficients in a 1d bounded domain. *To appear in ESAIM: Control, Optimisation and Calculus of Variations*, 2014.

[41] Thierry Champion and Luigi De Pascale. Asymptotic behaviour of nonlinear eigenvalue problems involving p-laplacian-type operators. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 137(06):1179–1195, 2007.

[42] Wen Chen and Jacob Fish. A dispersive model for wave propagation in periodic heterogeneous media based on homogenization with multiple spatial and temporal scales. *Journal of applied mechanics*, 68(2):153–161, 2001.

[43] Wen Chen, Jacob Fish. Uniformly valid multiple spatial-temporal scale modeling for wave propagation in heterogeneous media. *Mechanics of composite materials and structures*, 8(2):81–99, 2001.

[44] D. Cioranescu and P. Donato. *An introduction to Homogenization*. Oxford Lecture Series inMathematics and its applications 17, Oxford University Press, 1999.

[45] Doina Cioranescu, Alain Damlamian, and Georges Griso. Periodic unfolding and homogenization. *Comptes Rendus Mathematique*, 335(1):99–104, 2002.

[46] Doina Cioranescu, Alain Damlamian, and Georges Griso. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis*, 40(4):1585–1620, 2008.

[47] Ferruccio Colombini and Sergio Spagnolo. On the convergence of solutions of hyperbolic equations. *Communications in Partial Differential Equations*, 3(1):77–103, 1978.

[48] C Conca, R Orive, and M Vanninathan. Bloch approximation in homogenization on bounded domains. *Asymptotic Analysis*, 41(1):71–91, 2005.

[49] C Conca, J Planchard, and M Vanninathan. Limiting behaviour of a spectral problem in fluid-solid structures. *Asymptotic analysis*, 6(4):365–389, 1993.

[50] Carlos Conca, Rafael Orive, and Muthusamy Vanninathan. Bloch approximation in homogenization and applications. *SIAM journal on mathematical analysis*, 33(5):1166–1198, 2002.

[51] Carlos Conca and Muthusamy Vanninathan. Homogenization of periodic structures via bloch decomposition. *SIAM Journal on Applied Mathematics*, 57(6):1639–1659, 1997.

[52] M Conca and M Vanninathan. A spectral problem arising in fluid-solid structures. *Computer methods in applied mechanics and engineering*, 69(2):215–242, 1988.

[53] RV Craster, J Kaplunov, and AV Pichugin. High-frequency homogenization for periodic media. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2120):2341–2362, 2010.

[54] Wim Desmet and D Vandepitte. Mid-frequency vibro-acoustic modelling: challenges and potential solutions. In *Proceedings of ISMA*, volume 2, page 12. Citeseer, 2002.

[55] Hermann Douanla. *Two-scale Convergence and Homogenization of Some Partial Differential Equations.* Chalmers University of Technology, 2013.

[56] Jacob Fish and Wen Chen. Higher-order homogenization of initial/boundary-value problem. *Journal of engineering mechanics*, 127(12):1223–1230, 2001.

[57] Jacob Fish and Wen Chen. Space-time multiscale model for wave propagation in heterogeneous media. *Computer Methods in applied mechanics and engineering*, 193(45):4837–4856, 2004.

[58] Jacob Fish, Wen Chen, and Gakuji Nagai. Non-local dispersive model for wave propagation in heterogeneous media: multi-dimensional case. *International Journal for Numerical Methods in Engineering*, 54(3):347–363, 2002.

[59] Jacob Fish, Vasilina Filonova, and Sergey Kuznetsov. Micro-inertia effects in nonlinear heterogeneous media. *International Journal for Numerical Methods in Engineering*, 91(13):1406–1426, 2012.

[60] Gilles A Francfort and François Murat. Oscillations and energy densities in the wave equation. *Communications in partial differential equations*, 17(11-12):1785–1865, 1992.

[61] Jan Francu and Pavel Krejčí. Homogenization of scalar wave equations with hysteresis. *Continuum Mechanics and Thermodynamics*, 11(6):371–390, 1999.

[62] David Gérard-Varet and Nader Masmoudi. Homogenization and boundary layers. *Acta mathematica*, pages 1–46, 2012.

[63] Ulrich Hornung. *Homogenization and porous media*, volume 6. Springer, 1997.

[64] MN Ichchou, A Le Bot, and L Jezequel. Energy models of one-dimensional, multi-propagative systems. *Journal of Sound and Vibration*, 201(5):535–554, 1997.

[65] Mahamane Kader. Contributions á la modélisation et contrôle des systémes intelligents distribués: Application au contrôle de vibrations d'une poutre (phd thesis université de franche-comté). 2000.

[66] Satoshi Kaizu. Homogenization of eigenvalue problems for the laplace operator with nonlinear terms in domains in many tiny holes. *Nonlinear Analysis: Theory, Methods & Applications*, 28(2):377–391, 1997.

[67] SK Kanaun and VM Levin. Self-consistent methods in the problem of axial elastic shear wave propagation through fiber composites. *Archive of Applied Mechanics*, 73(1-2):105–130, 2003.

[68] SK Kanaun, VM Levin, and FJ Sabina. Propagation of elastic waves in composites with random set of spherical inclusions (effective medium approach). *Wave Motion*, 40(1):69–88, 2004.

[69] Srinivasan Kesavan. Homogenization of elliptic eigenvalue problems: Part 1. *Applied Mathematics and Optimization*, 5(1):153–167, 1979.

[70] Srinivasan Kesavan. Homogenization of elliptic eigenvalue problems: Part 2. *Applied Mathematics and Optimization*, 5(1):197–216, 1979.

[71] Peter Kuchment. On some spectral problems of mathematical physics. *Partial Differential Equations and Inverse Problems, Contemp. Math*, 362:241–276, 2004.

[72] Pierre Ladevèze, Lionel Arnaud, Philippe Rouch, and Claude Blanzé. The variational theory of complex rays for the calculation of medium-frequency vibrations. *Engineering Computations*, 18(1/2):193–214, 2001.

[73] Agnes Lamacz. Dispersive effective models for waves in heterogeneous media. *Mathematical Models and Methods in Applied Sciences*, 21(09):1871–1899, 2011.

[74] Y Lase, MN Ichchou, and L Jezequel. Energy flow analysis of bars and beams: theoretical formulations. *Journal of Sound and Vibration*, 192(1):281–305, 1996.

[75] Gilles Lebeau. The wave equation with oscillating density: observability at low frequency. *ESAIM: Control, Optimisation and Calculus of Variations*, 5:219–258, 2000.

[76] M Lenczner. Multiscale model for atomic force microscope array mechanical behavior. *Applied Physics Letters*, 90(9):091908–091908, 2007.

[77] Michel Lenczner. Homogénéisation d'un circuit électrique. *Comptes Rendus de l'Académie des Sciences-Series IIB-Mechanics-Physics-Chemistry-Astronomy*, 324(9):537–542, 1997.

[78] Michel Lenczner. Homogenization of linear spatially periodic electronic circuits. *Networks and Heterogeneous Media*, 1:467–494, 2006.

[79] Michel Lenczner and Ghouti Senouci-Bereksi. Homogenization of electrical networks including voltage-to-voltage amplifiers. *Mathematical Models and Methods in Applied Sciences*, 9(06):899–932, 1999.

[80] Michel Lenczner and Ralph C Smith. A two-scale model for an array of afm's cantilever in the static case. *Mathematical and computer modelling*, 46(5):776–805, 2007.

[81] Dag Lukkassen, Gabriel Nguetseng, and Peter Wall. Two-scale convergence. *Int. J. Pure Appl. Math*, 2(1):35–86, 2002.

[82] Shari Moskow and Michael Vogelius. First-order corrections to the homogenised eigenvalues of a periodic composite medium. a convergence proof. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 127(06):1263–1299, 1997.

[83] Shari Moskow and Michael Vogelius. First order corrections to the homogenized eigenvalues of a periodic composite medium. the case of neumann boundary conditions. *Indiana Univ. Math. J., to appear*, 1997.

[84] SA Nazarov. Asymptotics of negative eigenvalues of the dirichlet problem with the density changing sign. *Journal of Mathematical Sciences*, 163(2):151–175, 2009.

[85] SA Nazarov and AL Pyatnitskii. Homogenization of the spectral dirichlet problem for a system of differential equations with rapidly oscillating coefficients and changing sign density. *Journal of Mathematical Sciences*, 169(2):212–248, 2010.

[86] Sergey A Nazarov, Iryna L Pankratova, and Andrey L Piatnitski. Homogenization of the spectral problem for periodic elliptic operators with sign-changing density function. *Archive for rational mechanics and analysis*, 200(3):747–788, 2011.

[87] Sia Nemat-Nasser, John R Willis, Ankit Srivastava, and Alireza V Amirkhizi. Homogenization of periodic elastic composites and locally resonant sonic materials. *Physical Review B*, 83(10):104103, 2011.

[88] Maria Neuss-Radu. The boundary behavior of a composite material. *ESAIM: Mathematical Modelling and Numerical Analysis*, 35(03):407–435, 2001.

[89] Gabriel Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20(3):608–623, 1989.

[90] Gabriel Nguetseng. Asymptotic analysis for a stiff variational problem arising in mechanics. *SIAM journal on mathematical analysis*, 21(6):1394–1414, 1990.

[91] Gabriel Nguetseng. Homogenization structures and applications i. *Zeitschrift für Analysis und ihre Anwendungen*, 22(1):73–108, 2003.

[92] Gabriel Nguetseng. Homogenization in perforated domains beyond the periodic setting. *Journal of mathematical analysis and applications*, 289(2):608–628, 2004.

[93] Gabriel Nguetseng. Homogenization structures and applications ii. *Zeitschrift für Analysis und ihre Anwendungen*, 23(3):483–508, 2004.

[94] Thi Trang Nguyen, Michel Lenczner, and Matthieu Brassart. Homogenization of the one-dimensional wave equation. *arXiv preprint arXiv:1312.0654*, 2013.

[95] Thi Trang Nguyen, Michel Lenczner, and Matthieu Brassart. Homogenization of the spectral equation in one dimension. *International Journal for Multiscale Computational Engineering*, 12(5):423–450, 2014.

[96] Daniel Onofrei and Bogdan Vernescu. Asymptotic analysis of boundary layer correctors and applications. *arXiv preprint arXiv:0812.4788*, 2008.

[97] R Orive, E Zuazua, and AF Pazoto. Asymptotic expansion for damped wave equations with periodic coefficients. *Mathematical Models and Methods in Applied Sciences*, 11(07):1285–1310, 2001.

[98] I Pankratova and A Piatnitski. Homogenization of spectral problem for locally periodic elliptic operators with sign-changing density function. *Journal of Differential Equations*, 250(7):3088–3134, 2011.

[99] G Papanicolau, A Bensoussan, and J-L Lions. *Asymptotic analysis for periodic structures*, volume 5. North Holland, 1978.

[100] Christophe Prange. Asymptotic analysis of boundary layer correctors in periodic homogenization. *SIAM Journal on Mathematical Analysis*, 45(1):345–387, 2013.

[101] Christophe Prange. First-order expansion for the dirichlet eigenvalues of an elliptic system with oscillating coefficients. *Asymptotic Analysis*, 83(3):207–235, 2013.

[102] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics: Vol.: 4.: Analysis of Operators*. Academic Press, 1978.

[103] Federico J Sabina and JR Willis. A simple self-consistent analysis of wave propagation in particulate composites. *Wave Motion*, 10(2):127–142, 1988.

[104] Jacqueline Sanchez-Hubert and Évariste Sanchez-Palencia. *Vibration and coupling of continuous systems: asymptotic methods*. Springer Berlin, 1989.

[105] Fadil Santosa and William W Symes. A dispersive effective medium for wave propagation in periodic composites. *SIAM Journal on Applied Mathematics*, 51(4):984–1005, 1991.

[106] Fadil Santosa and Michael Vogelius. First-order corrections to the homogenized eigenvalues of a periodic composite medium. *SIAM Journal on Applied Mathematics*, 53(6):1636–1668, 1993.

[107] Fadil Santosa and Michael Vogelius. First-order corrections to the homogenized eigenvalues of a periodic composite medium. *SIAM Journal on Applied Mathematics*, 53(6):1636–1668, 1993.

[108] Philipp Emanuel Stelzig. *On problems in homogenization and two-scale convergence*. PhD thesis, University of Trento, 2012.

[109] Luc Tartar. *The general theory of homogenization: a personalized introduction*, volume 7. Springer, 2009.

[110] M Vanninathan. Homogenization of eigenvalue problems in perforated domains. In *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, volume 90, pages 239–271. Springer, 1981.

[111] Calvin H Wilcox. Theory of bloch waves. *Journal d'analyse mathématique*, 33(1):146–167, 1978.

Résumé :

Dans cette thèse, nous présentons des résultats d'homogénéisation périodique d'un problème spectral et de l'équation d'ondes avec des coefficients périodiques variant rapidement dans un domaine borné. Le comportement asymptotique est étudié en se basant sur une méthode d'homogénéisation par ondes de Bloch. Il permet de modéliser les ondes à basse et haute fréquences. La partie du modèle à basse fréquence est bien connu et n'est pas donc abordée dans ce travail. A contrario, la partie à haute fréquence du modèle, qui représente des oscillations aux échelles microscopiques et macroscopiques, est un problème laissé ouvert. En particulier, les conditions aux limites de l'équation macroscopique à hautes fréquences établies dans [36] n'étaient pas connues avant le début de la thèse. Ce dernier travail apporte trois contributions principales. Les deux premières contributions, portent sur le comportement asymptotique du problème d'homogénéisation périodique du problème spectral et de l'équation des ondes en une dimension. La troisième contribution consiste en une extension du modèle du problème spectral posé dans une bande mince bidimensionnelle et bornée. Le résultat d'homogénéisation comprend des effets de couche limite qui se produisent dans les conditions aux limites de l'équation macroscopique à haute fréquence.

Mots-clés : Homogénéisation, Ondes de Bloch, Décomposition en ondes de Bloch, Problème spectral, Equation des ondes, Transformée à deux-échelles, Convergence à deux échelles, Méthode d'éclatement périodique, Couches limites, Transformation à deux échelles pour des couche limites.

Abstract:

In this dissertation, we present the periodic homogenization of a spectral problem and the wave equation with periodic rapidly varying coefficients in a bounded domain. The asymptotic behavior is addressed based on a method of Bloch wave homogenization. It allows modeling both the low and high frequency waves. The low frequency part is well-known and it is not a new point here. In the opposite, the high frequency part of the model, which represents oscillations occurring at the microscopic and macroscopic scales, was not well understood. Especially, the boundary conditions of the high-frequency macroscopic equation established in [36] were not known prior to the commencement of thesis. The latter brings three main contributions. The first two contributions, are about the asymptotic behavior of the periodic homogenization of the spectral problem and wave equation in one-dimension. The third contribution consists in an extension of the model for the spectral problem to a thin two-dimensional bounded strip $\Omega = (0,\alpha)\times(0,\varepsilon) \subset \mathbb{R}^2$. The homogenization result includes boundary layer effects occurring in the boundary conditions of the high-frequency macroscopic equation.

Keywords: Homogenization, Bloch waves, Bloch wave decomposition, Spectral problem, Wave equation, Two-scale transform, Two-scale convergence, Unfolding method, Boundary layers, Boundary layer two-scale transform, Macroscopic equation, Microscopic equation, Boundary conditions.

SPIM

■ École doctorale SPIM 16 route de Gray F - 25030 Besançon cedex

■ tél. +33 (0)3 81 66 66 02 ■ **ed-spim@univ-fcomte.fr** ■ www.ed-spim.univ-fcomte.fr

UFC UNIVERSITÉ DE FRANCHE-COMTÉ